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PULLBACK DIAGRAM OF HILBERT MODULES OVER H^*-ALGEBRAS

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Abstract. In this paper, we generalize the construction of a pullback diagram in the framework of Hilbert modules over H^*-algebras. More precisely we prove that if a commutative diagram of Hilbert H^*-modules and morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \Big\downarrow{\scriptstyle \Psi_1} & & \Big\downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

is pullback and Ψ_2 is a surjection, then (i) Ψ_1 is a surjection and (ii) $\ker \Phi_1 \cap \ker \Psi_1 = \{0\}$. Conversely, if (i) and (ii) hold, $\psi_1(\tau(A_1))$ is τ_{A_2}-closed and Ψ_2 is injective, then the above diagram is pullback.

1. Introduction and Preliminaries

Pedersen [9] studied pullback diagrams of C^*-algebras. He found conditions under which a commutative diagram of C^*-algebras and morphisms is pullback. Then Amyari and Chakoshi [2] studied it in the framework of Hilbert C^*-modules. In reference [8], we study pullback diagram of H*-algebras and morphisms. We also find conditions for pullbackness such a commutative diagram and its underlying trace classes. In this paper, we generalized the notion of pullback diagram in the framework of Hilbert H^*-modules and describe some new relations between faithful Hilbert modules over commutative proper H^*-algebras and morphisms.

Some properties of pullback diagrams are stable under Hilbert modules over H^*-algebras. We use these properties to discover new ones for pullback diagram of Hilbert modules over H^*-algebras. An H^*-algebra, introduced by Ambrose [1] is a complex

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algebra A with a conjugate-linear mapping $*: A \to A$ and an inner product $\langle .,. \rangle$ such that it is a Hilbert space and satisfies $a^{**} = a, (ab)^* = b^*a^*, \langle ab, c \rangle = \langle a, cb^* \rangle$ and $\langle ab, c \rangle = \langle b, a^*c \rangle$ for all $a, b, c \in A$. Recall that $A_0 = \{a \in A : aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$ is called the annihilator ideal of A. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. The trace-class $\tau(A)$ of an H^*-algebra A is defined by the set $\tau(A) = \{ab : \ a, b \in A\}$. It is known that $\tau(A)$ is an ideal of A, which is a Banach algebra under a suitable norm $\tau_A(.)$. The norm τ_A is related to the given norm $\|.\|$ on A by $\|a\|^2 = \tau_A(a^*a)$ for each $a \in A$. By [1, Lemma 2.7], if A is proper, then $\tau(A)$ is dense in A. The trace functional tr on $\tau(A)$ is defined by $\mathrm{tr}(ab) = \langle b, a^* \rangle = \langle a, b^* \rangle = \mathrm{tr}(ba)$ for each $a, b \in A$, in particular $\mathrm{tr}(aa^*) = \langle a, a \rangle = \|a\|^2 = \tau_A(a^*a)$ for all $a \in A$.

A nonzero element $e \in A$ is called a projection, if it is self-adjoint and idempotent. In addition, if $eAe = \mathbb{C}e$, then it is called a minimal projection. Two idempotents e and e' are doubly orthogonal if $\langle e, e' \rangle = 0$ and $ee' = e'e = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotent.

Lemma 1.1. *Suppose that A is a commutative H^*-algebra. Then e is a minimal projection if and only if e is a primitive projection.*

Proof. Suppose that e is a minimal projection. Then $Ae = eAe = \mathbb{C}e$, but $\mathbb{C}e$ is a minimal ideal of A of rank one. So by [1, Lemma 3.4] e is primitive. Conversely, suppose e is a primitive projection in A. We will show that $eAe = \mathbb{C}e$ or $Ae^2 = \mathbb{C}e$. Obviously $e = e^2 \in Ae$. Therefore $\mathbb{C}e \subseteq Ae$. For the other side, on the contrary, suppose that there exists an element $a \in Ae$ such that $a \notin \mathbb{C}e$, so a and e are independent. Then Aa is a proper ideal of Ae, which contradicts minimality of Ae (Note that if e is primitive, then Ae is a minimal ideal). Hence $Ae \subseteq \mathbb{C}e$. □

Each simple H^*-algebra (an H^*-algebra without nontrivial closed two-sided ideals) contains minimal projections. It is known that all minimal projections in a simple H^*-algebra have equal norms [4]. Also note that if A and B are H^*-algebras, then $A \oplus B$ is an H^*-algebra with $\langle (a_1, b_1), (a_2, b_2) \rangle = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$. For more details on H^*-algebras, see [5, 10] and references cited therein.

Definition 1.1. Let A be a proper H^*-algebra. A Hilbert H^*-module [4, 7] is a right module X over A with a mapping $[\cdot|\cdot] : X \times X \to \tau(A)$, which satisfies the following conditions:

(i) $[x|\alpha y] = \alpha[x|y]$,
(ii) $[x + y|z] = [x|z] + [y|z]$,
(iii) $[x|ya] = [x|y]a$,
(iv) $[x|y]^* = [y|x]$,
(v) For each nonzero element x in X there is a nonzero element a in A such that $[x|x] = a^*a$,
(vi) X is a Hilbert space with the inner product $(x, y) = \mathrm{tr}([x|y])$,

for each $\alpha \in \mathbb{C}$, $x, y \in X$, $a \in A$.

The Hilbert H^*-module X is full if the ideal $I = [X, X] = \text{span}\{[x|y] : x, y \in X\}$ is dense in $\tau(A)$ under the norm $\tau_A(.)$.

Lemma 1.2. *Let X be a full Hilbert module over a proper H^*-algebra A and $a \in A$. Then $xa = 0$ for all $x \in X$ if and only if $a = 0$.*

Proof. If $a \in \tau(A)$ and $xa = 0$ for all $x \in X$, then $[xa|ya] = 0$ for all $x, y \in X$. Let $b \in \tau(A)$ be arbitrary. Since X is full, there exists a sequence $\{u_n\}$ in I such that $\lim_{n\to\infty}{}^{\tau_A} u_n = b$. Each u_n is of the form $\sum_{i=1}^{k_n} \alpha_i[x_i|y_i]$ in which $x_i, y_i \in X$ and $\alpha_i \in \mathbb{C}$. Hence $a^*ba = \lim_{n\to\infty}{}^{\tau_A} a^*u_na = \lim_{n\to\infty}{}^{\tau_A} a^* \left(\sum_{i=1}^{k_n} \alpha_i[x_i|y_i]\right) a = \lim_{n\to\infty}{}^{\tau_A} \sum_{i=1}^{k_n} \alpha_i[x_ia|y_ia] = 0$. Put $b = aa^*$. Therefore $\|a^*a\|^2 = \tau_A(a^*aa^*a) = \text{tr}(a^*aa^*a) = 0$. By [1, Lemma 2.2], $a = 0$.

Suppose that $a \in A - \tau(A)$ and $xa = 0$ for all $x \in X$. Let $b \in A$ be arbitrary. So $ab \in \tau(A)$ and $xab = 0$ for all $x \in X$. Recall that $\|xab\| \leq \|xa\|\|b\|$. By previous argument $ab = 0$ or $aA = \{0\}$. It implies that $a = 0$, since A is proper. □

Let X and Y be Hilbert modules over proper H^*-algebras A and B, respectively, and $\varphi : \tau(A) \to \tau(B)$ be a norm continuous $*$-homomorphism (morphism). A map $\Phi : X \to Y$ is said to be a φ-morphism if $[\Phi(x)|\Phi(y)] = \varphi([x|y])$ for all x, y in X. We can extend φ to a continuous morphism $\bar{\varphi} : A \to B$. Obviously, Φ is a $\bar{\varphi}$- morphism, i.e. $[\Phi(x)|\Phi(y)] = \bar{\varphi}([x|y])$ for each x, y in X. From now on we mean by a φ-morphism, a $\bar{\varphi}$-morphism. It is easy to see that each φ-morphism is necessarily a linear operator and a module mapping in the sense that $\Phi(xa) = \Phi(x)\varphi(a)$ for all $x \in X, a \in A$.

Let X be a Hilbert H^*-module over A and $a \in A$, the left translation $L_a{:}X \to X$ is defined by $L_a(x) = ax$ for $x \in X$. If $e \in A$ is a projection, then L_e is an orthogonal projection defined on the Hilbert space $(X, (.,.))$. Let us denote $X_e = L_eX$. The subspace X_e is a closed subspace of the Hilbert space $(X, (.,.))$ [4].

Theorem 1.1. (see [4, Lemma 2.7]) Let X be a Hilbert H^*-module over A and e be a minimal projection in A. Then $X_e = \{x \in X : [x|x] = \lambda e, \lambda \geq 0\}$. If A is a simple H^*-algebra, then the subspace X_e generates a dense submodule in X.

Remark 1.1. In the above theorem if A is a commutative, simple and proper H^*-algebra, then $X_e = X$. Recall that for each arbitrary minimal projection $e \in A$, we have $A = Ae = eAe = \mathbb{C}e$ [1, Theorem 4.1 and 4.2]. If x is a nonzero element in X, then there is a nonzero element a in A such that $[x|x] = a^*a$. Hence $[x|x] \in \tau(A) \subseteq A = \mathbb{C}e$. So there exists a positive number λ such that $[x|x] = \lambda e$. Therefore $x \in X_e$.

In this paper, we obtain some conditions under which a commutative diagram of Hilbert H^*-modules and morphisms is pullback.

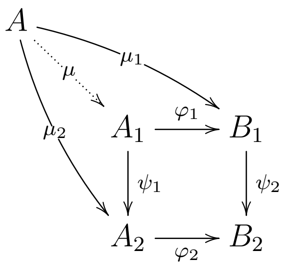
2. Pullback constructions in Hilbert modules over H^*-algebras

In this section we introduce a pullback diagram of H^*-algebras and investigate some properties of them. For this we need the following definition.

Definition 2.1. A commutative diagram of H^*-algebras and morphisms

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is pullback if $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$ and for any other pair of morphisms $\mu_1 : A \to B_1$ and $\mu_2 : A \to A_2$ from an H^*-algebra A that satisfy condition $\psi_2\mu_1 = \varphi_2\mu_2$, there is a unique morphism $\mu : A \to A_1$ such that $\mu_1 = \varphi_1\mu$ and $\mu_2 = \psi_1\mu$.



It follows that A_1 is isomorphic to the restricted direct sum $A_2 \bigoplus_{B_2} B_1 = \{(a_2, b_1) \in A_2 \bigoplus B_1 | \varphi_2(a_2) = \psi_2(b_1)\}$, so that φ_1 and ψ_1 can be identified with projections on first and second coordinates, respectively. In particular, the pullback exists for any triple of H^*-algebras A_2, B_1 and B_2 with linking morphisms φ_2 and ψ_2.

Theorem 2.1. *Suppose that*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array} \tag{2.1}$$

is a commutative diagram of H^*-algebras and morphisms. If $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$ *and* ψ_1, ψ_2 *are surjective and injective, respectively, then the above diagram is pullback.*

Proof. It is enough to show that the morphism $\varphi : A_1 \to A_2 \oplus_{B_2} B_1$ defined by $\varphi(a_1) = (\psi_1(a_1), \varphi_1(a_1))$ is an isomorphism. Let $(a_2, b_1) \in A_2 \oplus_{B_2} B_1$. Then $\psi_2(b_1) = \varphi_2(a_2)$. There exists $a_1 \in A_1$, such that $\psi_1(a_1) = a_2$, since ψ_1 is surjective. By the commutativity of the diagram and injectivity of ψ_2, we have $b_1 = {\psi_2}^{-1}\varphi_2(a_2) = {\psi_2}^{-1}\varphi_2\psi_1(a_1) = {\psi_2}^{-1}\psi_2\varphi_1(a_1) = \varphi_1(a_1)$. It proves the surjectivity of φ.

It is clear that if ψ_1 is injective, then so is φ. For injectivity of ψ_1, let $\psi_1(a_1) = 0$. Thus $\varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = 0$ and injectivity of ψ_2 implies that $\varphi_1(a_1) = 0$. Hence $a_1 \in \ker\varphi_1 \cap \ker\psi_1 = \{0\}$. $\square$

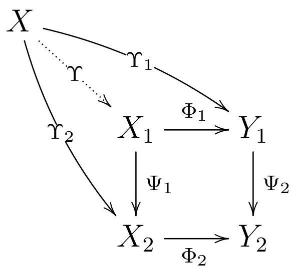
Lemma 2.1. *Suppose that $\Phi_2 : X_2 \to Y_2$ and $\Psi_2 : Y_1 \to Y_2$ are φ_2, ψ_2-morphisms of Hilbert H^*-modules, where $\varphi_2 : A_2 \to B_2$ and $\psi_2 : B_1 \to B_2$ are morphisms of underlying H^*-algebras. Denote by $X_2 \oplus_{Y_2} Y_1$ the set $\{(x_2, y_1) \in X_2 \oplus Y_1 : \Phi_2(x_2) = \Psi_2(y_1)\}$, then $X_2 \oplus_{Y_2} Y_1$ is a Hilbert module over H^*-algebra $A_2 \oplus_{B_2} B_1$ (with operations inherited from the Hilbert $A_2 \oplus B_1$-module $X_2 \oplus Y_1$). If X_2 and Y_1 are full, then $X_2 \oplus Y_1$ is a full $A_2 \oplus B_1$-module and one easily concludes that $X_2 \oplus_{Y_2} Y_1$ is also a full Hilbert module over $A_2 \oplus_{B_2} B_1$.*

Proof. Straightforward (see [3, Proposition 2.1]). □

Definition 2.2. A commutative diagram of Hilbert H^*-modules and morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

is pullback if $\ker(\Phi_1) \cap \ker(\Psi_1) = \{0\}$ and for any other pair of morphisms $\Upsilon_1 : X \to Y_1$ and $\Upsilon_2 : X \to X_2$ from a full Hilbert H^*-module X such that satisfy the condition $\Psi_2 \Upsilon_1 = \Phi_2 \Upsilon_2$, there exists a unique morphism $\Upsilon : X \to X_1$ such that $\Upsilon_1 = \Phi_1 \Upsilon$ and $\Upsilon_2 = \Psi_1 \Upsilon$.



It is easily verified that X_1 is isomorphic to $X_2 \oplus_{Y_2} Y_1$. The following proposition is proved in framework of Hilbert C^*-modules. It is easy to show that this proposition holds in the category of Hilbert H^*-modules. Density of trace class of a proper H^*-algebra in its own is useful in checking commutativity of the diagram of underlying H^*-algebras.

Proposition 2.1. (see [3, Proposition 2.3]) Let X_2, Y_1 and Y_2 be Hilbert modules over H^*-algebras with linking morphisms Φ_2 and Ψ_2. Then

$$\begin{array}{ccc} X_2 \oplus_{Y_2} Y_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

with the projections $\Phi_1(x_2, y_1) = y_1$ and $\Psi_1(x_2, y_1) = x_2$ is a pullback diagram of Hilbert modules over H^*-algebras, where Φ_1 is a φ_1-morphism and Ψ_1 is a ψ_1-morphism of Hilbert modules over H^*-algebras and $\varphi_1 : A_2 \oplus_{B_2} B_1 \to B_1$ and $\psi_1 : A_2 \oplus_{B_2} B_1 \to A_1$ are the corresponding projections.

$$\begin{array}{ccc} A_2 \oplus_{B_2} B_1 & \xrightarrow{\varphi_1} & B_1 \\ \Big\downarrow{\scriptstyle \psi_1} & & \Big\downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

Now we are ready to prove the main theorem of this paper.

Theorem 2.2. *Suppose that*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \Big\downarrow{\scriptstyle \Psi_1} & & \Big\downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array} \tag{2.2}$$

is a commutative diagram of full Hilbert H^*-modules X_1, X_2 and Y_1 and arbitrary Hilbert H^*-module Y_2 and continuous morphisms. If this diagram is pullback and Ψ_2 is surjective, then the following conditions hold

(i) $\ker \Phi_1 \cap \ker \Psi_1 = \{0\}$,
(ii) Ψ_1 *is surjective.*

Conversely, if (i) *and* (ii) *hold,* $\psi_1(\tau(A_1))$ *is* τ_{A_2}*-closed and* Ψ_2 *is injective, then* (2.2) *is pullback.*

Proof. Suppose that the above diagram is pullback. By the definition, (i) holds and there exists a unique isomorphism $\Phi : X_1 \to X_2 \oplus_{Y_2} Y_1$ defined by $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) = (x_2, y_1)$. We will show that the surjectivity of Ψ_2 implies surjectivity of Ψ_1. Let $x_2 \in X_2$. Then $\Phi_2(x_2) \in Y_2 = \Psi_2(Y_1)$. So $\Phi_2(x_2) = \Psi_2(y_1)$ for some $y_1 \in Y_1$. Thus $(x_2, y_1) \in X_2 \oplus_{Y_2} Y_1$. Therefore there exists $x_1 \in X_1$ such that $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) = (x_2, y_1)$, since Φ is onto. Hence Ψ_1 is surjective.

Conversely, suppose that conditions (i) and (ii) hold, $\psi_1(\tau(A_1))$ is τ_{A_2}-closed and Ψ_2 is injective and let (2.1) be the corresponding diagram of underlying H^*-algebras. Clearly Ψ_1, Ψ_2 are ψ_1, ψ_2-morphisms and Φ_1, Φ_2 are φ_1, φ_2-morphisms of corresponding Hilbert H^*-modules. We shall show that the three conditions of Theorem 2.1 hold for the diagram of underlying H^*-algebras. The diagram of H^*-algebras is commutative, since the diagram of their Hilbert modules is commutative. Note that density of trace class of a proper H^*-algebra implies commutativity of the diagram of underlying H^*-algebras.

(I) We want to show that $\ker \varphi_1 \cap \ker \psi_1 = \{0\}$. Let $a_1 \in \ker \varphi_1 \cap \ker \psi_1$ and $x_1 \in X_1$ be arbitrary. Then, we have that $[\Phi_1(x_1 a_1) | \Phi_1(x_1 a_1)] = \varphi_1([x_1 a_1 | x_1 a_1]) = \varphi_1(a_1^*)\varphi_1([x_1|x_1])\varphi_1(a_1) = 0$. Thus $\|\Phi_1(x_1 a_1)\|^2 = \mathrm{tr}([\Phi_1(x_1 a_1)|\Phi_1(x_1 a_1)]) = 0$, so $x_1 a_1 \in \ker \Phi_1$. Similarly $x_1 a_1 \in \ker \Psi_1$. Hence by (i), $x_1 a_1 = 0$, for all $x_1 \in X_1$. Since X_1 is full, by Lemma 1.2, $a_1 = 0$.

(II) Let $b_1 \in B_1$ such that $\psi_2(b_1) = 0$ and $y_1 \in Y_1$ be arbitrary. Then we have that $[\Psi_2(y_1b_1)|\Psi_2(y_1b_1)] = \psi_2(b_1^*)\psi_2([y_1|y_1])\psi_2(b_1) = 0$. So $\|\Psi_2(y_1b_1)\|^2 = \mathrm{tr}([\Psi_2(y_1b_1)|\Psi_2(y_1b_1)]) = 0$. Then $y_1b_1 = 0$ for each $y_1 \in Y_1$, since Ψ_2 is an injection. By the fullness of Y_1, $b_1 = 0$. Then ψ_2 is injective.

(III) We will show that Ψ_1 is surjective. First we show that ψ_1 is injective. If $a_1 \in \ker\psi_1$, then commutativity of (2.1), implies that $\psi_2\varphi_1(a_1) = \varphi_2\psi_1(a_1) = 0$. By (II), ψ_2 is injective, so $\varphi_1(a_1) = 0$ and by (I), $a_1 = 0$. Since X_2 is full, Ψ_1 is surjective and $\psi_1(\tau(A_1))$ is τ_{A_2}-closed, we have

$$\begin{aligned}\tau(A_2) &= \overline{[X_2|X_2]}^{\tau_{A_2}} = \overline{[\Psi_1(X_1)|\Psi_1(X_1)]}^{\tau_{A_2}} = \overline{\psi_1([X_1|X_1])}^{\tau_{A_2}} \\ &\subseteq \overline{\psi_1(\tau(A_1))}^{\tau_{A_2}} \\ &= \psi_1(\tau(A_1)).\end{aligned}$$

Clearly $\psi_1(\tau(A_1)) \subseteq \tau(A_2)$. Let $a_2 \in A_2$ be arbitrary, since A_2 is proper, then there exists a sequence $\{u_n\}$ in $\tau(A_2) = \psi_1(\tau(A_1))$ such that $\lim\limits_{n\to\infty} u_n = a_2$. Each u_n is of the form $\psi_1(a_nb_n)$ in which, $a_n, b_n \in A_1$. Since $\psi_1 : \tau(A_1) \to \tau(A_2)$ is a norm continuous isomorphism and the sequence $\{\psi_1(a_nb_n)\}$ is Cauchy in $\tau(A_2)$, then $\{\psi_1^{-1}(\psi_1(a_nb_n))\} = \{a_nb_n\}$ is Cauchy in $\tau(A_1) \subseteq A_1$. Hence this sequence is convergent in A_1 and $a_2 = \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \psi_1(a_nb_n) = \psi_1(\lim\limits_{n\to\infty}(a_nb_n)) \in \psi_1(A_1)$, i.e., $A_2 \subseteq \psi_1(A_1)$. Then ψ_1 is surjective, and by Theorem 2.1, diagram (2.1) is pullback. Therefore $\varphi : A_1 \to A_2 \oplus_{B_2} B_1$ is defined by $\varphi(a_1) = (\psi_1(a_1), \varphi_1(a_1))$, is an isomorphism.

Define $\Phi : X_1 \to X_2 \oplus_{Y_2} Y_1$ by $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1))$ and show that Φ is an isomorphism of Hilbert H^*-modules. Let $(x_2, y_1) \in X_2 \oplus_{Y_2} Y_1$. By the surjectivity of Ψ_1, $x_2 = \Psi_1(x_1)$ for some $x_1 \in X_1$. By the commutativity of the diagram (2), $\Psi_2\Phi_1(x_1) = \Phi_2\Psi_1(x_1) = \Phi_2(x_2) = \Psi_2(y_1)$. Since Ψ_2 is injective, we have $\Phi_1(x_1) = y_1$. So Φ is a surjection. Also (i) implies that Φ is an injection. On the other hand

$$\begin{aligned}[\Phi(x_1)|\Phi(x_1)] &= [(\Psi_1(x_1), \Phi_1(x_1))|(\Psi_1(x_1), \Phi_1(x_1))] \\ &= ([\Psi_1(x_1)|\Psi_1(x_1)], [\Phi_1(x_1)|\Phi_1(x_1)]) \\ &= (\psi_1([x_1|x_1]), \varphi_1([x_1|x_1])) = \varphi([x_1|x_1]).\end{aligned}$$

So Φ is a φ-morphism. Hence $X_1 \simeq X_2 \oplus_{Y_2} Y_1$. By Proposition 2.1, diagram (2) is a pullback diagram of Hilbert H^*-modules. □

Recall that a Hilbert H^*-module X over A is faithful if $\{a \in A : Xa = \{0\}\} = \{0\}$. By [4, Remark 1.6], for each faithful Hilbert H^*-module X over a proper H^*-algebra A there exists a family $\{X_i\}_{i\in I}$ of Hilbert H^*-modules, where each X_i is a Hilbert H^*-module over a simple H^*-algebra A_i such that X is equal to the mixed product of the family $\{X_i\}_{i\in I}$,

$$X = \bigotimes_{i\in I} X_i = \left\{ \{x_i\} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\|^2 < \infty \right\}.$$

Example 2.1. The Hilbert space $l^2 = \left\{ (a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$ is a commutative and proper H^*-algebra, where for each (a_n) and (b_n) in l^2, $(a_n)(b_n) = (a_n b_n)$ and $(a_n)^* = (\overline{a_n})$ [1, Example 3]. If $e_1 = (1, 0, 0, 0, \dots), e_2 = (0, 1, 0, 0, 0, \dots), \dots$, then $\{e_i\}_{i\in\mathbb{N}}$ is a maximal family of doubly orthogonal primitive elements of l^2. Put $I = \mathbb{C}e_i$ for each $i \in \mathbb{N}$. Then I is a simple and proper H^*-algebra. It is easy to verify that l^2 is a faithful Hilbert module over itself, under the inner product $[(a_j)|(b_j)] = (a_j\overline{b_j}) \in \tau(l^2)$. Since l^2 is also a proper H^*-algebra, then there exists the family $\{I_i\}_{i\in\mathbb{N}}$ of Hilbert H^*-modules, where each I_i is a Hilbert H^*-module over itself as a simple H^*-algebra. Hence $l^2 = \bigotimes_{i\in\mathbb{N}} I_i$.

Let A_1 and A_2 be simple and proper H^*-algebras and φ be a surjective morphism from A_1 into A_2. If e_1 is a minimal projection in A_1, then $\varphi(e_1)$ is a minimal projection in A_2, since

(i) $\varphi(e_1) = \varphi({e_1}^2) = (\varphi(e_1))^2$
(ii) $\varphi(e_1) = \varphi({e_1}^*) = (\varphi(e_1))^*$
(iii) $\varphi(e_1)A_2\varphi(e_1) = \varphi(e_1)\varphi(A_1)\varphi(e_1) = \varphi(e_1A_1e_1) = \varphi(\mathbb{C}e_1) = \mathbb{C}\varphi(e_1)$.

If A and B are commutative simple and proper H^*-algebras and $\varphi : A \to B$ is a nonzero morphism and e,e' are minimal projections in A and B, respectively, then for some complex number λ, $\varphi(\lambda e) = e'$. It implies that every nonzero morphism φ is a surjection. One can easily concludes that φ is an injection, too. Let (2.2) be a commutative diagram of Hilbert modules over commutative simple and proper H^*-algebras and morphisms and let (2.1) be its underlying diagram of H^*-algebras and morphisms. Then for an arbitrary minimal projection e_1 in A_1, there exist minimal projections ${e_1}' = \varphi_1(e_1)$ in B_1, $e_2 = \psi_1(e_1)$ in A_2 and ${e_2}' = \psi_2({e_1}')$ in B_2. Obviously, by the commutativity of diagram, $\varphi_2\psi_1(e_1) = \psi_2\varphi_1(e_1)$. So $\varphi_2(e_2) = {e_2}'$.

Suppose that $X_{1,e_1} = \{x_1 \in X_1 : [x_1|x_1] = \lambda e_1,\ \lambda \geq 0\}$ and $Y_{1,{e_1}'}$, X_{2,e_2}, $Y_{2,{e_2}'}$ are defined similarly. If $x_1 \in X_{1,e_1}$, then $[\Phi_1(x_1)|\Phi_1(x_1)] = \varphi_1([x_1|x_1]) = \varphi_1(\lambda e_1) = \lambda\varphi_1(e_1) = \lambda {e_1}'$ for some $\lambda \geq 0$. Therefore the φ_1-morphism $\Phi_1 : X_{1,e_1} \to Y_{1,{e_1}'}$ is well-defined.

Recall that if $\{e_i\}_{i\in I}$ is a maximal family of doubly orthogonal minimal projections in commutative proper H^*-algebra A, then A is the direct sum of the minimal left ideals Ae_i or the minimal right ideals e_iA [1, Theorem 4.1]. Also by [6, Lemma 34.14], we know that every minimal ideal in A is of the form Ae or eA for some minimal projection e.

Corollary 2.1. *Let* (2.2) *be a commutative diagram of faithful Hilbert modules over commutative proper H^*-algebras and morphisms. If their underlying H^*-algebras have the same cardinal of doubly orthogonal minimal projections and Ψ_1 is surjective, then* (2.2) *is pullback.*

Proof. Suppose that $\{e_{1,i}\}_{i\in I}$, $\{e_{2,i}\}_{i\in I} = \{\psi_1(e_{1,i})\}_{i\in I}$, $\{e'_{1,i}\}_{i\in I} = \{\varphi_1(e_{1,i})\}_{i\in I}$ and $\{e'_{2,i}\}_{i\in I} = \{\psi_2(e'_{1,i})\}_{i\in I}$ are the maximal family of doubly orthogonal minimal projections of A_1, A_2, B_1 and B_2, respectively. Note that these H^*-algebras have the same cardinal of doubly orthogonal minimal projections. Put $A_{1,i}(= A_1e_{1,i})$ for each $i \in I$. Then $\{A_{1,i}\}_{i\in I}$ is the family of minimal closed ideals of A_1. Also there exists a suitable family $\{X_{1,i}\}_{i\in I}$, of faithful Hilbert modules over simple H^*-algebras $A_{1,i}$, such that X_1 equals the mixed products of the family $\{X_{1,i}\}_{i\in I}$ [7, Theorem 2.3].

Similarly we can assume that X_2, Y_1, Y_2 are the mixed products of the family $\{X_{2,i}\}_{i\in I}$, $\{Y_{1,i}\}_{i\in I}$, $\{Y_{2,i}\}_{i\in I}$, respectively.

Now by Theorem 1.1, we can replace the above families of Hilbert modules over the simple and proper H^*-algebras $\{A_{1,i}\}_{i\in I}, \{B_{1,i}\}_{i\in I}, \{A_{2,i}\}_{i\in I}, \{B_{2,i}\}_{i\in I}$, by $\{X_{1,e_{1,i}}\}_{i\in I}$, $\{Y_{1,e'_{1,i}}\}_{i\in I}$, $\{X_{2,e_{2,i}}\}_{i\in I}$ and $\{Y_{2,e'_{2,i}}\}_{i\in I}$, respectively. By the assumption Ψ_1 is surjective, then $\Psi_{1,i} : X_{1,e_{1,i}} \to X_{2,e_{2,i}}$ is surjective, where $\Psi_{1,i} = \Psi_1|_{X_{1,e_{1,i}}}$, for each $i \in I$. Since for any arbitrary element $x_2 \in X_{2,e_{2,i}}$ and surjectivity of Ψ_1, there exists $x_1 \in X_1$ such that $[x_2|x_2] = [\Psi_1(x_1)|\Psi_1(x_1)] = \psi_1([x_1|x_1])$. Furthermore for some positive number λ, we have $[x_2|x_2] = \lambda e_{2,i} = \lambda\psi_1(e_{1,i}) = \psi_1(\lambda e_{1,i})$. Since ψ_1 is an isomorphism, then $[x_1|x_1] = \lambda e_{1,i}$. So $x_1 \in X_{1,e_{1,i}}$ and $\Psi_{1,i}$ is surjective. Now we are going to show the injectivity of $\Psi_{2,i}$. Let $y_1 \in Y_{1,e'_1}$ and $\Psi_{2,i}(y_1) = 0$. Then $[y_1|y_1] = \lambda e'_{1,i}$ for some positive number λ and $[\Psi_{2,i}(y_1)|\Psi_{2,i}(y_1)] = \psi_{2,i}(\lambda e'_{1,i}) = \lambda e'_{2,i} = 0$. Hence, $\lambda = 0$, so $[y_1|y_1] = 0$. By the definition of Hilbert H^*-module, $y_1 = 0$. We can prove the injectivity of other morphisms in a similar fashion. This implies that $\ker(\Phi_{1,i}) \cap \ker(\Psi_{1,i}) = \{0\}$. Finally one can easily verify the fullness of each of Hilbert H^*-modules in the following diagram. Hence by Theorem 2.2, the following diagram is pullback for each $i \in I$.

$$\begin{array}{ccc} X_{1,e_{1,i}} & \xrightarrow{\Phi_{1,i}} & Y_{1,e'_{1,i}} \\ \big\downarrow{\scriptstyle \Psi_{1,i}} & & \big\downarrow{\scriptstyle \Psi_{2,i}} \\ X_{2,e_{2,i}} & \xrightarrow{\Phi_{2,i}} & Y_{2,e'_{2,i}} \end{array}$$

In particular, the following diagram or (2.2) is pullback. (see [9, Proposition 4.8])

$$\begin{array}{ccc} \bigotimes X_{1,e_{1,i}} & \xrightarrow{\bigoplus \Phi_{1,i}} & \bigotimes Y_{1,e'_{1,i}} \\ \big\downarrow{\scriptstyle \bigoplus \Psi_{1,i}} & & \big\downarrow{\scriptstyle \bigoplus \Psi_{2,i}} \\ \bigotimes X_{2,e_{2,i}} & \xrightarrow{\bigoplus \Phi_{2,i}} & \bigotimes Y_{2,e'_{2,i}} \end{array}$$

□

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