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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS CONCERNING SHARED FUNCTIONS

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Abstract. It is mainly proved: Let $\mathfrak{F}$ be a family of meromorphic function in $\mathcal{D}$, $a(z)(\neq 0)$ and $b(z)(\not\equiv 0)$ be two holomorphic functions on $\mathcal{D}$. Suppose that admits the zeros of multiplicity at least 3 for each function $f \in \mathfrak{F}$. For each $f \in \mathfrak{F}$, if $f = a(z) \Leftrightarrow f' = b(z)$, then $\mathfrak{F}$ is normal in $\mathcal{D}$. Some example shows that the multiplicity of zeros of f is best in some sense. And the result of paper improve and supplement the result of Lei, Yang and Fang [J. Math. Anal. App. 364 (2010), 143–150].

1. Introduction and Main Results

Let $\mathcal{D}$ be a domain in $\mathbb{C}$, and $\mathfrak{F}$ be a family of meromorphic functions defined in the domain $\mathcal{D}$. $\mathfrak{F}$ is said to be normal in $\mathcal{D}$, in the sense of Montel, if for every sequence $\{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that f_{n_j} converges spherically uniformly to a meromorphic function $f(z)$ or ∞ (see [1]).

A family $\mathfrak{F}$ is said to be normal at a point $z_0 \in \mathcal{D}$ if there exists a neighborhood of z_0 in which $\mathfrak{F}$ is normal. It is well known that $\mathfrak{F}$ is normal in a domain $\mathcal{D}$ if and only if it is normal at each of its points (see [1]).

Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathcal{D}$ and $a, b \in \mathbb{C}$. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$.

In 2002, Fang and Zalcman [2] proved the following theorem.

Theorem 1.1. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and a, b be two nonzero complex numbers. Let k be a positive integer. Suppose that admits*

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the zeros of multiplicity at least $k+1$ for each function $f \in \mathfrak{F}$. For each $f \in \mathfrak{F}$, if $f = a \Leftrightarrow f^{(k)} = b$, then $\mathfrak{F}$ is normal in $\mathcal{D}$.

In 2010, Lei, Yang and Fang [3] extended the constants a, b in Theorem 1.1 to holomorphic functions $a(z)(\neq 0), b(z)(\not\equiv 0)$, as follows.

Theorem 1.2. [4] *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(z)(\neq 0), b(z)(\not\equiv 0)$ be two holomorphic functions. Let $k(\geq 2)$ be a positive integer. Suppose that admits the zeros of multiplicity at least $k+1$ for each function $f \in \mathfrak{F}$. For each $f \in \mathfrak{F}$, if $f = a(z) \Leftrightarrow f^{(k)} = b(z)$, then $\mathfrak{F}$ is normal in $\mathcal{D}$.*

Naturally, we pose the following question: Is the conclusion of Theorem 1.2 also true for $k = 1$.

First, we give the following counterexample.

Example 1.1. Let $\mathcal{D} = \{z : |z| < 1\}, a(z) = \frac{z^2+2}{2}, b(z) = z$. Let $\mathfrak{F} = \{f_n\}$ where $f_n(z) = \frac{(z^2-\frac{1}{n})^2}{2(z^2+\frac{1}{2n^2})}$, $z \in \mathcal{D}$ $(n = 1, 2, \ldots)$. Clearly, all the zeros of $f_n(z)$ are multiple, and $f_n(z) - a(z) = -\frac{\left(1+\frac{1}{2n}\right)^2 z^2}{z^2+\frac{1}{2n^2}}$, $f_n'(z) - b(z) = -\frac{\left(1+\frac{1}{2n}\right)^2 z}{n^2\left(z^2+\frac{1}{2n^2}\right)^2}$. Thus $f_n(z) - a(z) \neq 0$, $f_n'(z) - b(z) \neq 0$ in $\mathbb{C} \setminus \{0\}$. It follows that $f_n(z) - a(z) = 0 \Leftrightarrow f_n'(z) - b(z) = 0$, this is $f_n(z) = a(z) \Leftrightarrow f_n'(z) = b(z)$, however, $\mathfrak{F}$ fails to be normal in $\mathcal{D}$ since $f_n\left(\frac{1}{\sqrt{n}}\right) = 0, f_n\left(\frac{1}{\sqrt{2}in}\right) = \infty$ as $n \to \infty$.

Example 1.1 shows that the conclusion of Theorem 1.2 does not hold for $k = 1$. This suggests that some further investigation is necessary for the case $k = 1$. In the paper we take up this problem and prove the following result.

Theorem 1.3. *Let $\mathfrak{F}$ be a family of meromorphic function in $\mathcal{D}$, $a(z)(\neq 0)$ and $b(z)$ $(\not\equiv 0)$ be two holomorphic functions on $\mathcal{D}$. Suppose that admits the zeros of multiplicity at least* 3 *for each function $f \in \mathfrak{F}$. For each $f \in \mathcal{F}$, if $f = a(z) \Leftrightarrow f' = b(z)$, then $\mathfrak{F}$ is normal in $\mathcal{D}$.*

Example 1.2. Let $\mathcal{D} = \{z : |z| < 1\}, a(z) = \frac{z^2}{2}, b(z) = z$. Let $\mathfrak{F} = \{f_n\}$ where $f_n = \frac{z^4}{2\left(z^2-\frac{1}{n}\right)}$, $z \in D$ $(n = 1, 2, \ldots)$. Then $f_n(z) - \frac{1}{2}z^2 = \frac{\frac{z^2}{n}}{2\left(z^2-\frac{1}{n}\right)}$, $f_n'(z) - z = -\frac{\frac{z}{n^2}}{\left(z^2-\frac{1}{n}\right)^2}$. Clearly, $f_n(z) = a(z) \Leftrightarrow f_n'(z) = b(z)$, however, $\mathfrak{F}$ fails to be normal in $\mathcal{D}$ since $f_n\left(\frac{1}{\sqrt{n}}\right) = \infty, f_n(0) = 0$ as $n \to \infty$.

Remark 1.1. Example 1.1 shows that the condition that all zeros of f have multiplicity at least 3 in Theorems 1.3 is shape. Example 1.2 shows that the condition that $a(z) \neq 0$ is necessary.

2. Auxiliary Results

To prove our result, we require some preliminary results.

Lemma 2.1. [4] *Let $\mathfrak{F}$ be a family of functions meromorphic on a domain $\mathcal{D}$, all of whose zeros have multiplicity at least k. Suppose that there exists $A \geqslant 1$ such that $\left|f^{(k)}(z)\right| \leqslant A$ whenever $f(z)=0$. Then if $\mathfrak{F}$ is not normal at $z_0 \in \mathcal{D}$, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(i) *points $z_n,\ z_n \to z_0$;*
(ii) *functions $f_n \in \mathfrak{F}$; and*
(iii) *positive numbers $\rho_n \to 0^+$*

such that $\rho_n^{-\alpha} f_n(z_n+\rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros of $g(\xi)$ are of multiplicity at least k, and order at most 2.

Lemma 2.2. [5] *Let $f(z)$ be a transcendental meromorphic function with finite order, all of whose zeros are of multiplicity at least 2, and let $P(z)(\not\equiv 0)$ be a polynomial, then $f'(z)-P(z)$ has infinitely many zeros.*

Lemma 2.3. [5] *Let k be a positive integer, $f(z)$ be a meromorphic function with finite order, all of whose zeros have multiplicity at least $k+2$. If $f^{(k)} \neq 1$, then $f(z)$ is a constant function.*

Lemma 2.4. [6] *Let k, l be positive integers, $Q(z)$ be a non-constant rational function, all of whose zeros have multiplicity at least $k+2$. If $Q^{(k)}(z) \neq z^l$, then $l=1$ and $Q^{(k)}(z) = \frac{1}{(k+1)!}\frac{(z+c)^{k+2}}{(z+(k+2)c)}$, where c is a nonzero constant.*

Lemma 2.5. *Let $\{f_n\}$ be a family of meromorphic functions in a domain $\mathcal{D}$, all of whose zeros are of multiplicity at least 3, and let $a_n(z), b_n(z)$ be two sequences of analytic functions in $\mathcal{D}$ such that $a_n(z) \to a(z) \neq 0, b_n(z) \to b(z)$. If $f_n'(z) = b_n(z) \Rightarrow f_n(z) = a_n(z)$, then $\{f_n\}$ is normal in $\mathcal{D}$.*

Proof. Suppose that $\{f_n\}$ is not normal at $z_0 \in \mathcal{D}$. We may assume that $b(z_0)=1$. By Lemma 2.1, there exists a sequence of complex numbers $z_n \to z_0$, a sequence of functions $f_n \in \{f_n\}$ and a sequence of positive numbers $\rho_n \to 0$ such that $\rho_n^{-1} f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric to a non-constant meromorphic functions $g(\xi)$ on $\mathbb{C}$. Also the order of $g(\xi)$ does not exceed 2 and $g(\xi)$ has no zero of multiplicity less than 3.

We claim $g'(\xi) \neq 1$.

If this is not true, then $g'(\xi_0)=1$. So we have $g'(\xi) \not\equiv 1$, otherwise $g(\xi)$ must be a polynomial with $deg(g)=1$, which contradicts the fact that each zero of $g(\xi)$ has multiplicity at least 3. Since $g'(\xi_0) = 1 = b(\xi_0)$, then there exist $\xi_n \to \xi_0$ such that (for n sufficiently large) $f_n'(z_n+\rho_n\xi_n) = g_n'(\xi_n) = b(z_n+\rho_n\xi_n)$. It follows that $f_n(z_n+\rho_n\xi_n) = a(z_n+\rho_n\xi_n)$, then $g_n(\xi_n) = \rho_n^{-1} f_n(z_n+\rho_n\xi) = \rho_n^{-1} a_n(z_n+\rho_n\xi_n)$. Thus $g(\xi_0)=\infty$, which contradicts $g'(\xi_0)=1$. This proves $g'(\xi)\neq 1$. By Lemma 2.3 $g(\xi)$ is a constant, a contradiction. Thus $\{f_n\}$ is normal in $\mathcal{D}$. □

3. Proof of Theorem 1.3

Proof. For any point $z_0 \in \mathfrak{D}$, either $b(z_0) = 0$ or $b(z_0) \neq 0$. We consider two cases.

Case 1. $b(z_0) \neq 0$. Then by Lemma 2.5, we get that $\mathfrak{F}$ is normal at z_0.

Case 2. $b(z_0) = 0$. Let $z_0 = 0$, $\mathfrak{D} = \Delta = \{z : |z| < 1\}$ and $b(z) = z^m + a_{m+1}z^{m+1} + \cdots = z^m\phi(z)$, $\phi(0) = 1, \phi(z) \neq 1, z \in \{z : 0 < |z| < 1\}$.

Suppose that $\mathfrak{F}$ is not normal in $\mathfrak{D}$. Without loss of generality, we assume that $\mathfrak{F}$ is not normal at $z_0 = 0$.

Consider the family as follows

$$\mathfrak{G} = \left\{g(z) = \frac{f(z)}{z^m}, f \in \mathfrak{F}\right\}.$$

Then $\mathfrak{G}$ is not normal at $z_0 = 0$ in $\mathfrak{D}$. Applying Lemma 2.1, there exists a sequence of complex numbers $z_n \to z_0$, a sequence of functions $f_n \in \{f_n\}$ and a sequence of positive numbers $\rho_n \to 0$ such that

$$G_n(\xi) = \frac{g_n(z_n + \rho_n\xi)}{\rho_n} = \frac{f_n(z_n + \rho_n\xi)}{\rho_n(z_n + \rho_n\xi)^m} \to G(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric to a non-constant meromorphic functions $G(\xi)$ on $\mathbb{C}$. Also the order of $G(\xi)$ does not exceed 2 and $G(\xi)$ has no zero of multiplicity less than 3.

Next, we consider two cases.

Case 2.1. We may suppose that $\frac{z_n}{\rho_n} \to \infty$. From (3.1), we have

$$\widetilde{G}_n(\xi) := \frac{f_n(z_n + \rho_n\xi)}{\rho_n z_n^m} = \left(1 + \frac{\rho_n}{z_n}\right)^m G_n(\xi) \to G(\xi) \tag{3.2}$$

on $\mathbb{C}$, then we have $\widetilde{G}'_n \to G'$ on $\mathbb{C}\backslash G^{-1}(\infty)$. We claim $G' \neq 1$ on $\mathbb{C}$.

Suppose that $G'(\xi_0) = 1$ for $\xi_0 \in \mathbb{C}$. Then we get $G' \not\equiv 1$, otherwise $G(\xi) = \xi + c$, where c is a non-zero constant, which contradicts the fact each zero of $G(\xi)$ has multiplicity at least 3.

Since $\widetilde{G}'_n(\xi) - \left(1 + \frac{\rho_n}{z_n}\xi\right)^m \phi(z_n + \rho_n\xi) \to G'(\xi) - 1$ on $\mathbb{C}\backslash G^{-1}(\infty)$, then there exists $\xi_n \to \xi_0$, such that $\widetilde{G}'_n(\xi_n) - \left(1 + \frac{\rho_n}{z_n}\xi_n\right)^m \phi(z_n + \rho_n\xi_n) = 0$. Thus, for n sufficiently large, we obtain

$$f'_n(z_n + \rho_n\xi_n) = z_n^m\widetilde{G}'_n(\xi_n) = z_n^m\left(1 + \frac{\rho_n}{z_n}\xi_n\right)^m \phi(z_n + \rho_n\xi_n) = b(z_n + \rho_n\xi_n).$$

It follows that $f_n(z_n + \rho_n\xi_n) = a(z_n + \rho_n\xi_n)$. From (3.2) we have

$$G(\xi_0) = \lim_{n\to\infty} \widetilde{G}_n(\xi_n) = \lim_{n\to\infty} \frac{f_n(z_n + \rho_n\xi)}{\rho_n z_n^m} = \lim_{n\to\infty} \frac{a(z_n + \rho_n\xi)}{\rho_n z_n^m} = \infty,$$

which contradicts $G'(\xi_0) = 1$. So $G'(\xi) \neq 1$. By Lemma 2.3 G is a constant, a contradiction.

Case 2.2. We may suppose that $\frac{z_n}{\rho_n} \to \alpha$, a finite complex number. From (3.1), we have

$$\widehat{G}_n(\xi) := \frac{f_n(\rho_n\xi)}{\rho_n^{m+1}\xi^m} = G_n\left(\xi - \frac{z_n}{\rho_n}\right) \to G(\xi-\alpha) := \widehat{G}(\xi) \tag{3.3}$$

on $\mathbb{C}$. Then $\widehat{G}(\xi)$ has no a zero of multiplicity less than 3, and the pole of $\widehat{G}(\xi)$ at $\xi = 0$ has multiplicity at least m.

Now set $F_n(\xi) := \frac{f_n(\rho_n\xi)}{\rho_n^{m+1}}$, and $F(\xi) = \xi^m\widehat{G}(\xi)$. From (3.3), we can get $F_n(\xi) = \xi^m\widehat{G}_n(\xi) \to F(\xi)$. Clearly $F_n(\xi) \to F(\xi)$ on $\mathbb{C}$, $F(0) \neq 0$ and the zero of $F(\xi)$ has multiplicity at least 3.

We claim: (i) $F'(\xi) \neq \xi^m$, (ii) all the poles of $F(\xi)$ are multiple.

First, we now prove that $F'(\xi) \not\equiv \xi^m$. Otherwise $F(\xi) = \frac{1}{m+1}\xi^{m+1} + d$, where d is a constant. Since $F(0) \neq 0$, then $d \neq 0$. So $F(\xi)$ has only simple zeros, contradiction. Thus let $\xi_0 \in \mathbb{C}$ with $F'(\xi_0) = {\xi_0}^m$, then $F(\xi)$ is holomorphic at ξ_0. Therefore, we can obtain

$$\frac{f'{}_n(\rho_n\xi) - b(\rho_n\xi)}{\rho_n^m} = F_n'(\xi) - \xi^m\phi(\rho_n\xi) \to F'(\xi) - \xi^m.$$

By Hurwitz's theorem, there exists point $\xi_n \to \xi_0$ such that $f_n'(\rho_n\xi_n) - b(\rho_n\xi_n) = 0$ for n sufficiently large. So we have $f_n(\rho_n\xi_n) - a(\rho_n\xi_n) = 0$, this is $f_n(\rho_n\xi_n) = a(\rho_n\xi_n)$. Noting that $a(0) \neq 0$, thus

$$F(\xi_0) = \lim_{n\to\infty} F_n(\xi_n) = \lim_{n\to\infty}\frac{f_n(\rho_n\xi_n)}{\rho_n^{m+1}} = \lim_{n\to\infty}\frac{a(\rho_n\xi_n)}{\rho_n^{m+1}} = \infty,$$

which contradicts that $F(\xi)$ is holomorphic at ξ_0. This proves (i).

Next we prove (ii). Suppose $F(\xi_0) = \infty$. There exists a $\overline{\Delta} = \{\xi : |\xi - \xi_0| \leq \delta\}$ such that $\frac{1}{F(\xi)}$ is holomorphic and ξ_0 is the zero of $\frac{1}{F(\xi)}$. Hence $\frac{1}{F_n(\xi)} - \frac{\rho_n^{m+1}}{a(\rho_n\xi)} \to \frac{1}{F(\xi)}$ on $\overline{\Delta}$ and $\frac{1}{F(\xi)} \not\equiv 0$. It follows that there exists $\xi_n \to \xi_0$ such that $\frac{1}{F_n(\xi_n)} - \frac{\rho_n^{m+1}}{a(\rho_n\xi_n)} = 0$ for n sufficiently large. Therefore $f_n'(\rho_n\xi_n) = b(\rho_n\xi_n)$. So we get

$$F_n'(\xi_n) = \frac{f'{}_n(\rho_n\xi_n)}{\rho_n^m} = \frac{b(\rho_n\xi_n)}{\rho_n^m} = \phi(\rho_n\xi_n)\xi_n^m \to \xi_0^m.$$

Thus we have $\left(\frac{1}{F_n(\xi)}\right)'\Big|_{\xi=\xi_0} = -\frac{F_n{}'(\xi)}{F_n^2(\xi)}\Big|_{\xi=\xi_0} = 0$ and $\left(\frac{1}{F_n(\xi)}\right)' \to \left(\frac{1}{F(\xi)}\right)'$. It follows that ξ_0 is a multiple zero of $\left(\frac{1}{F(\xi)}\right)'$, this is, ξ_0 is a multiple pole of $F(\xi)$. This proves (ii).

By Lemma 2.2, $F(\xi)$ must be a rational function. Then by Lemma 2.4, we have $m = 1$ and $F(\xi) = \frac{1}{2}\frac{(\xi+c)^3}{(\xi+3c)}$, which contradicts the fact $F(\xi)$ has multiple poles. Hence we show that $\mathfrak{G}$ is normal at $z_0 = 0$.

Next, we show that $\mathfrak{F}$ is normal at $z_0 = 0$. Since $\mathfrak{G}$ is normal at $z_0 = 0$, let $g_n \to g$ in a neighborhood of 0, then there exist $\Delta_\delta = \{z : |z| < \delta\}$ and a subsequence of $\{g_n\}$ such that $\{g_n\}$ converges uniformly to a meromorphic function or ∞. Noting $g(0) = \infty$, we can find a ε with $0 < \varepsilon < \delta$ and $M > 0$ such that $|g(z)| > M$, $z \in \Delta_\varepsilon$. So, for sufficiently large n, we get $|g_n(z)| \geq \frac{M}{2}$. Therefore $f_n(z) \neq 0$ for sufficiently large n and $z \in \Delta_\varepsilon$.

Hence $\frac{1}{f_n}$ is analytic in Δ_ε. It follows that, for sufficiently large n,

$$\left|\frac{1}{f_n(z)}\right| = \left|\frac{1}{g_n(z)}\right| \frac{1}{|z|^m} \leq \left(\frac{2}{\varepsilon}\right)^m \frac{2}{M}, \quad |z| = \frac{\varepsilon}{2}.$$

By the Maximum Principle and Montel's theorem, $\mathfrak{F}$ is normal at $z_0 = 0$.

These shows that $\mathfrak{F}$ is normal in $\mathcal{D}$. □

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