CLUSTERS AND VARIOUS VERSIONS OF WIENER-TYPE INVARIANTS

MAHDIEH AZARI[1] AND ALI IRANMANESH[2]*

ABSTRACT. The Wiener type invariant $W^{(\lambda)}(G)$ of a simple connected graph G is defined as the sum of the terms $d(u,v\,|G)^\lambda$ over all unordered pairs $\{u,v\}$ of vertices of G, where $d(u,v\,|G)$ denotes the distance between the vertices u and v in G and λ is an arbitrary real number. The cluster $G_1\{G_2\}$ of a graph G_1 and a rooted graph G_2 is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by identifying the root vertex of the i-th copy of G_2 with the i-th vertex of G_1, for $i = 1, 2, \ldots, |V(G_1)|$. In this paper, we study the behavior of three versions of Wiener type invariant under the cluster product. Results are applied to compute several distance-based topological invariants of bristled and bridge graphs by specializing components in clusters.

1. INTRODUCTION

In this paper, we consider connected finite graphs without any loops or multiple edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\deg_G(u)$ the degree of the vertex u in G and by $V(e)$ the set of two end vertices of the edge e of G. The distance $d(u,v\,|G)$ between the vertices u and v of G is the length of any shortest path in G connecting them.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modelled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [11, 25]. The vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [26], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of

G is defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v \,|G)$. Details on the Wiener index can be found in [6, 9, 10, 18, 20, 28].

The definition of the Wiener index can be generalized by the following definition [10, 12]

$$W^{(\lambda)}(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v \,|G)^{\lambda},$$

where λ is an arbitrary real number. The index $W^{(\lambda)}(G)$ is called the vertex Wiener type invariant of G. It is easy to see that, $W^{(0)}(G) = \binom{|V(G)|}{2}$ and $W^{(1)}(G) = W(G)$. The so-called hyper-Wiener index $WW(G)$ of G [22] was shown [19] to be equal to $\frac{1}{2}W^{(2)}(G) + \frac{1}{2}W^{(1)}(G)$, and the so-called Tratch-Stankevich-Zefirov index $TSZ(G)$ of G [24] was shown [18] to be equal to $\frac{1}{6}W^{(3)}(G) + \frac{1}{2}W^{(2)}(G) + \frac{1}{3}W^{(1)}(G)$. Recall that, the hyper-Wiener and the Tratch-Stankevich-Zefirov indices were originally defined in terms completely different from the presently considered Wiener-type invariants; for details see [22, 24].

Edge versions of the Wiener index based on distance between all pairs of edges of a graph were introduced in 2009 [5, 15, 17]. Two possible distances between the edges $e = uv$ and $f = zt$ of a graph G can be considered. The first distance is denoted by $d_0(e, f \,|G)$ and defined as

$$d_0(e, f|G) = \begin{cases} d_1(e, f|G) + 1, & e \neq f, \\ 0, & e = f, \end{cases}$$

where $d_1(e, f \,|G) = \min\{d(u, z \,|G), d(u, t \,|G), d(v, z \,|G), d(v, t \,|G)\}$. It is easy to see that, $d_0(e, f \,|G) = d(e, f \,|L(G))$, where $L(G)$ is the line graph of G.
The second distance is denoted by $d_4(e, f \,|G)$ and defined as

$$d_4(e, f|G) = \begin{cases} d_2(e, f|G), & e \neq f, \\ 0, & e = f, \end{cases}$$

where $d_2(e, f \,|G) = \max\{d(u, z \,|G), d(u, t \,|G), d(v, z \,|G), d(v, t \,|G)\}$.
Related to these two distances, two edge versions of the Wiener index can be defined. The first and second edge-Wiener indices of G are denoted by $W_{e_0}(G)$ and $W_{e_4}(G)$, respectively and defined as

$$W_{e_i}(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e, f \,|G), \; i \in \{0, 4\}.$$

Obviously, $W_{e_0}(G) = W(L(G))$. For more information on the edge-Wiener indices see [1, 3, 21], and especially the recent survey [14].

The definitions of the edge-Wiener indices can be generalized by the following definition [16]

$$W_{e_i}^{(\lambda)}(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e, f \,|G)^{\lambda}, \; i \in \{0, 4\},$$

where λ is an arbitrary real number. The indices $W_{e_0}^{(\lambda)}(G)$ and $W_{e_4}^{(\lambda)}(G)$ are called the first and the second edge-Wiener type invariants of G, respectively. It is easy

to see that, $W_{e_i}^{(0)}(G) = \binom{|E(G)|}{2}$ and $W_{e_i}^{(1)}(G) = W_{e_i}(G)$, $i \in \{0, 4\}$. The first and the second edge hyper-Wiener indices of G are denoted by $WW_{e_0}(G)$ and $WW_{e_4}(G)$, respectively and defined as [16]

$$WW_{e_i}(G) = \frac{1}{2}W_{e_i}^{(1)}(G) + \frac{1}{2}W_{e_i}^{(2)}(G), \ i \in \{0, 4\}.$$

In analogy with definitions of the vertex version and edge versions of the Wiener index, vertex-edge versions of the Wiener index were introduced based on distance between vertices and edges in a graph [4, 17]. Two possible distances between a vertex u and an edge $e = ab$ of a graph G can be considered. The first distance is denoted by $D_1(u, e \,|G)$ and defined as

$$D_1(u, e \,|G) = \min\{d(u, a \,|G), d(u, b \,|G)\},$$

and the second one is denoted by $D_2(u, e \,|G)$ and defined as

$$D_2(u, e \,|G) = \max\{d(u, a \,|G), d(u, b \,|G)\}.$$

Corresponding to these two distances, two vertex-edge versions of the Wiener index can be introduced. The first and the second vertex-edge Wiener indices of G are denoted by $W_{ve_1}(G)$ and $W_{ve_2}(G)$, respectively and defined as

$$W_{ve_i}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_i(u, e \,|G), \ i \in \{1, 2\}.$$

The definitions of the vertex-edge Wiener indices can be generalized as follows

$$W_{ve_1}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G); u \notin V(e)} D_1(u, e \,|G)^{\lambda},$$

$$W_{ve_2}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_2(u, e \,|G)^{\lambda},$$

where λ is an arbitrary real number. The indices $W_{ve_1}^{(\lambda)}(G)$ and $W_{ve_2}^{(\lambda)}(G)$ are called the first and the second vertex-edge Wiener type invariants of G, respectively. Note that, if λ is a positive number then $W_{ve_1}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_1(u, e \,|G)^{\lambda}$. Obviously, $W_{ve_1}^{(0)}(G) = |E(G)| \, (|V(G)| - 2)$, $W_{ve_2}^{(0)}(G) = |V(G)| \, |E(G)|$, and $W_{ve_i}^{(1)}(G) = W_{ve_i}(G)$, $i \in \{1, 2\}$. We denote the first and the second vertex-edge hyper-Wiener indices of G by $WW_{ve_1}(G)$ and $WW_{ve_2}(G)$, respectively and define as

$$WW_{ve_i}(G) = \frac{1}{2}W_{ve_i}^{(1)}(G) + \frac{1}{2}W_{ve_i}^{(2)}(G), \quad i \in \{1, 2\}.$$

It is well-known that many graphs of general, and in particular of chemical interest, arise from simpler graphs via various graph operations sometimes known as graph products. It is, hence, important to understand how certain invariants of such graph operations are related to the corresponding invariants of their components. In this paper, we compute the vertex version, edge versions, and vertex-edge versions of the Wiener type invariant for an important graph product called cluster. Results are

applied for t-fold bristled graphs and bridge graphs by specializing components in clusters. We encourage the reader to consult [1, 2, 3, 7, 8, 13, 23, 27, 28] for more information on computing topological indices of graph products.

2. Main Results

The cluster $G_1\{G_2\}$ of a graph G_1 and a rooted graph G_2 is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by identifying the root vertex of the i-th copy of G_2 with the i-th vertex of G_1, for $i = 1, 2, \ldots, |V(G_1)|$. Some topological indices of clusters have been computed previously [2, 8, 28]. In this section, we determine the vertex Wiener, edge-Wiener, and vertex-edge Wiener type invariants of the cluster $G_1\{G_2\}$. Throughout this section, we denote the root vertex of G_2 by w, the degree of w in G_2 by ω, and the copy of G_2 whose root is identified with the vertex $u \in V(G_1)$ by G_2^u. Also for $i \in \{1, 2\}$, we denote the order and size of the graph G_i by n_i and m_i, respectively.

2.1. Wiener type invariant of clusters.
In this section, we determine the vertex Wiener type invariant of the cluster $G_1\{G_2\}$.

Let λ be a real number. For $u \in V(G)$, we define

$$d^{(\lambda)}(u\,|G) = \sum_{v \in V(G)-\{u\}} d(u, v\,|G)^{\lambda}.$$

It is easy to check that, $d^{(0)}(u\,|G) = |V(G)| - 1$.

Theorem 2.1. *If λ is a positive integer, then the vertex Wiener type invariant of $G_1\{G_2\}$ is given by*

$$W^{(\lambda)}(G_1\{G_2\}) = W^{(\lambda)}(G_1) + n_1 W^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1)\left[2d^{(\lambda-i)}(w\,|G_2)\right.$$

$$\left. + \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w\,|G_2)d^{(\lambda-i-j)}(w\,|G_2)\right].$$

Proof. We partition the sum in the definition of $W^{(\lambda)}(G_1\{G_2\})$ into four sums as follows.

The first sum S_1 consists of contributions to $W^{(\lambda)}(G_1\{G_2\})$ of pairs of vertices from G_1. For vertices $x, y \in V(G_1)$, $d(x, y\,|G_1\{G_2\}) = d(x, y\,|G_1)$. Taking this into account, we have

$$S_1 = \sum_{\{x,y\}\subseteq V(G_1)} d(x, y\,|G_1)^{\lambda} = W^{(\lambda)}(G_1).$$

The second sum S_2 is taken over all pairs of vertices in G_2^u, and then over all $u \in V(G_1)$. For vertices $x, y \in V(G_2^u)$, $d(x, y\,|G_1\{G_2\}) = d(x, y\,|G_2)$. Taking this

into account, we have

$$S_2 = \sum_{u \in V(G_1)} \sum_{\{x,y\} \subseteq V(G_2^u)} d(x,y \,|G_2)^\lambda = n_1 W^{(\lambda)}(G_2).$$

The third sum S_3 is taken over all pairs of vertices $x, y \in V(G_1\{G_2\})$ such that $x \in V(G_1)$ and $y \in V(G_2^u) - \{w\}$, where $u \in V(G_1)$ and $u \neq x$. In this case, $d(x, y \,|G_1\{G_2\}) = d(x, u \,|G_1) + d(y, w \,|G_2)$. So,

$$S_3 = \sum_{x \in V(G_1)} \sum_{u \in V(G_1)-\{x\}} \sum_{y \in V(G_2^u)-\{w\}} [d(x, u \,|G_1) + d(y, w \,|G_2)]^\lambda$$

$$= 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) d^{(\lambda-i)}(w \,|G_2).$$

The last sum S_4 is taken over all pairs of non-root vertices from different copies of G_2. For such a pair $x \in V(G_2^u) - \{w\}$ and $y \in V(G_2^v) - \{w\}$,

$$d(x, y \,|G_1\{G_2\}) = d(x, w \,|G_2) + d(u, v \,|G_1) + d(y, w \,|G_2).$$

So,

$$S_4 = \sum_{\{u,v\} \subseteq V(G_1)} \sum_{x \in V(G_2^u)-\{w\}} \sum_{y \in V(G_2^v)-\{w\}} [d(x, w \,|G_2) + d(u, v \,|G_1) + d(w, y \,|G_2)]^\lambda$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w \,|G_2) d^{(\lambda-i-j)}(w \,|G_2).$$

The formula for $W^{(\lambda)}(G_1\{G_2\})$ follows upon addition of the quantities S_1, S_2, S_3, and S_4. $\qquad \square$

By Theorem 2.1, we can reproduce the result of Theorem 5 in [28] about the Wiener index of the cluster of two graphs.

Corollary 2.1. *The Wiener index of $G_1\{G_2\}$ is given by*

$$W(G_1\{G_2\}) = n_2^2 W(G_1) + n_1 W(G_2) + 2n_2 \binom{n_1}{2} d^{(1)}(w \,|G_2).$$

Using Theorem 2.1, we can also get the formulae for the hyper-Wiener index and TSZ index of $G_1\{G_2\}$.

Corollary 2.2. *The hyper-Wiener index of $G_1\{G_2\}$ is given by*

$$WW(G_1\{G_2\}) = n_2^2 WW(G_1) + n_1 WW(G_2) + 2n_2 d^{(1)}(w \,|G_2) W(G_1)$$

$$+ \binom{n_1}{2} \left[n_2 d^{(2)}(w \,|G_2) + d^{(1)}(w \,|G_2)^2 + n_2 d^{(1)}(w \,|G_2) \right].$$

Corollary 2.3. *The Tratch-Stankevich-Zefirov index of $G_1\{G_2\}$ is given by*

$$
\begin{aligned}
TSZ(G_1\{G_2\}) &= n_2^2 TSZ(G_1) + n_1 TSZ(G_2) + 2n_2 d^{(1)}(w\,|G_2)WW(G_1) \\
&\quad + \left[n_2 d^{(2)}(w\,|G_2) + d^{(1)}(w\,|G_2)^2 + n_2 d^{(1)}(w\,|G_2) \right] W(G_1) \\
&\quad + \frac{1}{3} n_2 \binom{n_1}{2} \left[d^{(3)}(w\,|G_2) + 3d^{(2)}(w\,|G_2) + 2d^{(1)}(w\,|G_2) \right] \\
&\quad + \binom{n_1}{2} d^{(1)}(w\,|G_2) \left[d^{(2)}(w\,|G_2) + d^{(1)}(w\,|G_2) \right].
\end{aligned}
$$

2.2. **Edge-Wiener type invariants of clusters.** In this section, we determine the edge-Wiener type invariants of the cluster $G_1\{G_2\}$.

Let λ be a real number and let $u \in V(G)$. We define

$$
\begin{aligned}
D_1^{(\lambda)}(u\,|G) &= \sum_{e\in E(G);\, u\notin V(e)} D_1(u,e\,|G)^\lambda, \\
D_2^{(\lambda)}(u\,|G) &= \sum_{e\in E(G)} D_2(u,e\,|G)^\lambda.
\end{aligned}
$$

Note that, if λ is a positive number then $D_1^{(\lambda)}(u\,|G) = \sum_{e\in E(G)} D_1(u,e\,|G)^\lambda$. In particular for $\lambda = 0$, $D_1^{(0)}(u\,|G) = |E(G)| - \deg_G(u)$ and $D_2^{(0)}(u\,|G) = |E(G)|$.

Theorem 2.2. *If λ is a positive integer, then the edge-Wiener type invariants of $G_1\{G_2\}$ are given by*

(i) $W_{e_0}^{(\lambda)}(G_1\{G_2\}) = W_{e_0}^{(\lambda)}(G_1) + n_1 W_{e_0}^{(\lambda)}(G_2) + 2m_1\omega$

$$
\begin{aligned}
&+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[\omega^2 W^{(i)}(G_1) + \omega W_{ve_1}^{(i)}(G_1) + 2m_1 D_1^{(i)}(w\,|G_2) \right] \\
&+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} \left[\left(W_{ve_1}^{(j)}(G_1) + 2\omega W^{(j)}(G_1)\right) D_1^{(i-j)}(w\,|G_2) \right. \\
&\left. + W^{(j)}(G_1) \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w\,|G_2) D_1^{(i-j-k)}(w\,|G_2) \right],
\end{aligned}
$$

(ii) $W_{e_4}^{(\lambda)}(G_1\{G_2\}) = W_{e_4}^{(\lambda)}(G_1) + n_1 W_{e_4}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_2}^{(i)}(G_1) D_2^{(\lambda-i)}(w\,|G_2)$

$$
+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_2^{(j)}(w\,|G_2) D_2^{(\lambda-i-j)}(w\,|G_2).
$$

Proof. **(i)** We partition the sum in the definition of $W_{e_0}^{(\lambda)}(G_1\{G_2\})$ into four sums as follows.

The first sum S_1 consists of contributions to $W_{e_0}^{(\lambda)}(G_1\{G_2\})$ of pairs of edges from G_1. For edges $e, f \in E(G_1)$, $d_0(e, f \,|G_1\{G_2\}) = d_0(e, f \,|G_1)$. Taking this into account, we have

$$S_1 = \sum_{\{e,f\} \subseteq E(G_1)} d_0(e, f \,|G_1)^\lambda = W_{e_0}^{(\lambda)}(G_1).$$

The second sum S_2 is taken over all pairs of edges in G_2^u, and then over all $u \in V(G_1)$. For edges $e, f \in E(G_2^u)$, $d_0(e, f \,|G_1\{G_2\}) = d_0(e, f \,|G_2)$. Taking this into account, we have

$$S_2 = \sum_{u \in V(G_1)} \sum_{\{e,f\} \subseteq E(G_2^u)} d_0(e, f \,|G_2)^\lambda = n_1 W_{e_0}^{(\lambda)}(G_2).$$

The third sum S_3 is taken over all pairs of edges $e, f \in E(G_1\{G_2\})$ such that $e \in E(G_1)$ and $f \in E(G_2^u)$, where $u \in V(G_1)$. In this case,

$$d_0(e, f \,|G_1\{G_2\}) = D_1(u, e \,|G_1) + D_1(w, f \,|G_2) + 1.$$

So,

$$S_3 = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{f \in E(G_2^u)} [D_1(u, e \,|G_1) + D_1(w, f \,|G_2) + 1]^\lambda.$$

In order to compute the sum S_3, we partition it into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows. The sum S_{31} is equal to

$$S_{31} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1); u \notin V(e)} \sum_{f \in E(G_2^u); w \notin V(f)} [D_1(u, e \,|G_1) + D_1(w, f \,|G_2) + 1]^\lambda$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{ve_1}^{(j)}(G_1) D_1^{(i-j)}(w \,|G_2).$$

The sum S_{32} is equal to

$$S_{32} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1); u \notin V(e)} \sum_{f \in E(G_2^u); w \in V(f)} [D_1(u, e \,|G_1) + 1]^\lambda = \omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_1}^{(i)}(G_1).$$

The sum S_{33} is equal to

$$S_{33} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1); u \in V(e)} \sum_{f \in E(G_2^u); w \notin V(f)} [D_1(w, f \,|G_1) + 1]^\lambda$$

$$= 2m_1 \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1^{(i)}(w \,|G_2).$$

The sum S_{34} is equal to

$$S_{34} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1); u \in V(e)} \sum_{f \in E(G_2^u); w \in V(f)} 1^\lambda = 2m_1\omega.$$

By adding the quantities S_{31}, S_{32}, S_{33} and S_{34}, we obtain

$$S_3 = 2m_1\omega + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[\omega W_{ve_1}^{(i)}(G_1) + 2m_1 D_1^{(i)}(w\,|G_2) \right]$$

$$+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{ve_1}^{(j)}(G_1) D_1^{(i-j)}(w\,|G_2).$$

The last sum S_4 is taken over all pairs of edges from different copies of G_2. For such a pair $e \in E(G_2^u)$ and $f \in E(G_2^v)$,

$$d_0(e, f\,|G_1\{G_2\}) = D_1(w, e\,|G_2) + D_1(w, f\,|G_2) + d(u, y\,|G_1) + 1.$$

So,

$$S_4 = \sum_{\{u,v\}\subseteq V(G_1)} \sum_{e\in E(G_2^u)} \sum_{f\in E(G_2^v)} [D_1(w, e\,|G_2) + D_1(w, f\,|G_2) + d(u, v\,|G_1) + 1]^{\lambda}.$$

Similar to the previous case, we partition the sum S_4 into four sums S_{41}, S_{42}, S_{43} and S_{44} as follows. The sum S_{41} is equal to

$$S_{41} = \sum_{\{u,v\}\subseteq V(G_1)} \sum_{e\in E(G_2^u);w\notin V(e)} \sum_{f\in E(G_2^v);w\notin V(f)} [D_1(w, e\,|G_2) + D_1(w, f\,|G_2)$$

$$+ d(u, v\,|G_1) + 1]^{\lambda}$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W^{(j)}(G_1) \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w\,|G_2) D_1^{(i-j-k)}(w\,|G_2).$$

The sum S_{42} is equal to

$$S_{42} = \sum_{\{u,v\}\subseteq V(G_1)} \sum_{e\in E(G_2^u);w\notin V(e)} \sum_{f\in E(G_2^v);w\in V(f)} [D_1(w, e\,|G_2) + d(u, v\,|G_1) + 1]^{\lambda}$$

$$= \omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W^{(j)}(G_1) D_1^{(i-j)}(w\,|G_2).$$

The sum S_{43} is equal to

$$S_{43} = \sum_{\{u,v\}\subseteq V(G_1)} \sum_{e\in E(G_2^u);w\in V(e)} \sum_{f\in E(G_2^v);w\notin V(f)} [D_1(w, f\,|G_2) + d(u, v\,|G_1) + 1]^{\lambda}.$$

It is clear that, $S_{43} = S_{42}$. So,

$$S_{43} = \omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W^{(j)}(G_1) D_1^{(i-j)}(w\,|G_2).$$

The sum S_{44} is equal to

$$S_{44} = \sum_{\{u,v\} \subseteq V(G_1)} \sum_{e \in E(G_2^u); w \in V(e)} \sum_{f \in E(G_2^v); w \in V(f)} [d(u,v \,|G_1) + 1]^\lambda = \omega^2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1).$$

By adding the quantities S_{41}, S_{42}, S_{43} and S_{44}, we obtain

$$S_4 = \omega^2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W^{(j)}(G_1) \left[2\omega D_1^{(i-j)}(w \,|G_2) \right.$$
$$\left. + \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w \,|G_2) D_1^{(i-j-k)}(w \,|G_2) \right].$$

Now, the formula for $W_{e0}^{(\lambda)}(G_1\{G_2\})$ is obtained by adding the quantities S_1, S_2, S_3 and S_4.

(ii) Using the similar argument as in the proof of part *(i)*, we can get the desired result. $\qquad\square$

Using Theorem 2.2, we can get the formulae for the edge-Wiener and edge hyper-Wiener indices of the cluster $G_1\{G_2\}$.

Corollary 2.4. *The first and second edge-Wiener indices of $G_1\{G_2\}$ are given by*

(i) $W_{e_0}(G_1\{G_2\}) = W_{e_0}(G_1) + n_1 W_{e_0}(G_2) + m_2 W_{ve_1}(G_1) + m_2^2 W(G_1)$
$$+ \left[2m_2 \binom{n_1}{2} + n_1 m_1 \right] D_1^{(1)}(w \,|G_2) + \binom{n_1}{2} m_2^2 + n_1 m_1 m_2,$$
(ii) $W_{e_4}(G_1\{G_2\}) = W_{e_4}(G_1) + n_1 W_{e_4}(G_2) + m_2 W_{ve_2}(G_1) + m_2^2 W(G_1)$
$$+ \left[2m_2 \binom{n_1}{2} + n_1 m_1 \right] D_2^{(1)}(w \,|G_2).$$

Corollary 2.5. *The first and second edge hyper-Wiener indices of $G_1\{G_2\}$ are given by*

(i) $WW_{e_0}(G_1\{G_2\}) = WW_{e_0}(G_1) + n_1 WW_{e_0}(G_2) + m_2 WW_{ve_1}(G_1) + n_1 m_1 m_2$
$$+ m_2^2 WW(G_1) + \left[m_2 + D_1^{(1)}(w \,|G_2) \right] W_{ve_1}(G_1) + m_2^2 \binom{n_1}{2}$$
$$+ m_2 \left[m_2 + 2D_1^{(1)}(w \,|G_2) \right] W(G_1) + \binom{n_1}{2} D_1^{(1)}(w \,|G_2)^2$$
$$+ \left[m_2 \binom{n_1}{2} + \frac{1}{2} n_1 m_1 \right] [D_1^{(2)}(w \,|G_2) + 3D_1^{(1)}(w \,|G_2)],$$

(ii) $WW_{e_4}(G_1\{G_2\}) = WW_{e_4}(G_1) + n_1 WW_{e_4}(G_2) + m_2 WW_{ve_2}(G_1)$

$$+ m_2^2 WW(G_1) + D_2^{(1)}(w\,|G_2)[W_{ve_2}(G_1) + 2m_2 W(G_1)]$$

$$+ \left[m_2 \binom{n_1}{2} + \frac{1}{2} n_1 m_1 \right] \times \left[D_2^{(2)}(w\,|G_2) + D_2^{(1)}(w\,|G_2) \right]$$

$$+ \binom{n_1}{2} D_2^{(1)}(w\,|G_2)^2.$$

2.3. **Vertex-edge Wiener type invariants of clusters.** In this section, we determine the vertex-edge Wiener type invariants of the cluster $G_1\{G_2\}$.

Theorem 2.3. *If λ is a positive integer, then the vertex-edge Wiener type invariants of $G_1\{G_2\}$ are given by:*

(i) $W_{ve_1}^{(\lambda)}(G_1\{G_2\}) = W_{ve_1}^{(\lambda)}(G_1) + n_1 W_{ve_1}^{(\lambda)}(G_2) + 2\omega W^{(\lambda)}(G_1) + 2m_1 d^{(\lambda)}(w\,|G_2)$

$$+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_1}^{(i)}(G_1) d^{(\lambda-i)}(w\,|G_2)$$

$$+ 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[D_1^{(\lambda-i)}(w\,|G_2) + \omega d^{(\lambda-i)}(w\,|G_2) \right.$$

$$\left. + \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w\,|G_2) D_1^{(\lambda-i-j)}(w\,|G_2) \right],$$

(ii) $W_{ve_2}^{(\lambda)}(G_1\{G_2\}) = W_{ve_2}^{(\lambda)}(G_1) + n_1 W_{ve_2}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_2}^{(i)}(G_1) d^{(\lambda-i)}(w\,|G_2)$

$$+ 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[D_2^{(\lambda-i)}(w\,|G_2) \right.$$

$$\left. + \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w\,|G_2) D_2^{(\lambda-i-j)}(w\,|G_2) \right].$$

Proof. **(i)** We partition the sum in the definition of $W_{ve_1}^{(\lambda)}(G_1\{G_2\})$ into three sums as follows. The first sum S_1 is taken over all vertices $u \in V(G_2^x)$ and edges $e \in E(G_2^x)$, where $x \in V(G_1)$. In this case, $D_1(u, e\,|G_1\{G_2\}) = D_1(u, e\,|G_2)$. So,

$$S_1 = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)} \sum_{e \in E(G_2^x)} D_1(u, e\,|G_2)^{\lambda} = n_1 W_{ve_1}^{(\lambda)}(G_2).$$

The second sum S_2 is taken over all vertices $u \in V(G_2^x)$ and edges $e \in E(G_1)$, where $x \in V(G_1)$. In this case, $D_1(u, e\,|G_1\{G_2\}) = d(u, w\,|G_2) + D_1(x, e\,|G_1)$. So,

$$S_2 = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)} \sum_{e \in E(G_1)} [d(u, w\,|G_2) + D_1(x, e\,|G_1)]^{\lambda}.$$

We partition the sum S_2 into three sums S_{21}, S_{22} and S_{23} as follows. The sum S_{21} is equal to $S_{21} = \sum_{x \in V(G_1)} \sum_{u=w} \sum_{e \in E(G_1)} D_1(x, e \,|G_1)^\lambda = W_{ve_1}^{(\lambda)}(G_1)$. The sum S_{22} is equal to $S_{22} = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)-\{w\}} \sum_{e \in E(G_1); x \in V(e)} d(u, w \,|G_2)^\lambda = 2m_1 d^{(\lambda)}(w \,|G_2)$. The sum S_{23} is equal to

$$S_{23} = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)-\{w\}} \sum_{e \in E(G_1); x \notin V(e)} [d(u, w \,|G_2) + D_1(x, e \,|G_1)]^\lambda$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_1}^{(i)}(G_1) d^{(\lambda-i)}(w \,|G_2).$$

By adding the quantities S_{21}, S_{22} and S_{23}, we obtain

$$S_2 = W_{ve_1}^{(\lambda)}(G_1) + 2m_1 d^{(\lambda)}(w \,|G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_1}^{(i)}(G_1) d^{(\lambda-i)}(w \,|G_2).$$

The third sum S_3 is taken over all vertices $u \in V(G_2^x)$ and edges $e \in E(G_2^y)$, where $x, y \in V(G_1)$ and $x \neq y$. In this case, $D_1(u, e \,|G_1\{G_2\}) = d(u, w \,|G_2) + d(x, y \,|G_1) + D_1(w, e \,|G_2)$. So,

$$S_3 = \sum_{x \in V(G_1)} \sum_{y \in V(G_1)-\{x\}} \sum_{u \in V(G_2^x)} \sum_{e \in E(G_2^y)} [d(u, w \,|G_2) + d(x, y \,|G_1) + D_1(w, e \,|G_2)]^\lambda.$$

We partition the sum S_3 into four sums S_{31}, S_{32}, S_{33} and S_{34} as follows. The sum S_{31} is equal to $S_{31} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1)-\{x\}} \sum_{u=w} \sum_{e \in E(G_2^y); w \in V(e)} d(x, y \,|G_1)^\lambda = 2\omega W^{(\lambda)}(G_1)$. The sum S_{32} is equal to

$$S_{32} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1)-\{x\}} \sum_{u=w} \sum_{e \in E(G_2^y); w \notin V(e)} [d(x, y \,|G_1) + D_1(w, e \,|G_2)]^\lambda$$

$$= 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) D_1^{(\lambda-i)}(w \,|G_2).$$

The sum S_{33} is equal to

$$S_{33} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1)-\{x\}} \sum_{u \in V(G_2^x)-\{w\}} \sum_{e \in E(G_2^y); w \in V(e)} [d(u, w \,|G_2) + d(x, y \,|G_1)]^\lambda$$

$$= 2\omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) d^{(\lambda-i)}(w \,|G_2).$$

The sum S_{34} is equal to

$$S_{34} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1)-\{x\}} \sum_{u \in V(G_2^x)-\{w\}} \sum_{e \in E(G_2^y); w \notin V(e)} [d(u,w \,|G_2) + d(x,y \,|G_1)$$

$$+ D_1(w,e \,|G_2)]^\lambda$$

$$= 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w \,|G_2) D_1^{(\lambda-i-j)}(w \,|G_2).$$

By adding the quantities S_{31}, S_{32}, S_{33} and S_{34}, we obtain

$$S_3 = 2\omega W^{(\lambda)}(G_1) + 2\sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \Big[D_1^{(\lambda-i)}(w \,|G_2) + \omega d^{(\lambda-i)}(w \,|G_2)$$

$$+ \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w \,|G_2) D_1^{(\lambda-i-j)}(w \,|G_2) \Big].$$

The formula for $W_{ve_1}^{(\lambda)}(G_1\{G_2\})$ is obtained by adding the quantities S_1, S_2 and S_3.
(ii) Using the similar argument as in the proof of part (i), we can get the desired
result. $\square$

Using Theorem 2.3, the formulae for the vertex-edge Wiener and vertex-edge hyper-Wiener indices of $G_1\{G_2\}$ are obtained at once.

Corollary 2.6. *For $i \in \{1,2\}$, the first and second vertex-edge Wiener indices of $G_1\{G_2\}$ are given by*

$$W_{ve_i}(G_1\{G_2\}) = n_2 W_{ve_i}(G_1) + n_1 W_{ve_i}(G_2) + 2n_2 m_2 W(G_1) + 2n_2 \binom{n_1}{2} D_i^{(1)}(w \,|G_2)$$

$$+ \Big[2m_2 \binom{n_1}{2} + n_1 m_1 \Big] d^{(1)}(w \,|G_2).$$

Corollary 2.7. *For $i \in \{1,2\}$, the first and second vertex-edge hyper-Wiener indices of $G_1\{G_2\}$ are given by*

$$WW_{ve_i}(G_1\{G_2\}) = n_2 WW_{ve_i}(G_1) + n_1 WW_{ve_i}(G_2) + n_2 m_2 WW(G_1)$$

$$+ d^{(1)}(w \,|G_2) W_{ve_i}(G_1) + 2W(G_1)\big[n_2 D_i^{(1)}(w \,|G_2)$$

$$+ m_2 d^{(1)}(w \,|G_2)\big] + \Big[m_2 \binom{n_1}{2} + \frac{1}{2} n_1 m_1 \Big] \big[d^{(1)}(w \,|G_2)$$

$$+ d^{(2)}(w \,|G_2)\big] + n_2 \binom{n_1}{2} \Big[D_i^{(1)}(w \,|G_2) + D_i^{(2)}(w \,|G_2) \Big]$$

$$+ 2\binom{n_1}{2} d^{(1)}(w \,|G_2) D_i^{(1)}(w \,|G_2).$$

TABLE 1. Some distance-based topological invariants of path and star.

Graph G	P_n	S_n
$W(G)$	$\binom{n+1}{3}$	$(n-1)^2$
$WW(G)$	$\binom{n+2}{4}$	$\frac{1}{2}(n-1)(3n-4)$
$TSZ(G)$	$\binom{n+3}{5}$	$(n-1)(2n-3)$
$W_{e_0}(G)$	$\binom{n}{3}$	$\binom{n-1}{2}$
$WW_{e_0}(G)$	$2\binom{n+1}{4}$	$\binom{n-1}{2}$
$W_{e_4}(G)$	$\frac{n+3}{3}\binom{n-1}{2}$	$2\binom{n-1}{2}$
$WW_{e_4}(G)$	$\frac{n^2+5n+12}{6}\binom{n-1}{2}$	$3\binom{n-1}{2}$
$W_{ve_1}(G)$	$2\binom{n}{3}$	$2\binom{n-1}{2}$
$WW_{ve_1}(G)$	$2\binom{n+1}{4}$	$2\binom{n-1}{2}$
$W_{ve_2}(G)$	$2\binom{n+1}{3}$	$2(n-1)^2$
$WW_{ve_2}(G)$	$2\binom{n+2}{4}$	$(n-1)(3n-4)$

3. EXAMPLES AND COROLLARIES

In this section, our results are illustrated by some examples. Interesting classes of graphs can be obtained by specializing components in clusters. Let P_n and S_n denote the n vertex path and star, respectively. Some distance-based topological invariants of these graphs are listed in Table 1.

Our first example is about the t-fold bristled graph of a given graph. For a given graph G, its t-fold bristled graph $Brs_t(G)$ is obtained by attaching t pendant vertices to each vertex of G. This graph can be represented as the cluster of G and S_{t+1}, where the root vertex w of S_{t+1} is on its vertex of degree t. The t-fold bristled graph of a given graph is also known as its t-thorny graph. It is easy to see that,

$$d^{(1)}(w\,|S_{t+1}) = d^{(2)}(w\,|S_{t+1}) = d^{(3)}(w\,|S_{t+1}) = D_2^{(1)}(w\,|S_{t+1}) = D_2^{(2)}(w\,|S_{t+1}) = t,$$

$$D_1^{(1)}(w\,|S_{t+1}) = D_1^{(2)}(w\,|S_{t+1}) = 0.$$

Using these results, the results obtained in the previous section and Table 1, we can compute several Wiener-like topological invariants of t-fold bristled graphs.

Corollary 3.1. *Let G be a graph of order n and size m and let t be a positive integer. Then*

(i) $W(Brs_t(G)) = (t+1)^2 W(G) + nt(nt+n-1)$;

(ii) $WW(Brs_t(G)) = (t+1)^2 WW(G) + 2t(t+1)W(G) + \frac{1}{2}nt(3nt+2n-3)$;

(iii) $TSZ(Brs_t(G)) = (t+1)^2 TSZ(G) + 2t(t+1)WW(G) + t(3t+2)W(G)$
$\qquad + nt(2nt+n-2)$;

(iv) $W_{e_0}(Brs_t(G)) = W_{e_0}(G) + tW_{ve_1}(G) + t^2 W(G) + \frac{1}{2}nt(nt+2m-1)$;

(v) $W_{e_4}(Brs_t(G)) = W_{e_4}(G) + tW_{ve_2}(G) + t^2 W(G) + nt(nt+m-1)$;

(vi) $WW_{e_0}(Brs_t(G)) = WW_{e_0}(G) + tWW_{ve_1}(G) + t^2WW(G) + tW_{ve_1}(G)$

$$+ t^2W(G) + \frac{1}{2}nt(nt + 2m - 1);$$

(vii) $WW_{e_4}(Brs_t(G)) = WW_{e_4}(G) + tWW_{ve_2}(G) + t^2WW(G) + tW_{ve_2}(G)$

$$+ 2t^2W(G) + \frac{1}{2}nt(3nt + 2m - 3);$$

(viii) $W_{ve_1}(Brs_t(G)) = (t + 1)W_{ve_1}(G) + 2t(t + 1)W(G) + nt(nt + m - 1);$

(ix) $W_{ve_2}(Brs_t(G)) = (t + 1)W_{ve_2}(G) + 2t(t + 1)W(G) + nt(2nt + n + m - 1);$

(x) $WW_{ve_1}(Brs_t(G)) = (t + 1)WW_{ve_1}(G) + t(t + 1)WW(G) + tW_{ve_1}(G)$

$$+ 2t^2W(G) + nt(nt + m - 1);$$

(xi) $WW_{ve_2}(Brs_t(G)) = (t + 1)WW_{ve_2}(G) + t(t + 1)WW(G) + tW_{ve_2}(G)$

$$+ 2t(2t + 1)W(G) + nt(3nt + n + m - 2).$$

Our next example is about the bridge graph constructed on a given graph. Let G be a graph rooted at vertex w and let t be a positive integer. The bridge graph $B_t(G, w)$ is the graph obtained by taking t copies of G and by connecting the vertex w of the i-th copy of G to the vertex w of the $i + 1$-th copy of G by an edge for $i = 1, 2, ..., t - 1$. The bridge graph $B_t(G, w)$ can be considered as the cluster of the t-vertex path P_t and the graph G. So, using the results of the previous section and Table 1, we get the following results for bridge graphs.

Corollary 3.2. *Let G be a rooted graph of order n and size m and let w be its root vertex. For positive integer t, the following hold*

(i) $W(B_t(G, w)) = tW(G) + 2n\binom{t}{2}d^{(1)}(w \,|G) + n^2\binom{t+1}{3};$

(ii) $WW(B_t(G, w)) = tWW(G) + 2n\binom{t+1}{3}d^{(1)}(w \,|G) + n^2\binom{t+2}{4}$

$$+ \binom{t}{2}\left[nd^{(1)}(w \,|G) + d^{(1)}(w \,|G)^2 + nd^{(2)}(w \,|G)\right];$$

(iii) $TSZ(B_t(G, w)) = tTSZ(G) + 2n\binom{t+2}{4}d^{(1)}(w \,|G) + \binom{t+1}{3}\left[nd^{(2)}(w \,|G)\right.$

$$\left. + d^{(1)}(w \,|G)^2 + nd^{(1)}(w \,|G)\right] + \frac{1}{3}n\binom{t}{2}\left[d^{(3)}(w \,|G)\right.$$

$$\left. + 3d^{(2)}(w \,|G) + 2d^{(1)}(w \,|G)\right] + \binom{t}{2}d^{(1)}(w \,|G)\left[d^{(1)}(w \,|G)\right.$$

$$\left. + d^{(2)}(w \,|G)\right] + n^2\binom{t+3}{5};$$

(iv) $W_{e_0}(B_t(G, w)) = tW_{e_0}(G) + 2(m + 1)\binom{t}{2}D_1^{(1)}(w \,|G) + m^2\left[\binom{t+1}{3} + \binom{t}{2}\right]$

$$+ (2m + 1)\binom{t}{3} + 2m\binom{t}{2};$$

(v) $W_{e_4}(B_t(G, w)) = tW_{e_4}(G) + 2(m+1)\binom{t}{2}D_2^{(1)}(w\,|G) + m(m+2)\binom{t+1}{3}$

$$+ \frac{1}{6}(t-1)(t-2)(t+3);$$

(vi) $WW_{e_0}(B_t(G, w)) = tWW_{e_0}(G) + \binom{t}{2}\Big[(m+1)D_1^{(2)}(w\,|G) + D_1^{(1)}(w\,|G)^2$

$$+ (m+1)D_1^{(1)}(w\,|G)\Big] + 2\Big[m\binom{t+1}{3} + \binom{t}{3}$$

$$+ (m+1)\binom{t}{2}\Big]D_1^{(1)}(w\,|G) + m^2\Big[\binom{t+2}{4} + \binom{t+1}{3}\Big]$$

$$+ 2(m+1)\binom{t+1}{4} + 2m\binom{t}{3} + m(m+2)\binom{t}{2};$$

(vii) $WW_{e_4}(B_t(G, w)) = tWW_{e_4}(G) + \binom{t}{2}\Big[(m+1)D_2^{(2)}(w\,|G) + D_2^{(1)}(w\,|G)^2$

$$+ (m+1)D_2^{(1)}(w\,|G)\Big] + 2(m+1)\binom{t+1}{3}D_2^{(1)}(w\,|G)$$

$$+ m(m+2)\binom{t+2}{4} + \frac{1}{12}(t-1)(t-2)(t^2+5t+12);$$

(viii) $W_{ve_1}(B_t(G, w)) = tW_{ve_1}(G) + 2\binom{t}{2}\Big[(m+1)d^{(1)}(w\,|G) + nD_1^{(1)}(w\,|G)\Big]$

$$+ 2n\Big[m\binom{t+1}{3} + \binom{t}{3}\Big];$$

(ix) $W_{ve_2}(B_t(G, w)) = tW_{ve_2}(G) + 2\binom{t}{2}\Big[(m+1)d^{(1)}(w\,|G) + nD_2^{(1)}(w\,|G)\Big]$

$$+ 2n(m+1)\binom{t+1}{3};$$

(x) $WW_{ve_1}(B_t(G, w)) = tWW_{ve_1}(G) + 2\binom{t}{3}d^{(1)}(w\,|G) + 2\binom{t+1}{3}\Big[nD_1^{(1)}(w\,|G)$

$$+ md^{(1)}(w\,|G)\Big] + (m+1)\binom{t}{2}\Big[d^{(1)}(w\,|G) + d^{(2)}(w\,|G)\Big]$

$$+ n\binom{t}{2}\Big[D_1^{(1)}(w\,|G) + D_1^{(2)}(w\,|G)\Big] + nm\binom{t+2}{4}$$

$$+ 2\binom{t}{2}d^{(1)}(w\,|G)D_1^{(1)}(w\,|G) + 2n\binom{t+1}{4};$$

$$(\text{xi}) \quad WW_{ve_2}(B_t(G,w)) = 2\binom{t+1}{3}\left[nD_2^{(1)}(w\,|G) + (m+1)d^{(1)}(w\,|G)\right]$$

$$+ tWW_{ve_2}(G) + (m+1)\binom{t}{2}\left[d^{(1)}(w\,|G) + d^{(2)}(w\,|G)\right]$$

$$+ n\binom{t}{2}\left[D_2^{(1)}(w\,|G) + D_2^{(2)}(w\,|G)\right]$$

$$+ 2\binom{t}{2}d^{(1)}(w\,|G)D_2^{(1)}(w\,|G) + n(m+2)\binom{t+2}{4}.$$

The results of parts (i) and (ii) of Corollary 3.2 have also been given in [20].

REFERENCES

[1] M. Azari and A. Iranmanesh, *Computation of the edge Wiener indices of the sum of graphs*, Ars Combin. **100** (2011), 113–128.

[2] M. Azari and A. Iranmanesh, *Computing the eccentric-distance sum for graph operations*, Discrete Appl. Math. **161**(18) (2013), 2827–2840.

[3] M. Azari and A. Iranmanesh, *The second edge-Wiener index of some composite graphs*, Miskolc. Math. Notes. **15**(2) (2014), 305–316.

[4] M. Azari, A. Iranmanesh and A. Tehranian, *Two topological indices of three chemical structures*, MATCH Commun. Math. Comput. Chem. **69** (2013), 69–86.

[5] P. Dankelman, I. Gutman, S. Mukwembi and H. C. Swart, *The edge Wiener index of a graph*, Discrete Math. **309** (2009), 3452–3457.

[6] M. R. Darafsheh and M. H. Khalifeh, *Calculation of the Wiener, Szeged, and PI indices of a certain nanostar dendrimer*, Ars Combin. **100** (2011), 289–298.

[7] T. Došlić, *Splices, links and their degree-weighted Wiener polynomials*, Graph Theory Notes N. Y. **48** (2005), 47–55.

[8] T. Došlić, *Vertex-Weighted Wiener polynomials for composite graphs*, Ars Math. Contemp. **1** (2008), 66–80.

[9] B. Furtula, I. Gutman and H. Lin, *More trees with all degrees odd having extremal Wiener index*, MATCH Commun. Math. Comput. Chem. **70** (2013), 293–296.

[10] I. Gutman, *A property of the Wiener number and its modifications*, Indian J. Chem. **36**(A) (1997), 128–132.

[11] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, 1986.

[12] I. Gutman, D. Vidović and L. Popović, *On graph representation of organic molecules-Cayley's plerograms vs. his kenograms*, J. Chem. Soc. Faraday Trans. **94** (1998), 857–860.

[13] S. Hossein-Zadeh, A. Hamzeh and A. R. Ashrafi, *Wiener-type invariants of some graph operations*, Filomat, **23**(3) (2009), 103–113.

[14] A. Iranmanesh and M. Azari, *Edge-Wiener descriptors in chemical graph theory: A survey*, Curr. Org. Chem. **19**(3) (2015), 219–239.

[15] A. Iranmanesh, I. Gutman, O. Khormali and A. Mahmiani, *The edge versions of the Wiener index*, MATCH Commun. Math. Comput. Chem. **61** (2009), 663–672.

[16] A. Iranmanesh, A. Soltani Kafrani and O. Khormali, *A new version of Hyper-Wiener index*, MATCH Commun. Math. Comput. Chem. **65** (2011), 113–122.

[17] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi and S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin. **30** (2009), 1149–1163.

[18] D. J. Klein and I. Gutman, *Wiener-number-related sequences*, J. Chem. Inf. Comput. Sci. **39** (1999), 534–536.

[19] D. J. Klein, I. Lukovits and I. Gutman, *On the definition of the hyper-Wiener index for cycle-containing structures*, J. Chem. Inf. Comput. Sci. **35** (1995), 50–52.

[20] T. Mansour and M. Schork, *Wiener, hyper-Wiener, detour and hyper-detour indices of bridge and chain graphs*, J. Math. Chem. **581** (2009), 59–69.

[21] M. J. Nadjafi-Arani, H. Khodashenas and A. R. Ashrafi, *Relationship between edge Szeged and edge Wiener indices of graphs*, Glas. Mat. Ser. III. **47**(1) (2012), 21–29.

[22] M. Randić, *Novel molecular descriptor for structure-property studies*, Chem. Phys. Lett. **211** (1993), 478–483.

[23] D. Stevanović, *Hosoya polynomials of composite graphs*, Discrete Math. **235** (2001), 237–244.

[24] S. S. Tratch, M. I. Stankevich and N. S. Zefirov, *Combinatorial models and algorithms in chemistry. The expanded Wiener number - A novel topological index*, J. Comput. Chem. **11** (1990), 899–908.

[25] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, FL, 1992.

[26] Wiener, H. *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. **69** (1947), 17–20.

[27] Z. Yarahmadi, T. Došlić and A. R. Ashrafi, *The bipartite edge frustration of composite graphs*, Discrete Appl. Math. **158** (2010), 1551–1558.

[28] Y. N. Yeh and I. Gutman, *On the sum of all distances in composite graphs*, Discrete Math. **135** (1994), 359–365.

[1]DEPARTMENT OF MATHEMATICS,
KAZERUN BRANCH, ISLAMIC AZAD UNIVERSITY,
P. O. BOX: 73135-168, KAZERUN, IRAN
E-mail address: `azari@kau.ac.ir`

[2]DEPARTMENT OF MATHEMATICS,
FACULTY OF MATHEMATICAL SCIENCES,
TARBIAT MODARES UNIVERSITY,
P. O. BOX: 14115-137, TEHRAN, IRAN
E-mail address: `iranmanesh@modares.ac.ir`