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ON THE APOLAR POLYNOMIALS

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Abstract. We give generalizations of theorems of Aziz on apolar polynomials. Our proofs are based on direct methods.

1. Introduction

Let $p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k$ and $q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k$ be two complex polynomials, where $m \leq n$. Let us define $A(p,q) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_{m-k} B_k$.

Classical theorem of Grace states that if the polynomials p and q are apolar, that is if $A(p,q) = 0$ and $m = n$, then any circular region that contains all zeros of one of these polynomials contains at least one zero of the other one.

The following theorem is due to Aziz, see [1].

Theorem 1.1. *Let $p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k$ and $q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k$ where $m \leq n$. Assume $A(p,q) = 0$. Then*

(i) *If $q(z)$ has all zeros in the disc $|z| \leq r$, then $p(z)$ has at least one zero in the same disc.*
(ii) *If $p(z)$ has all zeros in the region $|z| \geq r$, then $q(z)$ has at least one zero in the same region.*

A different proof of this theorem is given in [5]. The following theorem is proved in [2].

Theorem 1.2. *Let $p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k$ and $q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k$ where $m \leq n$. Let us assume*

$$\binom{m}{0} B_0 A_n - \binom{m}{1} B_1 A_{n-1} + \binom{m}{2} B_2 A_{n-2} + \cdots + (-1)^m \binom{m}{m} B_m A_{n-m} = 0. \tag{1.1}$$

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Then

(i) *If $q(z)$ has all zeros in the region $|z-a| \geq r$, then $p(z)$ has at least one zero in the same region.*
(ii) *If $p(z)$ has all zeros in the disc $|z-a| \leq r$, then $q(z)$ has at least one zero in the same disc.*

We also note that the both above theorems are proved in [3] also in the case of certain half-planes. Our goal is to give generalizations of these results.

2. Main results

Our first result is a generalization of Theorem 1.1.

Theorem 2.1. *Let $p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k$ and $q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k$, where $m \leq n$ and $A(p,q) = 0$. Then any circular region K that contains 0 and all zeros of $q(z)$ contains at least one zero of $p(z)$.*

Proof. Let us set $Q(z) = z^{n-m}q(z)$. Then $A(p,q) = 0$ implies that $p(z)$ and $Q(z)$ are apolar. All the zeros of $Q(z)$ are contained in K. By theorem of Grace, at least one zero of $p(z)$ is in K. □

Also, if all the zeros of $p(z)$ are outside K, then at least one zero of $q(z)$ is also outside K. This is an equivalent form of stating the above theorem.

We are going to use the following result.

Theorem 2.2. [4] *For polynomials $f(z) = a_0 + a_1 z + \cdots + z^n$ and $g(z) = b_0 + b_1 z + \cdots + z^n$ the following holds: if all the zeros of $f(z)$ are contained in some disc of radius r, then a zero of $g(z)$ is contained in concentric disc of radius $r + |A(f,g)|^{1/n}$.*

If K is a disc, we can generalize Theorem 2.1 to include the case $A(p,q) \neq 0$. Namely, we have the following theorem, which can be deduced from Theorem 2.2.

Theorem 2.3. *Let $p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k$ and $q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k$, where $m \leq n$ and $A_n B_m \neq 0$. Assume a disc K of radius r contains 0 and all zeros of $q(z)$. Then the concentric disc of radius $r + \left|\frac{A(f,g)}{A_n B_m}\right|^{1/n}$ contains at least one zero of $p(z)$.*

Proof. We note that polynomials $p(z)/A_n$ and $Q(z)/B_m$ are monic, where $Q(z) = z^{n-m}q(z)$, and that all the zeros of $Q(z)/B_m$ are contained in K. Since $\deg Q = \deg p = n$, Theorem 2.2 implies that at least one zero of $p(z)$ is contained in the concentric disc of radius $r + |A(p/A_n, Q/B_m)|^{1/n}$. However, by a simple calculation, $|A(p/A_n, Q/B_m)| = |A(p,q)/A_n B_m|$ and this completes the proof. □

Our last theorem generalizes Theorem 1.2.

Theorem 2.4. *Assume polynomials $p(z)$ and $q(z)$ are as in Theorem 1.2. Assume condition* (1.1) *is satisfied, and let K be a closed disc or a closed half-plane. Then*

(i) *If all the zeros of* $p(z)$ *are in* K*, then at least one zero of* $q(z)$ *is in* K*.*
(ii) *If all the zeros of* $q(z)$ *are outside* K*, then at least one of the zeros of* $p(z)$ *is outside* K*.*

Proof. At first let us note that the two assertions are logically equivalent, so it suffices to prove the first one. We assume all the zeros of $p(z)$ are in K. Then, by Gauss-Lucas theorem, the same is true for the polynomial $p^{(n-m)}(z)$. Condition (1.1) tells us that in fact $p^{(n-m)}(z)$ is apolar to $q(z)$, and the proof is finished by invoking the theorem of Grace. □

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