Kragujevac Journal of Mathematics
Volume 39(2) (2015), Pages 193–205.

LAPLACIAN ENERGY OF UNION AND CARTESIAN PRODUCT AND LAPLACIAN EQUIENERGETIC GRAPHS

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Abstract. The Laplacian energy of a graph G with n vertices and m edges is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|$, where $\mu_1, \mu_2, \dots, \mu_n$ are the Laplacian eigenvalues of G. If two graphs G_1 and G_2 have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. Otherwise, this identity is violated. We determine a term Ξ, such that $LE(G_1) + LE(G_2) - \Xi \leq LE(G_1 \cup G_2) \leq LE(G_1) + LE(G_2) + \Xi$ holds for all graphs. Further, by calculating LE of the Cartesian product of some graphs, we construct new classes of Laplacian non-cospectral, Laplacian equienergetic graphs.

1. Introduction

Let G be a finite, simple, undirected graph with n vertices $v_1, v_2, \dots, v_n$ and m edges. In what follows, we say that G is an (n,m)-graph. Let $A(G)$ be the adjacency matrix of G and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues.

Let $D(G)$ be the diagonal matrix whose (i,i)-th entry is the degree of a vertex v_i. The matrix $C(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The *Laplacian polynomial* of G is defined as $\psi(G,\mu) = \det[\mu I - C(G)]$, where I is an identity matrix. The eigenvalues of $C(G)$, denoted by $\mu_i = \mu_i(G)$, $i = 1, 2, \dots, n$, are called the *Laplacian eigenvalues* of G [16]. Two graphs are said to be *Laplacian cospectral* if they have same Laplacian eignevalues. The adjacency eigenvalues and Laplacian eigenvalues satisfies the following conditions:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \mu_i = 2m.$$

Key words and phrases. Laplacian spectrum, Laplacian energy, Cartesian product, Laplacian equienergetic graphs.

2010 *Mathematics Subject Classification.* Primary: 05C50, 05C75
Received: September 22, 2015.
Accepted: November 11, 2015.

The *energy* of a graph G is defined as

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

It was introduced by one of the present authors in the 1970s, and since then has been much studied in both chemical and mathematical literature. For details see the book [15] and the references cited therein.

The *Laplacian energy* of a graph was introduced a few years ago [13] and is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i(G) - \frac{2m}{n} \right|.$$

This definition is chosen so as to preserve the main features of the ordinary graph energy $\mathcal{E}$, see [18]. Basic properties and other results on Laplacian energy can be found in the survey [1], the recent papers [6–8, 11, 12, 17, 19], and the references cited therein.

2. Laplacian Energy of Union of Graphs

Let G_1 and G_2 be two graphs with disjoint vertex sets. Let for $i = 1, 2$, the vertex and edges sets of G_i be, respectively, V_i and E_i. The *union* of G_1 and G_2 is a graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If G_1 is an (n_1, m_1)-graph and G_2 is an (n_2, m_2)-graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges. It is easy to see that the Laplacian spectrum of $G_1 \cup G_2$ is the union of the Laplacian spectra of G_1 and G_2.

In [13] it was proven that if G_1 and G_2 have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. If the average vertex degrees are not equal, that is $\frac{2m_1}{n_1} \neq \frac{2m_2}{n_2}$, then it may be either $LE(G_1 \cup G_2) > LE(G_1) + LE(G_2)$ or $LE(G_1 \cup G_2) < LE(G_1) + LE(G_2)$ or, exceptionally, $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$ [13].

In this section we study the Laplacian establish some additional relations between $LE(G_1 \cup G_2)$ and $LE(G_1) + LE(G_2)$.

Theorem 2.1. *Let G_1 be an (n_1, m_1)-graph and G_2 be an (n_2, m_2)-graph, such that $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$. Then*

$$\begin{aligned} LE(G_1) + LE(G_2) - \frac{4(n_2 m_1 - n_1 m_2)}{n_1 + n_2} &\leq LE(G_1 \cup G_2) \\ &\leq LE(G_1) + LE(G_2) + \frac{4(n_2 m_1 - n_1 m_2)}{n_1 + n_2}. \end{aligned} \tag{2.1}$$

Proof. For the sake of simplicity, denote $G_1 \cup G_2$ by G. Then G is an (n_1+n_2, m_1+m_2)-graphs. By the definition of Laplacian energy,

$$\begin{aligned}
LE(G_1 \cup G_2) &= \sum_{i=1}^{n_1+n_2} \left| \mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2} \right| + \sum_{i=n_1+1}^{n_1+n_2} \left| \mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2(m_1+m_2)}{n_1+n_2} \right| + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} + \frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&\quad + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} + \frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&\leq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| + n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2} \right| \\
&\quad + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} \right| + n_2 \left| \frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2} \right| . \qquad (2.2)
\end{aligned}$$

Since $n_2 m_1 > n_1 m_2$, Eq. (2.2) becomes

$$\begin{aligned}
LE(G_1 \cup G_2) &\leq LE(G_1) + n_1 \left(\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2} \right) \\
&\quad + LE(G_2) + n_2 \left(-\frac{2m_2}{n_2} + \frac{2(m_1+m_2)}{n_1+n_2} \right) \\
&= LE(G_1) + LE(G_2) + \frac{4(n_2 m_1 - n_1 m_2)}{n_1+n_2}
\end{aligned}$$

which is an upper bound.

To obtain the lower bound, we just have to note that in full analogy to the above arguments,

$$LE(G_1 \cup G_2) \geq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| - n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2} \right|$$

$$+\sum_{i=1}^{n_2}\left|\mu_i(G_2)-\frac{2m_2}{n_2}\right|-n_2\left|\frac{2m_2}{n_2}-\frac{2(m_1+m_2)}{n_1+n_2}\right|. \tag{2.3}$$

Since $n_2m_1 > n_1m_2$, the Eq. (2.3) becomes

$$\begin{aligned}LE(G_1\cup G_2) &\geq LE(G_1)-n_1\left(\frac{2m_1}{n_1}-\frac{2(m_1+m_2)}{n_1+n_2}\right)\\ &\quad + LE(G_2)-n_2\left(-\frac{2m_2}{n_2}+\frac{2(m_1+m_2)}{n_1+n_2}\right)\\ &= LE(G_1)+LE(G_2)-\frac{4(n_2m_1-n_1m_2)}{n_1+n_2}\end{aligned}$$

which is a lower bound. □

Corollary 2.1. [13] *Let G_1 be an (n_1, m_1)-graph and G_2 be (n_2, m_2)-graph such that $\frac{2m_1}{n_1}=\frac{2m_2}{n_2}$. Then*

$$LE(G_1\cup G_2)=LE(G_1)+LE(G_2).$$

Corollary 2.2. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an r_2-regular graph on n_2 vertices, such that $r_1 > r_2$. Then*

$$\begin{aligned}LE(G_1)+LE(G_2)-\frac{2n_1n_2(r_1-r_2)}{n_1+n_2} &\leq LE(G_1\cup G_2)\\ &\leq LE(G_1)+LE(G_2)+\frac{2n_1n_2(r_1-r_2)}{n_1+n_2}.\end{aligned}$$

Proof. Result follows by setting $m_1 = n_1r_1/2$ and $m_2 = n_2r_2/2$ into Theorem 2.1. □

Theorem 2.2. *Let G be an (n, m)-graph and $\overline{G}$ be its complement, and let $m > n(n-1)/4$. Then*

$$LE(G)+LE(\overline{G})-\big[4m-n(n-1)\big] \leq LE(G\cup\overline{G}) \leq LE(G)+LE(\overline{G})+\big[4m-n(n-1)\big].$$

Proof. $\overline{G}$ is a graph with n vertices and $n(n-1)/2-m$ edges. Substituting this into Eq. (2.1), the result follows. □

Theorem 2.3. *Let G be an (n, m)-graph and G' be the graph obtained from G by removing k edges, $0 \leq k \leq m$. Then*

$$LE(G)+LE(G')-2k \leq LE(G\cup G') \leq LE(G)+LE(G')+2k.$$

Proof. The number of vertices of G' is n and the number of edges is $m-k$. Substituting this in Eq. (2.1), the result follows. □

3. Laplacian Energy of Cartesian Product

Let G be a graph with vertex set V_1 and H be a graph with vertex set V_2. The *Cartesian product* of G and H, denoted by $G \times H$ is a graph with vertex set $V_1 \times V_2$, such that two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H or $v_1 = v_2$ and u_1 is adjacent to u_2 in G [14].

Lemma 3.1. [9] *Let* $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ *be a symmetric* 2×2 *block matrix. Then the spectrum of* A *is the union of the spectra of* $A_0 + A_1$ *and* $A_0 - A_1$.

Theorem 3.1. *If* $\mu_1, \mu_2, \ldots, \mu_n$ *are the Laplacian eigenvalues of a graph* G*, then the Laplacian eigenvalues of* $G \times K_2$ *are* $\mu_1, \mu_2, \ldots, \mu_n$ *and* $\mu_1 + 2, \mu_2 + 2, \ldots, \mu_n + 2$.

Proof. The Laplacian matrix of $G \times K_2$ is

$$C(G \times K_2) = \begin{bmatrix} C(G) + I & -I \\ -I & C(G) + I \end{bmatrix} = \begin{bmatrix} C_0 & C_1 \\ C_1 & C_0 \end{bmatrix}$$

where $C(G)$ is the Laplacian matrix of G and I is an identity matrix of order n. By Lemma 3.1, the Laplacian spectrum of $G \times K_2$ is the union of the spectra of $C_0 + C_1$ and $C_0 - C_1$.

Here $C_0 + C_1 = C(G)$. Therefore, the eigenvalues of $C_0 + C_1$ are the Laplacian eigenvalues of G.

Because $C_0 - C_1 = C(G) + 2I$, the characteristic polynomial of $C_0 - C_1$ is

$$\begin{aligned} \psi(C_0 - C_1, \mu) &= \det\left[\mu I - (C_0 - C_1)\right] = \det\left[\mu I - (C(G) + 2I)\right] \\ &= \det\left[(\mu - 2)I - C(G))\right] = \psi(G, \mu - 2). \end{aligned}$$

Therefore the eigenvalues of $C_0 - C_1$ are $\mu_i + 2$, $i = 1, 2, \ldots, n$. ☐

The Laplacian eigenvalues of the complete graph K_n are n ($n-1$ times) and 0. The Laplacian eigenvalues of the complete bipartite regular graph $K_{k,k}$ are $2k$, k ($2k - 2$ times) and 0. The Laplacian eigenvalues of the cocktail party graph $CP(k)$ (the regular graph on $n = 2k$ vertices and of degree $2k - 2$) are $2k$ ($k - 1$ times), $2k - 2$ (k times) and 0 [16]. Applying Theorem 3.1, we directly arrive at the following example.

Example 3.1.

$$LE(K_n \times K_2) = \begin{cases} 4n - 4, & \text{if } n > 2, \\ 2n, & \text{if } n \leq 2, \end{cases}$$

$$LE(K_{k,k} \times K_2) = \begin{cases} 8k - 4, & \text{if } k > 1, \\ 6k - 2, & \text{if } k = 1, \end{cases}$$

$$LE(CP(k) \times K_2) = 10k - 8, \quad \text{if } k \geq 2.$$

Theorem 3.2. *Let* G *be an* (n, m)*-graph. Then*

$$2[LE(G) - n] \leq LE(G \times K_2) \leq 2[LE(G) + n].$$

Proof. Let $\mu_1, \mu_2, \dots, \mu_n$ be the Laplacian eigenvalues of G. Then by Theorem 3.1, the Laplacian eigenvalues of $G \times K_2$ are μ_i, $i = 1, 2, \dots, n$ and $\mu_i + 2$, $i = 1, 2, \dots, n$. The graph $G \times K_2$ has $2n$ vertices and $2m + n$ edges. Therefore,

$$LE(G \times K_2) = \sum_{i=1}^{n} \left| \mu_i - \frac{2(2m+n)}{2n} \right| + \sum_{i=1}^{n} \left| \mu_i + 2 - \frac{2(2m+n)}{2n} \right|$$

$$= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} - 1 \right| + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} + 1 \right| . \tag{3.1}$$

Equation (3.1) can be rewritten as

$$LE(G \times K_2) \leq \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| + n + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| + n = 2LE(G) + 2n$$

which is an upper bound.

For lower bound, Eq. (3.1) can be rewritten as

$$LE(G \times K_2) \geq \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| - n + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| - n = 2LE(G) - 2n.$$

□

Theorem 3.3. *For a graph G with n vertices, $LE(G \times K_2) \geq 2n$.*

Proof. From Eq. (3.1)

$$LE(G \times K_2) \geq \left| \sum_{i=1}^{n} \left(\mu_i - \frac{2m}{n} - 1 \right) \right| + \left| \sum_{i=1}^{n} \left(\mu_i - \frac{2m}{n} + 1 \right) \right|$$

$$= \left| 2m - 2m - n \right| + \left| 2m - 2m + n \right| = 2.$$

□

4. Laplacian Equienergetic Graphs

Two graphs G_1 and G_2 are said to be *equienergetic* if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$ [2]. For details see the book [15]. In analogy to this, two graphs G_1 and G_2 are said to be *Laplacian equienergetic* if $LE(G_1) = LE(G_2)$.

Obviously Laplacian cospectral graphs are Laplacian equienergetic. Therefore we are interested in Laplacian non-cospectral graphs with equal number of vertices, having equal Laplacian energies. Stevanović [24] has constructed Laplacian equienergetic threshold graphs. Fritscher et al. [10] discovered a family of Laplacian equienergetic unicyclic graphs. We now report some additional classes of such graphs.

The *line graph* of the graph G, denoted by $L(G)$, is a graph whose vertices corresponds to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [14]. The k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$. If G is a regular graph of order n_0 and of degree r_0, then $L(G)$ is a regular graph of order $n_1 = n_0 r_0/2$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [3, 4]:

$$n_k = \frac{1}{2} n_{k-1} r_{k-1} \qquad \text{and} \qquad r_k = 2r_{k-1} - 2$$

where n_i and r_i stand for the order and degree of $L^i(G)$, $i = 0, 1, 2, \ldots$. Therefore [3,4],

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} \left(2^i r_0 - 2^{i+1} + 2\right) \tag{4.1}$$

and

$$r_k = 2^k r_0 - 2^{k+1} + 2. \tag{4.2}$$

Theorem 4.1. [23] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\begin{array}{rll} \lambda_i + r - 2 & i = 1, 2, \ldots, n, & \text{and} \\ -2 & n(r-2)/2 \text{ times.} & \end{array}$$

Theorem 4.2. [22] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

Theorem 4.3. [16] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then its Laplacian eigenvalues are $r - \lambda_i$, $i = 1, 2, \ldots, n$.*

For G being a regular graph of degree $r \geq 3$, and for $k \geq 2$, expressions for $\mathcal{E}\big(L^k(G)\big)$ and $\mathcal{E}\big(\overline{L^k(G)}\big)$ were reported in [20, 21].

Theorem 4.4. *If G is a regular graph of order n and of degree $r \geq 4$, then*

$$LE(L^2(G) \times K_2) = 4nr(r-2).$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of G . Then by Theorem 4.1, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} \lambda_i + r - 2 & i = 1, 2, \ldots, n, & \text{and} \\ -2 & n(r-2)/2 \text{ times.} & \end{array}\right\} \tag{4.3}$$

Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, by Eq. (4.3) the adjacency eigenvalues of $L^2(G)$ are

$$\left.\begin{array}{rll} \lambda_i + 3r - 6 & i = 1, 2, \ldots, n, & \text{and} \\ 2r - 6 & n(r-2)/2 \text{ times,} & \text{and} \\ -2 & nr(r-2)/2 \text{ times.} & \end{array}\right\} \tag{4.4}$$

Since $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and of degree $4r-2$, by Theorem 4.3 and Eq. (4.4), the Laplacian eigenvalues of $L^2(G)$ are

$$\left.\begin{array}{rll} r-\lambda_i & i=1,2,\ldots,n, & \text{and} \\ 2r & n(r-2)/2 \text{ times}, & \text{and} \\ 4r-4 & nr(r-2)/2 \text{ times}. & \end{array}\right\} \tag{4.5}$$

Using Theorem 3.1 and Eq. (4.5), the Laplacian eigenvalues of $L^2(G) \times K_2$ are

$$\left.\begin{array}{rll} r-\lambda_i & i=1,2,\ldots,n, & \text{and} \\ 2r & n(r-2)/2 \text{ times}, & \text{and} \\ 4r-4 & nr(r-2)/2 \text{ times}, & \text{and} \\ r-\lambda_i+2 & i=1,2,\ldots,n, & \text{and} \\ 2r+2 & n(r-2)/2 \text{ times}, & \text{and} \\ 4r-2 & nr(r-2)/2 \text{ times}. & \end{array}\right\} \tag{4.6}$$

The graph $L^2(G) \times K_2$ is a regular graph of order $nr(r-1)$ and of degree $4r-5$. By Eq. (4.6), the Laplacian energy of $L^2(G) \times K_2$ is computed as

$$\begin{aligned} LE(L^2(G) \times K_2) &= \sum_{i=1}^{n} \big|r-\lambda_i-(4r-5)\big| + \big|2r-(4r-5)\big|\frac{n(r-2)}{2} \\ &\quad + \big|4r-4-(4r-5)\big|\frac{nr(r-2)}{2} + \sum_{i=1}^{n} \big|r-\lambda_i+2-(4r-5)\big| \\ &\quad + \big|2r+2-(4r-5)\big|\frac{n(r-2)}{2} + \big|4r-2-(4r-5)\big|\frac{nr(r-2)}{2} \\ &= \sum_{i=1}^{n} \big|-\lambda_i-3r+5\big| + \big|-2r+5\big|\frac{n(r-2)}{2} \\ &\quad + \big|1\big|\frac{nr(r-2)}{2} + \sum_{i=1}^{n} \big|-\lambda_i-3r+7\big| \\ &\quad + \big|-2r+7\big|\frac{n(r-2)}{2} + \big|3\big|\frac{nr(r-2)}{2}. \end{aligned} \tag{4.7}$$

If d_{max} is the greatest vertex degree of a graph, then all its adjacency eigenvalues belongs to the interval $[-d_{max}, +d_{max}]$ [5]. In particular, the adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$, $i=1,2,\ldots,n$.

If $r \geq 4$ then $\lambda_i + 3r - 5 > 0$, $\lambda_i + 3r - 7 > 0$, $2r - 5 > 0$, and $2r - 7 > 0$. Therefore by Eq. (4.7), and bearing in mind that $\sum_{i=1}^{n} \lambda_i = 0$,

$$\begin{aligned} LE(L^2(G) \times K_2) &= \sum_{i=1}^{n} \lambda_i + n(3r-5) + (2r-5)\frac{n(r-2)}{2} + \frac{nr(r-2)}{2} \\ &\quad + \sum_{i=1}^{n} \lambda_i + n(3r-7) + (2r-7)\frac{n(r-2)}{2} + \frac{3nr(r-2)}{2} \\ &= 4nr(r-2). \end{aligned}$$

□

Corollary 4.1. *Let G be a regular graph of order n_0 and of degree $r_0 \geq 4$. Let n_k and r_k be the order and degree, respectively of the k-th iterated line graph $L^k(G)$ of G, $k \geq 2$. Then*

$$\begin{aligned} & LE(L^k(G) \times K_2) = 4n_{k-2}r_{k-2}(r_{k-2} - 2) = 4n_{k-1}(r_{k-1} - 2), \\ (4.8) \qquad & LE(L^k(G) \times K_2) = 4n_0(r_0 - 2) \prod_{i=0}^{k-2} (2^i r_0 - 2^{i+1} + 2), \\ & LE(L^k(G) \times K_2) = 8(n_k - n_{k-1}) = 8n_k \left(\frac{r_k - 2}{r_k + 2} \right). \end{aligned}$$

From Eq. (4.8) we see that the energy of $L^k(G) \times K_2$, $k \geq 2$ is fully determined by the order n and degree $r \geq 4$ of G.

Theorem 4.5. *If G is a regular graph of order n and of degree $r \geq 3$, then*

$$LE\left(\overline{L^2(G)} \times K_2\right) = (nr-4)(4r-6) - 4.$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of a regular graph G of order n and of degree $r \geq 3$. Then the adjacency eigenvalues of $L^2(G)$ are as given by Eq. (4.4).

Since $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and of degree $4r-2$, by Theorem 4.2 and Eq. (4.4), the adjacency eigenvalues of $\overline{L^2(G)}$ are

$$(4.9) \qquad \left. \begin{array}{rll} -\lambda_i - 3r + 5 & i = 2, 3, \ldots, n, & \text{and} \\ -2r + 5 & n(r-2)/2 \text{ times}, & \text{and} \\ 1 & nr(r-2)/2 \text{ times}, & \text{and} \\ (nr(r-1)/2) - 4r + 5. & & \end{array} \right\}$$

Since $\overline{L^2(G)}$ is a regular graph of order $nr(r-1)/2$ and of degree $(nr(r-1)/2)-4r+5$, by Theorem 4.3 and Eq. (4.9), the Laplacian eigenvalues of $\overline{L^2(G)}$ are

$$\left.\begin{array}{rll} (nr(r-1)/2)-r-\lambda_i & i=2,3,\ldots,n, & \text{and} \\ (nr(r-1)/2)-2r & n(r-2)/2 \text{ times}, & \text{and} \\ (nr(r-1)/2)-4r+4 & nr(r-2)/2 \text{ times}, & \text{and} \\ 0. & & \end{array}\right\} \tag{4.10}$$

Using Theorem 3.1 and Eq. (4.10), the Laplacian eigenvalues of $\overline{L^2(G)} \times K_2$ are

$$\left.\begin{array}{rll} (nr(r-1)/2)-r-\lambda_i & i=2,3,\ldots,n, & \text{and} \\ (nr(r-1)/2)-2r & n(r-2)/2 \text{ times}, & \text{and} \\ (nr(r-1)/2)-4r+4 & nr(r-2)/2 \text{ times}, & \text{and} \\ 0 & 1 \text{ time}, & \text{and} \\ (nr(r-1)/2)-r-\lambda_i+2 & i=2,3,\ldots,n, & \text{and} \\ (nr(r-1)/2)-2r+2 & n(r-2)/2 \text{ times}, & \text{and} \\ (nr(r-1)/2)-4r+6 & nr(r-2)/2 \text{ times}, & \text{and} \\ 2. & & \end{array}\right\} \tag{4.11}$$

The graph $\overline{L^2(G)} \times K_2$ is a regular graph of order $nr(r-1)$ and of degree $(nr(r-1)/2)-4r+6$. By Eq. (4.11),

$$\begin{aligned} LE\left(\overline{L^2(G)} \times K_2\right) = & \sum_{i=2}^{n} \left| \frac{nr(r-1)}{2} - r + \lambda_i - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ & + \left| \frac{nr(r-1)}{2} - 2r - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \\ & + \left| \frac{nr(r-1)}{2} - 4r + 4 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{nr(r-2)}{2} \\ & + \left| 0 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ & + \sum_{i=2}^{n} \left| \frac{nr(r-1)}{2} - r + \lambda_i + 2 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ & + \left| \frac{nr(r-1)}{2} - 2r + 2 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \end{aligned}$$

$$+\left|\frac{nr(r-1)}{2}-4r+6-\left(\frac{nr(r-1)}{2}-4r+6\right)\right|\frac{nr(r-2)}{2}$$

$$+\left|2-\left(\frac{nr(r-1)}{2}-4r+6\right)\right|$$

$$=\sum_{i=2}^{n}\left|\lambda_i+3r-6\right|+\left|2r-6\right|\frac{n(r-2)}{2}$$

$$+\left|-2\right|\frac{nr(r-2)}{2}+\left|\frac{-nr(r-1)}{2}+4r-6\right|+\sum_{i=2}^{n}\left|\lambda_i+3r-4\right|$$

$$+\left|2r-4\right|\frac{n(r-2)}{2}+\left|0\right|\frac{nr(r-2)}{2}+\left|\frac{-nr(r-1)}{2}+4r-4\right|. \tag{4.12}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$, $i=1,2,\ldots,n$ [5]. Therefore if $r \geq 3$, then $\lambda_i+3r-6 \geq 0$, $\lambda_i+3r-4 \geq 0$, $2r-6 \geq 0$, $2r-4 \geq 0$, $(-nr(r-1)/2)+4r-6 < 0$, and $(-nr(r-1)/2)+4r-4 < 0$. Then from Eq. (4.12), and bearing in mind that $\sum_{i=2}^{n}\lambda_i = -r$, we get

$$LE\left(\overline{L^2(G)}\times K_2\right)=\sum_{i=2}^{n}\lambda_i+(n-1)(3r-6)+(r-3)n(r-2)+nr(r-2)$$

$$+\frac{nr(r-1)}{2}-4r+6+\sum_{i=2}^{n}\lambda_i+(n-1)(3r-4)$$

$$+(r-2)n(r-2)+\frac{nr(r-1)}{2}-4r+4$$

$$=2(nr-4)(2r-3)-4.$$

□

Corollary 4.2. *Let G be a regular graph of order n_0 and of degree $r_0 \geq 3$. Let n_k and r_k be the order and degree, respectively of the k-th iterated line graph $L^k(G)$ of G, $k \geq 2$. Then*

$$LE\left(\overline{L^k(G)}\times K_2\right)=(n_{k-2}r_{k-2}-4)(4r_{k-2}-6)-4$$

$$=(2n_{k-1}-4)(2r_{k-1}-2)-4,$$

$$LE\left(\overline{L^k(G)}\times K_2\right)=\left[\frac{n_0}{2^{k-2}}\prod_{i=0}^{k-2}(2^ir_0-2^{i+1}+2)-4\right](2^kr_0-2^{k+1}+2)-4, \tag{4.13}$$

$$LE\left(\overline{L^k(G)} \times K_2\right) = \frac{8n_k r_k}{2+r_k} - 4(r_k+1).$$

From Eq. (4.13) we see that the energy of $\overline{L^k(G)} \times K_2$, $k \geq 2$ is fully determined by the order n and degree $r \geq 3$ of G.

Theorem 4.6. *Let G_1 and G_2 be two Laplacian non-cospectral, regular graphs of the same order and of the same degree $r \geq 4$. Then for any $k \geq 2$, $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.*

Proof. If G is any graph with n vertices and m edges, then $G \times K_2$ has $2n$ vertices and $2m+n$ edges. Hence by repeated applications of Eqs. (4.1) and (4.2), $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ have same number of vertices and same number of edges. By Eqs. (4.5) and (4.6), if G_1 and G_2 are not Laplacian cospectral, then $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ are not Laplacian cospectral for all $k \geq 1$. Finally, Eq. (4.8) implies that $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ are Laplacian equienergetic. □

Theorem 4.7. *Let G_1 and G_2 be two Laplacian non-cospectral, regular graphs of the same order and of the same degree $r \geq 3$. Then for any $k \geq 2$, $\overline{L^k(G_1)} \times K_2$ and $\overline{L^k(G_2)} \times K_2$ is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.*

Proof. The proof is similar to that of Theorem 4.6 by using Eqs. (4.10), (4.11), and (4.13). □

Acknowledgments. H. S. Ramane is thankful to the University Grants Commission (UGC), Govt. of India, for support through research grant under UPE FAR-II grant No. F 14-3/2012 (NS/PE). G. A. Gudodagi is thankful to the Karnatak University, Dharwad, for support through UGC-UPE scholarship No. KU/SCH/UGC-UPE/2014-15/901.

References

[1] E. O. D. Andriantiana, *Laplacian energy*, in: I. Gutman, X. Li (Eds.), *Graph Energies – Theory and Applications*, Univ. Kragujevac, Kragujevac, 2016, pp. 49–80.
[2] R. Balakrishnan, *The energy of a graph*, Linear Algebra Appl. **387** (2004), 287–295.
[3] F. Buckley, *Iterated line graphs*, Congr. Numer. **33** (1981), 390–394.
[4] F. Buckley, *The size of iterated line graphs*, Graph Theory Notes N. Y., **25** (1993), 33–36.
[5] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.
[6] G. Dahl, *The Laplacian energy of threshold graphs and majorization*, Linear Algebra Appl. **469** (2015), 518–530.
[7] K. C. Das , S. A. Mojallal, *Relation between energy and (signless) Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem. **74** (2015), 359–366.
[8] K. C. Das, S. A. Mojallal, I. Gutman, *On Laplacian energy in terms of graph invariants*, Appl. Math. Comput. **268** (2015), 83–92.
[9] P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.

[10] E. Fritscher, C. Hoppen, V. Trevisan, *Unicyclic graphs with equal Laplacian energy*, Lin. Multilinear Algebra **62** (2014), 180–194.

[11] H. A. Ganie, S. Pirzada, E. T. Baskoro, *On energy, Laplacian energy and p-fold graphs*, El. J. Graph Theory Appl. **3** (2015), 94–107.

[12] I. Gutman, E. Milovanović, I. Milovanović, *Bounds for Laplacian–type graph energies*, Miskolc Math. Notes **16** (2015), 195–203.

[13] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Linear Algebra Appl. **414** (2006) 29–37.

[14] F. Harary, *Graph Theory*, Addison–Wesley, Reading, 1969.

[15] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[16] B. Mohar, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Ollermann, A. J. Schwenk (Eds.), *Graph Theory, Combinatorics and Applications*, Wiley, (1991), pp. 871–898.

[17] S. Pirzada, H. A. Ganie, *On the Laplacian eigenvalues of a graph and Laplacian energy*, Linear Algebra Appl. **486** (2015), 454–468.

[18] S. Radenković, I. Gutman, *Total π-electron energy and Laplacian energy: How far the analogy goes?*, J. Serb. Chem. Soc. **72** (2007), 1343–1350.

[19] H. S. Ramane, S. B. Gudimani, *Laplacian polynomial and Laplacian energy of some cluster graphs*, Int. J. Sci. Innov. Math. Res. **2** (2014) 448–452.

[20] H. S. Ramane, I. Gutman, H. B. Walikar, S. B. Halkarni, *Equienergetic complement graphs*, Kragujevac J. Sci. **27** (2005), 67–74.

[21] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, *Spectra and energies of iterated line graphs of regular graphs*, Appl. Math. Lett. **18** (2005), 679–682.

[22] H. Sachs, *Über selbstkomplementare Graphen*, Publ. Math. Debrecen **9** (1962), 270–288.

[23] H. Sachs, *Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II*, Wiss. Z. TH Ilmenau **13** (1967), 405–412.

[24] D. Stevanović, *Large sets of noncospectral graphs with equal Laplacian energy*, MATCH Commun. Math. Comput. Chem. **61** (2009), 463–470.

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