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BAZILEVIČ P-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. The aim of this paper is to introduce and study a new class of Bazilevič p-valent function of order β by using the subordination concept between this function and a generalized derivative operator. Some interesting properties are also obtained.

1. Introduction

Let $\mathcal{A}_p$ the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad (z \in \mathbb{U}, p \in \mathbb{N}),$$

which are analytic and p-valent in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. For $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that f is subordinate to g if there exists an analytic function ω in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now, we define new generalized differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q)$ of analytic p-valent functions as follows.

Definition 1.1. Let f be in the class $\mathcal{A}_p$, then we have

$$D^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q) f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)n + b}{p + \lambda_2 n + b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{a_{p+n} z^{p+n}}{n!},$$

where $p \in \mathbb{N}, m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$ $(i = 1, \ldots, r,\ q = 1, \ldots, s)$ and $r \leq s + 1$; $r, s \in \mathbb{N}_0$, and $(x)_n$ is the Pochhammer

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symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n=0, \\ x(x+1)\cdots(x+n-1), & n=\{1,2,3,\ldots\}. \end{cases}$$

It follows from the above definition that

$$(p+\lambda_2 n+b)D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z) = (p+\lambda_2 n-p\lambda_1+b)D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z) + p\lambda_1 z\left(D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)\right)'. \tag{1.1}$$

Remark 1.1. It should be remarked that the linear operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)$ is a generalization of many operators considered earlier. Let us see some of the examples:

- For $\lambda_2 = b = 0$, the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f$ reduces to the operator was given by Selvaraj and Karthikeyan [18].
- For $m = 0$, the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f$ reduces to the operator was given by El-Ashwah [9].
- For $m = 0$ and $p = 1$, the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f$ reduces to the well-known operator introduced by Dziok and Srivastava [8].
- For $m = 0, r = 2, s = 1$ and $p = 1$, we obtain the operator which was given by Hohlov [12].
- For $r = 1, s = 0, a_1 = 1, \lambda_1 = 1, \lambda_2 = b = 0$ and $p = 1$, we get the operator introduced by Sălăgean [17].
- For $r = 1, s = 0, a_1 = 1, \lambda_2 = b = 0$ and $p = 1$, we get the generalized Sălăgean derivative operator introduced by Al-Oboudi [1].
- For $m = 0, r = 1, s = 0, a_1 = \delta + 1$ and $p = 1$, we obtain the operator introduced by Ruscheweyh [16].
- For $r = 1, s = 0, a_1 = \delta + 1$ and $p = 1$, we obtain the operator studied by El-Yagubi and Darus [10], [11].
- For $m = 0, r = 2$ and $s = 1, a_2 = 1$ and $p = 1$, we obtain the operator studied by Carlson and Shaffer [4].
- For $r = 1, s = 0, a_1 = 1, \lambda_2 = 0$ and $p = 1$, we get the operator introduced by Cátás [5].

By making use of the differential operator $D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)$ and the principle of subordination between Bazilevič p-valent functions, we introduce and investigate the following subclass of $\mathcal{A}_p$.

Definition 1.2. Let $f \in \mathcal{A}_p$ is said to be in the class $S^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q,\gamma,\beta,A,B)$ if it satisfies the following subordination condition

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} + \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz} \quad (p\in\mathbb{N}, z\in\mathbb{U}),$$

where $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \backslash \{0, -1, -2, \dots\}$ $(i = 1, \dots, r,\ q = 1, \dots, s)$, $r \leq s + 1; r, s \in \mathbb{N}_0$, $-1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Clearly, if we put $p = 1, m = b = 0, \lambda_1 = \lambda_2 = 1, r = 1, s = 0$ and $a_1 = 1$ in the class $S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$, then we obtain the class of Bazilevič functions studied by Liu and Noor [14].

To prove our main result, the following lemmas are required.

Lemma 1.1. [1] *Let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = 1$. Assume also the function $\wp(z)$ given by*

$$\wp(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \tag{1.2}$$

be analytic in $\mathbb{U}$. If

$$\wp(z) + \frac{z\wp'(z)}{\delta} \prec h(z) \quad \{\Re(\delta) \geq 0; \delta \neq 0, z \in \mathbb{U}\},$$

then

$$\wp(z) \prec \psi(z) = \frac{\delta}{n} z^{-(\frac{\delta}{n})} \int_0^z t^{(\frac{\delta}{n})-1} h(t)dt \prec h(z) \quad (z \in \mathbb{U}), \tag{1.3}$$

and ψ is the best dominant.

Lemma 1.2. [19] *Let $q(z)$ be a convex univalent function in $\mathbb{U}$ and let $\sigma, \eta \in \mathbb{C}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.3. [15] *Let q be convex univalent in $\mathbb{U}$ and $k \in \mathbb{C}$. Further assume that $\Re(k) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + kzp'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + kzq'(z) \prec p(z) + kzp'(z)$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subdominant.

2. Main Results

In what follows we aim to study some interesting properties of the class $S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$.

Theorem 2.1. *Let $f(z) \in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$, then*

$$\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du$$
$$\prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$ $(i = 1, \ldots, r, q = 1, \ldots, s)$, $r \leq s+1; r, s \in \mathbb{N}_0$, $-1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Proof. Let

$$p(z) = \left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} (z \in \mathbb{U}). \tag{2.1}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By taking the derivative in the both sides in equality (2.1) and using (1.1), we get

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} + \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$$
$$= p(z) + \frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)}zp'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{2.2}$$

By applying Lemma1.1 in the last equation, we get

$$\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}z^{-\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}}\int_0^z t^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}\frac{1+At}{1+Bt}dt$$
$$= \frac{\zeta}{n}\int_0^1 u^{\frac{\zeta}{n}-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{2.3}$$

where $\zeta = \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1\gamma}$. □

Theorem 2.2. *Let $q(z)$ be univalent in $\mathbb{U}$. Suppose also that $q(z)$ satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1\gamma}\right)\right\}. \tag{2.4}$$

If $f(z) \in \mathcal{A}_p$ is satisfying the following subordination

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} + \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$$
$$\prec q(z) + \frac{p\gamma\lambda_1}{(p+\lambda_2 n+b)\beta}zq'(z), \tag{2.5}$$

then $\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec q(z)$, *and* $q(z)$ *is the best dominant.*

Proof. Let $p(z)$ be defined by (2.1). We know that (2.2) is true. Combining (2.2) and (2.5), we see that

$$p(z) + \frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)} zp'(z) \prec q(z) + \frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)} zq'(z). \tag{2.6}$$

By using Lemma 1.2 and (2.6), we get the assertion of Theorem 2.2. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2.2, we get the following result.

Corollary 2.1. *Let* $\gamma \in \mathbb{C}$ *and* $-1 \leq B < A \leq 1$. *Suppose also that* $\frac{1+Az}{1+Bz}$ *satisfies the condition* (2.4). *If* $f(z) \in \mathcal{A}_p$ *satisfies the following subordination*

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} + \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\beta}$$
$$\prec \frac{1+Az}{1+Bz} + \frac{p\lambda_1\gamma(A-B)z}{(p+\lambda_2 n+b)\beta(1+Bz)^2},$$

then $\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz}$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 2.3. *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$, $\gamma \in \mathbb{C}$, *with* $\Re(\gamma) > 0$. *Also let* $\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \in H[q(0),1] \cap Q$ *and*

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} + \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$$

be univalent in $\mathbb{U}$. *If*

$$q(z) + \frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)} zq'(z) \prec (1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$$
$$+ \gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta},$$

then $q(z) \prec \left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$ *and* $q(z)$ *is the best subdominant.*

Proof. Let $p(z)$ be defined by (2.1). Then

$$\begin{aligned} q(z)+\frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)}zq'(z) \prec (1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \\ +\gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \\ = p(z)+\frac{p\lambda_1\gamma}{\beta(p+\lambda_2 n+b)}zq'(z). \end{aligned}$$

An application of Lemma 1.3 yields the assertion of Theorem 2.3. □

Corollary 2.2. *Let $q(z)$ be convex univalent in $\mathbb{U}$ and $-1\le B<A\le 1,\gamma\in\mathbb{C}$ with $\Re(\gamma)>0$. Also let $\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\in H[q(0),1]\cap Q$ and*

$$(1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}+\gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$$

be univalent in $\mathbb{U}$. If

$$\begin{aligned} \frac{1+Az}{1+Bz}+\frac{p\lambda_1\gamma(A-B)z}{(p+\lambda_2 n+b)\beta(1+Bz)^2} \prec (1-\gamma)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \\ +\gamma\left(\frac{D^{m+1,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}\right)\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}, \end{aligned}$$

then $\frac{1+Az}{1+Bz}\prec\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}$ and $\frac{1+Az}{1+Bz}$ is the best subdominant.

Theorem 2.4. *Let $f(z)\in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q,\gamma,\beta,A,B)$, then*

$$\begin{aligned} \inf_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\} \\ <\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\} \\ <\sup_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\}. \end{aligned}$$

where $\gamma\in\mathbb{C}$, $\Re(\beta)>0$, $p\in\mathbb{N}$, $m,b\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda_2\ge\lambda_1\ge 0$, $a_i\in\mathbb{C}$, $b_q\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}(i=1,\ldots,r,q=1,\ldots,s)$, $r\le s+1;r,s\in\mathbb{N}_0$, $-1\le B\le 1$ and $A\ne B\in\mathbb{N}_0$.

Proof. Suppose that $f(z) \in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$, then from Theorem 2.1 we know that

$$\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du.$$

Therefore, from the definition of the subordination, we have

$$\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\} > \inf_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\}$$

and

$$\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\}$$
$$< \sup_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\}. \qquad \square$$

Corollary 2.3. *Let* $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$ $(i = 1,\ldots,r, q = 1,\ldots,s)$, $r \leq s+1; r, s \in \mathbb{N}_0$ *and* $-1 \leq B < A \leq 1$. *If* $f(z) \in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$. *then*

$$\begin{aligned}\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du &< \Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\}\\ &< \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du \quad (z\in\mathbb{U}).\end{aligned} \tag{2.7}$$

Proof. Suppose that $f(z) \in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i, b_q, \gamma, \beta, A, B)$, then from Theorem 2.1 we know that

$$\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du.$$

Therefore, from the definition of the subordination and $A > B$, we have

$$\begin{aligned}\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\} &> \inf_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\}\\ &\geq \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \inf_{z\in\mathbb{U}}\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\\ &> \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du,\end{aligned}$$

and

$$\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\} < \sup_{z\in\mathbb{U}}\Re\left\{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right\}$$

$$\leq \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \sup_{z\in\mathbb{U}}\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du$$

$$< \frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du,$$

which proves the result. □

Corollary 2.4. *Let $\gamma\in\mathbb{C}$, $\Re(\beta)>0$, $p\in\mathbb{N}$, $m,b\in\mathbb{N}_0$, $\lambda_2\geq\lambda_1\geq 0$, $a_i\in\mathbb{C}$, $b_q\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ $(i=1,\ldots,r, q=1,\ldots,s)$, $r\leq s+1; r,s\in\mathbb{N}_0$ and $-1\leq B<A\leq 1$. If $f(z)\in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q,\gamma,\beta,A,B)$, then*

$$\left(\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right)^{\frac{1}{2}} < \Re\left\{\left(\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right)^{\frac{1}{2}}\right\}$$

$$< \left(\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right)^{\frac{1}{2}} \quad (z\in\mathbb{U}). \tag{2.8}$$

Proof. Suppose that $f(z)\in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q,\gamma,\beta,A,B)$, then from Theorem 2.1, we have

$$\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz} \quad (z\in\mathbb{U}).$$

Since $-1\leq B<A\leq 1$, we have

$$0\leq\frac{1-A}{1-B}<\Re\left\{\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right\}<\frac{1+A}{1+B}.$$

Thus, from the inequality (2.7), we can get the inequality (2.8). □

Corollary 2.5. *Let $\gamma\in\mathbb{C}$, $\Re(\beta)>0$, $p\in\mathbb{N}$, $m,b\in\mathbb{N}_0,\lambda_2\geq\lambda_1\geq 0$, $a_i\in\mathbb{C}$, $b_q\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ $(i=1,\ldots,r, q=1,\ldots,s)$, $r\leq s+1; r,s\in\mathbb{N}_0$ and $-1\leq A<B\leq 1$. If $f(z)\in S^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q,\gamma,\beta,A,B)$, then*

$$\left(\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right)^{\frac{1}{2}} < \Re\left\{\left(\left(\frac{D^{m,b}_{\lambda_1,\lambda_2,p}(a_i,b_q)f(z)}{z^p}\right)^{\beta}\right)^{\frac{1}{2}}\right\}$$

$$< \left(\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{(p+\lambda_2 n+b)\beta}{p\lambda_1 n\gamma}-1}du\right)^{\frac{1}{2}} \quad (z\in\mathbb{U}).$$

Proof. By applying similar method as in Corollary 2.4, we get the required result. □

Note that other work related to classes of Bazilevič functions can be found in [2], [3], [6], [7], [13].

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