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ODD SUM LABELING OF SOME SUBDIVISION GRAPHS

S. AROCKIARAJ[1], P. MAHALAKSHMI[2], AND P. NAMASIVAYAM[3]

Abstract. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$, for all $uv \in E(G)$, is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q - 1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper, we have studied the odd sum property of the subdivision of the triangular snake, quadrilateral snake, slanting ladder, $C_p \odot K_1$, $H \odot K_1$, $C_m@C_n$, the grid graph $P_m \times P_n$, duplication of a vertex of a path and duplication of a vertex of a cycle.

1. Introduction

Throughout this paper, by a graph, we mean a finite, undirected simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [1].

A path on p vertices is denoted by P_p and a cycle on p vertices is denoted by C_p. If m number of pendant vertices are attached at each vertex of G, then the resultant graph obtained from G is the graph $G \odot mK_1$. When $m = 1, G \odot K_1$ is the corona of G. A triangular (quadrilateral) snake is obtained from a path by identifying each edge of the path with an edge of the cycle $C_3(C_4)$. The graph $C_m@C_n$ is obtained by identifying an edge of C_m with an edge of C_n. A graph which can be obtained from a given graph by breaking up each edge into one or more segments by inserting intermediate vertices between its two ends. If each edge of a graph G is broken into two by exactly one vertex, then the resultant graph is taken as $S(G)$. The slanting ladder SL_n is a graph obtained from two paths $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ by joining each u_i with $v_{i+1}, 1 \leq i \leq n - 1$.

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Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words, a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.

In [2], an odd edge labeling of a graph is defined as follows: A labeling $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ is called an odd edge labeling of G if for the edge labeling f^+ on $E(G)$ defined by $f^+(uv) = f(u) + f(v)$ for any edge $uv \in E(G)$, for a connected graph G, the edge labeling is not necessarily injective. In [5], the concept of pair sum labeling was introduced. An injective function $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e : E(G) \to \mathcal{Z} - \{0\}$ defined by $f_e(uv) = f(u)+f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm \frac{k_q}{2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm \frac{k_{q-1}}{2}\} \cup \{\frac{k_{q+1}}{2}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph. In [6], the concept of mean labeling was introduced. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is said to be a mean labeling if the induced edge labeling f^* defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is injective and $f^*(E(G)) = \{1, 2, \ldots, q\}$. A graph G is said to be odd mean graph if there exists an injective function f from $V(G)$ to $\{0, 1, 2, 3, \ldots, 2q-1\}$ such that the induced map f^* from $E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection [4].

Motivated by these, we introduce a new concept called odd sum labeling. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$, for all $uv \in E(G)$ is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling.

In this paper, we have studied the odd sum property of the subdivision of the triangular snake, quadrilateral snake, slanting ladder, $C_p \odot K_1, H \odot K_1, C_m@C_n$, the grid graph $P_m \times P_n$, duplication of a vertex of path and cycle by a vertex.

2. Main Results

Proposition 2.1. *$S(T_n)$ is an odd sum graph, when n is even.*

Proof. Let $u_1, u_2, \ldots, u_n, u_{n+1}$ the vertices on the path of length n in T_n and let $v_i, 1 \le i \le n$ be the vertices of T_n in which v_i is adjacent to u_i and u_{i+1}. Let x_i, y_i and z_i be the vertices which subdivide the edges u_iu_{i+1}, u_iv_i and v_iu_{i+1} respectively for each $i, 1 \le i \le n$.

Define $f : V(S(T_n)) \to \{0, 1, 2, \dots, 6n\}$ as follows:

$$f(u_i) = 6i - 6, \quad 1 \leq i \leq n + 1$$

$$f(v_i) = \begin{cases} 6i - 4, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 6i - 2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 6i + 1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 6i - 7, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 6i - 5, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 6i - 3, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(z_i) = \begin{cases} 6i - 3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 6i - 1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_i x_i) = \begin{cases} 12i - 5, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 13, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(x_i u_{i+1}) = \begin{cases} 12i + 1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 7, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_i y_i) = \begin{cases} 12i - 11, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 9, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(y_i v_i) = \begin{cases} 12i - 9, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 5, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_i z_i) = \begin{cases} 12i - 7, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 3, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f^*(z_i u_{i+1}) = \begin{cases} 12i - 3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12i - 1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Thus, f is an odd sum labeling of $S(T_n)$ where n is even. □

Proposition 2.2. *The graph $S(Q_n)$ is an odd sum graph for any n.*

Proof. Let $u_1, u_2, \dots, u_n, u_{n+1}$ be the vertices on the path of length n in Q_n and let v_i and w_i be the vertices of Q_n in which v_i is adjacent to u_i and w_i is adjacent to u_{i+1}, for each $i, 1 \leq i \leq n$. Let t_i, x_i, y_i, z_i be the vertices which subdivide the edges $u_i u_{i+1}, u_i v_i, v_i w_i$ and $w_i u_{i+1}$ respectively for each $i, 1 \leq i \leq n$.

We define $f : V(S(Q_n)) \to \{0, 1, 2, \dots, 8n\}$ as follows: $f(u_i) = 8i - 8$, $1 \leq i \leq n+1$, $f(v_i) = 8i - 6$, $1 \leq i \leq n$ $f(w_i) = 8i - 2$, $1 \leq i \leq n$, $f(t_i) = 8i - 1$, $1 \leq i \leq n$, $f(x_i) = 8i - 7$, $1 \leq i \leq n$ $f(y_i) = 8i - 5$, $1 \leq i \leq n$ and $f(z_i) = 8i - 3$, $1 \leq i \leq n$. Then the induced edge labeling is obtained as follows: $f^*(u_i x_i) = 16i - 15$, $1 \leq i \leq n$, $f^*(x_i v_i) = 16i - 13$, $1 \leq i \leq n$, $f^*(v_i y_i) = 16i - 11$, $1 \leq i \leq n$, $f^*(y_i w_i) = 16i - 7$, $1 \leq i \leq n$, $f^*(w_i z_i) = 16i - 5$, $1 \leq i \leq n$, $f^*(z_i u_{i+1}) = 16i - 3$, $1 \leq i \leq n$, $f^*(u_i t_i) = 16i - 9$, $1 \leq i \leq n$ and $f^*(t_i u_{i+1}) = 16i - 1$, $1 \leq i \leq n$.

Thus, f is an odd sum labeling of $S(Q_n)$. □

Proposition 2.3. *The subdivision graph of slanting ladder $S(SL_n)$ is an odd sum graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length $n-1$. Let x_i, y_i and z_i be the vertices subdivided the edges u_iu_{i+1}, v_iv_{i+1} and u_iv_{i+1} respectively for each $i, 1 \le i \le n-1$.

Case (i). n is even.

We define $f : V(S(SL_n)) \to \{1, 2, \ldots, 6(n-1)\}$ as follows:

$$f(u_i) = \begin{cases} 6i-2, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 6i-4, & 1 \le i \le n-2 \text{ and } i \text{ is even} \\ 6(n-1), & i = n \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 6i-10, & 2 \le i \le n \text{ and } i \text{ is even} \\ 6i-8, & 2 \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

$$f(x_i) = \begin{cases} 6i-1, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ 6i+3, & 1 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 6i-3, & 2 \le i \le n-1 \text{ and } i \text{ is even} \\ 6i-7, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \end{cases} \quad \text{and}$$

$$f(z_i) = \begin{cases} 3, & i = 1 \\ 6i-5, & 2 \le i \le n-1. \end{cases}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_ix_i) = \begin{cases} 12i-13, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ 12i-1, & 1 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(x_iu_{i+1}) = \begin{cases} 12i+1, & 1 \le i \le n-2 \text{ and } i \text{ is odd} \\ 12i+7, & 1 \le i \le n-2 \text{ and } i \text{ is even} \\ 12n-13, & i = n-1 \end{cases}$$

$$f^*(v_iy_i) = \begin{cases} 1, & i = 1 \\ 12i-13, & 2 \le i \le n-1 \text{ and } i \text{ is even} \\ 12i-15, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \end{cases}$$

$$f^*(y_iv_{i+1}) = \begin{cases} 3, & i = 1 \\ 12i-15, & 2 \le i \le n-1 \text{ and } i \text{ is even} \\ 12i-11, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \end{cases}$$

$$f^*(u_iz_i) = \begin{cases} 7, & i = 1 \\ 12i-9, & 2 \le i \le n-1 \text{ and } i \text{ is even} \\ 12i-7, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \end{cases}$$

$$\text{and } f^*(z_iv_{i+1}) = \begin{cases} 5, & i = 1 \\ 12i - 9, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \\ 12i - 7, & 2 \le i \le n-1 \text{ and } i \text{ is even.} \end{cases}$$

Thus, f is an odd sum labeling of $S(SL_n)$, for $n \ge 4$.

When $n = 2$, an odd sum labeling of the graph is given below.

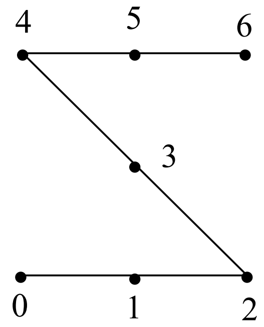

FIGURE 1. $S(SL_2)$

Case (ii). n is odd.

We define $f : V(SL_n) \to \{1, 2, \ldots, 6(n-1)\}$ as follows:

$$f(u_i) = \begin{cases} 4, & i = 1 \\ 12, & i = 2 \\ 6i - 4, & 3 \le i \le n-1 \text{ and } i \text{ is odd} \\ 6i - 2, & 3 \le i \le n-1 \text{ and } i \text{ is even} \\ 6(n-1), & i = n \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 2, & i = 2 \\ 6, & i = 3 \\ 6i - 8, & 4 \le i \le n \text{ and } i \text{ is even} \\ 6i - 10, & 4 \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

$$f(x_i) = \begin{cases} 5, & i = 1 \\ 6i - 1, & 2 \le i \le n-1 \text{ and } i \text{ is even} \\ 6i + 3, & 2 \le i \le n-1 \text{ and } i \text{ is odd} \end{cases}$$

$$f(y_i) = \begin{cases} 1, & i = 1 \\ 9, & i = 2 \\ 6i - 3, & 3 \le i \le n-1 \text{ and } i \text{ is odd} \\ 6i - 7, & 3 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(z_i) = \begin{cases} 3, & i = 1 \\ 6i - 5, & 2 \le i \le n-1. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(u_ix_i) = \begin{cases} 9, & i=1 \\ 23, & i=2 \\ 12i-1, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 12i-3, & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(x_iu_{i+1}) = \begin{cases} 17, & i=1 \\ 12i+1, & 2 \leq i \leq n-2 \text{ and } i \text{ is even} \\ 12i+7, & 2 \leq i \leq n-2 \text{ and } i \text{ is odd} \\ 12n-13, & i=n-1 \end{cases}$$

$$f^*(v_iy_i) = \begin{cases} 1, & i=1 \\ 11, & i=2 \\ 21, & i=3 \\ 12i-15, & 4 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 12i-13, & 4 \leq i \leq n-1 \text{ and } i \text{ is odd} \end{cases}$$

$$f^*(y_iv_{i+1}) = \begin{cases} 3, & i=1 \\ 15, & i=2 \\ 12i-5, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 12i-11, & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_iz_i) = \begin{cases} 7, & i=1 \\ 19, & i=2 \\ 12i-9, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 12i-7, & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f^*(z_iv_{i+1}) = \begin{cases} 5, & i=1 \\ 13, & i=2 \\ 12i-7, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 12i-9, & 3 \leq i \leq n-1 \text{ and } i \text{ is even.} \end{cases}$$

Thus, f is an odd sum labeling for $n \geq 5$.

When $n = 3$, an odd sum labeling of the graph is given below.

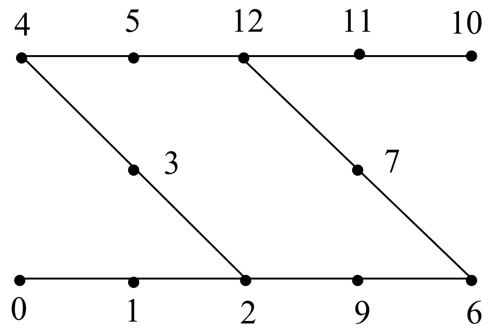

FIGURE 2

Hence, $S(SL_n)$ is an odd sum graph. □

Proposition 2.4. *The graph $S(C_p \odot K_1)$ is an odd sum graph.*

Proof. Let $u_1, v_1, u_2, v_2, \ldots, u_p$ and v_p be the vertices on the cycle and $u_i y_i x_i$ be the path on 3 vertices attached at each u_i.
Case (i). p is even.
Define $f : V(S(C_p \odot K_1)) \to \{0, 1, 2, \ldots, 4p\}$ as follows:

$$f(u_i) = \begin{cases} 4i-3, & 1 \le i \le p \text{ and } i \text{ is odd} \\ 4i-1, & 1 \le i \le p \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 4i+2, & 1 \le i \le \frac{p}{2}-1 \text{ and } i \text{ is odd} \\ 4i, & 1 \le i \le \frac{p}{2}-1 \text{ and } i \text{ is even} \\ 4i+4, & \frac{p}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 4i+2, & \frac{p}{2} \le i \le p-1 \text{ and } i \text{ is even} \\ 0, & i = p \end{cases}$$

$$f(x_i) = \begin{cases} 4i-1, & 1 \le i \le p \text{ and } i \text{ is odd} \\ 4i-3, & 1 \le i \le p \text{ and } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 4i-2, & 1 \le i \le \frac{p}{2} \text{ and } i \text{ is odd} \\ 4i-4, & 1 \le i \le \frac{p}{2} \text{ and } i \text{ is even} \\ 4i, & \frac{p}{2}+1 \le i \le p-1 \text{ and } i \text{ is odd} \\ 4i-2, & \frac{p}{2}+1 \le i \le p-1 \text{ and } i \text{ is even} \\ 0, & i = p. \end{cases}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_i v_i) = \begin{cases} 8i-1, & 1 \le i \le \frac{p}{2}-1 \\ 8i+1, & \frac{p}{2} \le i \le p-1 \\ 4p-1, & i = p \end{cases}$$

$$f^*(v_iu_{i+1}) = \begin{cases} 8i+5, & 1 \le i \le \frac{p}{2}-1 \text{ and } i \text{ is odd} \\ 8i+1, & 1 \le i \le \frac{p}{2}-1 \text{ and } i \text{ is even} \\ 8i+7, & \frac{p}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 8i+3, & \frac{p}{2} \le i \le p-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_iy_i) = \begin{cases} 8i-5, & 1 \le i \le \frac{p}{2} \\ 8i-3, & \frac{p}{2}+1 \le i \le p \end{cases}$$

$$f^*(y_ix_i) = \begin{cases} 8i-3, & 1 \le i \le \frac{p}{2} \text{ and } i \text{ is odd} \\ 8i-7, & 1 \le i \le \frac{p}{2} \text{ and } i \text{ is even} \\ 8i-1, & \frac{p}{2}+1 \le i \le p \text{ and } i \text{ is odd} \\ 8i-5, & \frac{p}{2}+1 \le i \le p \text{ and } i \text{ is even} \end{cases}$$

and $f^*(v_pu_1) = 1$.

Thus, f is an odd sum labeling of $S(C_p \odot K_1)$.

Case (ii). p is odd, $p \equiv 1(mod\ 4)$.

The labeling $f : V(S(C_p \odot K_1)) \to \{0, 1, 2, \dots, 4p\}$ is defined as follows:

$$f(u_i) = \begin{cases} 4i-2, & 1 \le i \le p \text{ and } i \text{ is even} \\ 4i-4, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is odd} \\ 4i, & \frac{p-1}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 4p-2, & i = p \end{cases}$$

$$f(x_i) = \begin{cases} 4i-4, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is even} \\ 4i, & \frac{p+3}{2} \le i \le p \text{ and } i \text{ is even} \\ 4i-2, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is odd} \\ 4i-4, & i = \frac{p+1}{2} \\ 4i-2, & \frac{p+3}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 4p, & i = p \end{cases}$$

$$f(y_i) = \begin{cases} 4i-5, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is even} \\ 4i-3, & \frac{p+3}{2} \le i \le p \text{ and } i \text{ is even} \\ 4i-3, & 1 \le i \le \frac{p+1}{2} \text{ and } i \text{ is odd} \\ 4i-5, & \frac{p+3}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 4p-1, & i = p \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 4i+1, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is odd} \\ 4i-1, & \frac{p+1}{2} \le i \le p-2 \text{ and } i \text{ is odd} \\ 4i-1, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is even} \\ 4i+1, & \frac{p+3}{2} \le i \le p-1 \text{ and } i \text{ is even} \\ 4p-5, & i = p. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(u_iv_i) = \begin{cases} 8i-3, & 1 \le i \le \frac{p-1}{2} \\ 8i-1, & \frac{p+1}{2} \le i \le p-1 \\ 8p-7, & i = p \end{cases}$$

$$f^*(v_iu_{i+1}) = \begin{cases} 8i+3, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is odd} \\ 8i+1, & \frac{p+1}{2} \le i \le p-2 \text{ and } i \text{ is odd} \\ 8i-1, & 1 \le i \le \frac{p-5}{2} \text{ and } i \text{ is even} \\ 8i+3, & i = \frac{p-1}{2} \\ 8i+5, & \frac{p+3}{2} \le i \le p-3 \text{ and } i \text{ is even} \\ 8p-5, & i = p-1 \end{cases}$$

$$f^*(v_pu_1) = 4p-5$$

$$f^*(u_iy_i) = \begin{cases} 8i-7, & 1 \le i \le \frac{p-1}{2} \\ 8i-3, & i = \frac{p+1}{2} \\ 8i-5, & \frac{p+3}{2} \le i \le p-1 \\ 8p-3, & i = p \end{cases}$$

$$\text{and } f^*(y_ix_i) = \begin{cases} 8i-5, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is odd} \\ 8i-7, & \frac{p+1}{2} \le i \le p-2 \text{ and } i \text{ is odd} \\ 8p-1, & i = p \\ 8i-9, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is even} \\ 8i-3, & \frac{p+3}{2} \le i \le p-1 \text{ and } i \text{ is even.} \end{cases}$$

Thus, f is an odd sum labeling of $S(C_p \odot K_1)$.
Case (iii). p is odd and $p \equiv 3(mod\ 4)$.
The labeling f is defined as follows:

$$f(u_i) = \begin{cases} 4i-4, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is odd} \\ 4i-2, & \frac{p+3}{2} \le i \le p \text{ and } i \text{ is odd} \\ 4i-2, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is even} \\ 4i, & \frac{p+1}{2} \le i \le p \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 4i+1, & 1 \le i \le p-2 \text{ and } i \text{ is odd} \\ 4p-3, & i = p \\ 4i+3, & 1 \le i \le p-1 \text{ and } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 4i-2, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is odd} \\ 4i, & \frac{p+3}{2} \le i \le p \text{ and } i \text{ is odd} \\ 4i-4, & 1 \le i \le \frac{p+1}{2} \text{ and } i \text{ is even} \\ 4i-2, & \frac{p+5}{2} \le i \le p \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(y_i) = \begin{cases} 4i-3, & 1 \le i \le p-2 \text{ and } i \text{ is odd} \\ 4p-1, & i = p \\ 4i-5, & 1 \le i \le p-1 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(u_iv_i) = \begin{cases} 8i-3, & 1 \le i \le \frac{p-1}{2} \\ 8i-1, & \frac{p+1}{2} \le i \le p-1 \\ 8p-5, & i = p \end{cases}$$

$$f^*(v_iu_{i+1}) = \begin{cases} 8i+3, & 1 \le i \le \frac{p-5}{2} \text{ and } i \text{ is odd} \\ 8i+5, & \frac{p-1}{2} \le i \le p-2 \text{ and } i \text{ is odd} \\ 8i-1, & 1 \le i \le \frac{p-3}{2} \text{ and } i \text{ is even} \\ 8i+1, & \frac{p+1}{2} \le i \le p-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_pu_1) = 4p-3$$

$$f^*(u_iy_i) = \begin{cases} 8i-7, & 1 \le i \le \frac{p-1}{2} \\ 8i-5, & \frac{p+1}{2} \le i \le p-1 \\ 8p-3, & i = p \end{cases}$$

$$\text{and } f^*(y_ix_i) = \begin{cases} 8i-5, & 1 \le i \le \frac{p-1}{2} \text{ and } i \text{ is odd} \\ 8i-3, & \frac{p+3}{2} \le i \le p-1 \text{ and } i \text{ is odd} \\ 8p-1, & i = p \\ 8i-9, & 1 \le i \le \frac{p+1}{2} \text{ and } i \text{ is even} \\ 8i-7, & \frac{p+5}{2} \le i \le p-1 \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is an odd sum labeling of $S(C_p \odot K_1)$. □

Proposition 2.5. *The graph $S(H_n \odot K_1)$ is an odd sum graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the paths of length $n-1$. Let $a_{1,i}a_{2,i}u_i$ be the path attached at each $u_i, 1 \le i \le n$ and $b_{1,i}b_{2,i}v_i$ be the path attached at each $v_i, 1 \le i \le n$. Each edge u_iu_{i+1} is subdivided by a vertex $x_i, 1 \le i \le n-1$ and each edge v_iv_{i+1} is subdivided by a vertex $y_i, 1 \le i \le n-1$. The edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ is divided by a vertex z when n is odd. The edge $u_{\frac{n+2}{2}}v_{\frac{n}{2}}$ is divided by a vertex z when n is even.

Case (i). n is odd.

The labeling $f : V(S(H_n \odot K_1)) \to \{0, 1, 2, \ldots, 8n-2\}$ is defined as follows:

$$f(u_i) = \begin{cases} 4i-2, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-4, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 4(n+i)-4, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4(n+i)-2, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(a_{1,i}) = \begin{cases} 4i-4, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-2, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(a_{2,i}) = \begin{cases} 1, & i = 1 \\ 4i-5, & 2 \le i \le n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(b_{1,i}) = \begin{cases} 4(n+i)-2, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4(n+i)-4, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(b_{2,i}) = \begin{cases} 4(n+i)-5, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is odd} \\ 4(n+i)-3, & \frac{n+1}{2} \le i \le n \text{ and } i \text{ is odd} \\ 4(n+i)-7, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is even} \\ 4(n+i)-5, & \frac{n+1}{2} \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 4i-1, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ 4i+1, & 1 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 4(n+i)-1, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is odd} \\ 4(n+i)+1, & \frac{n+1}{2} \le i \le n-1 \text{ and } i \text{ is odd} \\ 4(n+i)-3, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is even} \\ 4(n+i)-1, & \frac{n+1}{2} \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

and $f(z) = 6n-3$.

The induced edge labeling is obtained as follows:

$$f^*(u_i x_i) = 8i-3, \quad 1 \le i \le n-1$$

$$f^*(x_i u_{i+1}) = \begin{cases} 8i-1, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ 8i+3, & 1 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_i y_i) = \begin{cases} 8(n+i)-5, & 1 \le i \le \frac{n-1}{2} \\ 8(n+i)-3, & \frac{n+1}{2} \le i \le n-1 \end{cases}$$

$$f^*(y_i v_{i+1}) = \begin{cases} 8(n+i)+1, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is odd} \\ 8(n+i)+3, & \frac{n+1}{2} \le i \le n-1 \text{ and } i \text{ is odd} \\ 8(n+i)-3, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is even} \\ 8(n+i)-1, & \frac{n+1}{2} \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(a_{1,i} a_{2,i}) = \begin{cases} 1, & i = 1 \\ 8i-9, & 2 \le i \le n \text{ and } i \text{ is odd} \\ 8i-5, & 2 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(a_{2,i} u_i) = \begin{cases} 3, & i = 1 \\ 8i-7, & 2 \le i \le n \end{cases}$$

$$f^*(b_{2,i} b_{1,i}) = \begin{cases} 8(n+i)-7, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is odd} \\ 8(n+i)-5, & \frac{n+1}{2} \le i \le n \text{ and } i \text{ is odd} \\ 8(n+i)-11, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is even} \\ 8(n+i)-9, & \frac{n+1}{2} \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_i b_{2,i}) = \begin{cases} 8(n+i)-9, & 1 \le i \le \frac{n-1}{2} \\ 8(n+i)-7, & \frac{n+1}{2} \le i \le n \end{cases}$$

$$f^*(u_{\frac{n+1}{2}} z) = 8n-3$$

and $f^*(z v_{\frac{n+1}{2}}) = 12n-5$.

Case (ii). n is even.

The labeling $f : V(S(H_n \odot K_1)) \to \{0, 1, 2, \ldots, 8n-2\}$ is defined as follows:

$$f(u_i) = \begin{cases} 4i-2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-4, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(a_{1,i}) = \begin{cases} 4i-4, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(a_{2,i}) = \begin{cases} 1, & i = 1 \\ 4i-5, & 2 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 4(n+i)-2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4(n+i)-4, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(b_{1,i}) = \begin{cases} 4(n+i)-4, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4(n+i)-2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(b_{2,i}) = \begin{cases} 4(n+i)-5, & i = 1 \\ 4(n+i)-7, & 2 \leq i \leq \frac{n}{2} \text{ and } i \text{ is odd} \\ 4(n+i)-5, & \frac{n+2}{2} \leq i \leq n \text{ and } i \text{ is odd} \\ 4(n+i)-5, & 2 \leq i \leq \frac{n-2}{2} \text{ and } i \text{ is even} \\ 4(n+i)-3, & \frac{n}{2} \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 4i-1, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 4i+1, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 4(n+i)-3, & 1 \leq i \leq \frac{n-2}{2} \text{ and } i \text{ is odd} \\ 4(n+i)-1, & \frac{n}{2} \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 4(n+i)-1, & 1 \leq i \leq \frac{n-2}{2} \text{ and } i \text{ is even} \\ 4(n+i)+1, & \frac{n}{2} \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(z) = \begin{cases} 6n-3, & n \equiv 2(mod\ 4) \\ 6n-5, & n \equiv 0(mod\ 4). \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(u_i x_i) = 8i-3, \quad 1 \leq i \leq n-1$$

$$f^*(x_i u_{i+1}) = \begin{cases} 8i-1, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 8i+3, & 1 \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_i y_i) = \begin{cases} 8(n+i)-5, & 1 \leq i \leq \frac{n-2}{2} \\ 8(n+i)-3, & \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(y_i v_{i+1}) = \begin{cases} 8(n+i)-3, & 1 \leq i \leq \frac{n-2}{2} \text{ and } i \text{ is odd} \\ 8(n+i)-1, & \frac{n}{2} \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 8(n+i)+1, & 1 \leq i \leq \frac{n-2}{2} \text{ and } i \text{ is even} \\ 8(n+i)+3, & \frac{n}{2} \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(a_{1,i}a_{2,i}) = \begin{cases} 1, & i = 1 \\ 8i-9, & 2 \le i \le n \text{ and } i \text{ is odd} \\ 8i-5, & 2 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(a_{2,i}u_i) = \begin{cases} 3, & i = 1 \\ 8i-7, & 2 \le i \le n \end{cases}$$

$$f^*(b_{2,i}b_{1,i}) = \begin{cases} 8(n+i)-9, & i = 1 \\ 8(n+i)-11, & 2 \le i \le \frac{n}{2} \text{ and } i \text{ is odd} \\ 8(n+i)-9, & \frac{n+2}{2} \le i \le n \text{ and } i \text{ is odd} \\ 8(n+i)-7, & 2 \le i \le \frac{n-2}{2} \text{ and } i \text{ is even} \\ 8(n+i)-5, & \frac{n}{2} \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_ib_{2,i}) = \begin{cases} 8(n+i)-7, & i = 1 \\ 8(n+i)-9, & 2 \le i \le \frac{n}{2} \text{ and } i \text{ is odd} \\ 8(n+i)-7, & \frac{n+2}{2} \le i \le n \text{ and } i \text{ is odd} \\ 8(n+i)-9, & 2 \le i \le \frac{n-2}{2} \text{ and } i \text{ is even} \\ 8(n+i)-7, & \frac{n}{2} \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_{\frac{n+2}{2}}z) = 8n-3$$

$$\text{and } f^*(zv_{\frac{n}{2}}) = \begin{cases} 12n-5, & n \equiv 2(mod\ 4) \\ 12n-9, & n \equiv 0(mod\ 4). \end{cases}$$

Hence, f is an odd sum labeling. □

Proposition 2.6. *The graph $S(C_m@C_n)$ is an odd sum graph for any positive integers $m, n \ge 3$.*

Proof. In $S(C_m@C_n), 2(m+n-2)$ vertices lies on the circle and one vertex lies on a chord. Let $v_1, v_2, \ldots, v_{2(m+n-2)}$ be the vertices on the cycle in $S(C_m@C_n)$ and $v_{2(m+n-2)+1}$ be the vertex having neighbours v_{2n-2} and $v_{2(m+n-2)}$.

Case (i). m and n are odd with $n \ge m$.

$$f(v_i) = \begin{cases} 2m+2n-4, & i = 1 \\ i-1, & 2 \le i \le m+2n-4 \text{ and } i \text{ is even} \\ i+1, & m+2n-2 \le i \le 2m+2n-4 \text{ and } i \text{ is even} \\ i-3, & 3 \le i \le m+n-1 \text{ and } i \text{ is odd} \\ i-1, & m+n-3 \le i \le 2m+2n-4 \text{ and } i \text{ is odd} \\ 2m+2n-2, & i = 2m+2n-3. \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 2m+2n-3, & i = 1 \\ 2i-3, & 2 \le i \le m+n-1 \\ 2i-1, & m+n \le i \le m+2n-3 \\ 2i+1, & m+2n-2 \le i \le 2m+2n-5 \end{cases}$$

$f^*(v_{2n-2}v_{2m+2n-3}) = 2m+4n-5$ and

$f^*(v_{2m+2n-4}v_{2m+2n-3}) = 4m+4n-5.$

Case (ii). m is odd, n is even and $n \geq m+3$.

$$f(v_i) = \begin{cases} 2m+2n-4, & i = 1 \\ i-1, & 2 \leq i \leq m+n-1 \text{ and } i \text{ is even} \\ i+1, & m+n+1 \leq i \leq 2m+2n-4 \text{ and } i \text{ is even} \\ i-3, & 3 \leq i \leq m+2n-2 \text{ and } i \text{ is odd} \\ i-1, & m+2n-4 \leq i \leq 2m+2n-4 \text{ and } i \text{ is even} \\ 2m+2n-2, & i = 2m+2n-3 \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2m+2n-3, & i = 1 \\ 2i-3, & 2 \leq i \leq m+n-1 \\ 2i-1, & m+n \leq i \leq m+2n-1 \\ 2i+1, & m+2n \leq i \leq 2m+2n-5 \end{cases}$$

$$f^*(v_{2n-2}v_{2m+2n-3}) = 2m+4n-3 \text{ and}$$

$$f^*(v_{2m+2n-4}v_{2m+2n-3}) = 4m+4n-5$$

Case (iii). m is odd $n = m+1$.

$$f(v_i) = \begin{cases} i, & 1 \leq i \leq 2m+2n-3 \text{ and } i \text{ is odd} \\ i-2, & 2 \leq i \leq 2n-2 \text{ and } i \text{ is even} \\ i, & 2n \leq i \leq m+2n-5 \text{ and } i \text{ is even} \\ i+2, & m+2n-4 \leq i \leq 2m+2n-4 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2i-1, & 1 \leq i \leq 2n-2 \\ 2i+1, & 2n-1 \leq i \leq m+2n-5 \\ 2i+3, & m+2n-4 \leq i \leq 2m+2n-5 \end{cases}$$

$$f^*(v_{2m+2n-4}v_1) = 2m+2n-1$$

$$f^*(v_{2n-2}v_{2m+2n-3}) = 2m+4n-7 \text{ and}$$

$$f^*(v_{2m+2n-4}v_{2m+2n-3}) = 4m+4n-5.$$

Case (iv). m is even and $n \geq m+1$ is even.

$$f(v_i) = \begin{cases} i-1, & 1 \leq i \leq m+n-2 \text{ and } i \text{ is odd} \\ i+1, & m+n \leq i \leq 2m+2n-3 \text{ and } i \text{ is odd} \\ i-1, & 2 \leq i \leq m+2n-4 \text{ and } i \text{ is even} \\ i+1, & m+2n-5 \leq i \leq 2m+2n-4 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2i-1, & 1 \leq i \leq m+n-2 \\ 2i+1, & m+n-1 \leq i \leq m+2n-4 \\ 2i+3, & m+2n-3 \leq i \leq 2m+2n-4 \end{cases}$$

$$f^*(v_{2m+2n-4}v_1) = 2m + 2n - 3 \text{ and}$$
$$f^*(v_{2n-2}v_{2m+2n-3}) = 2m + 4n - 5.$$

Case (v). m is even and $n \geq m + 2$ is even.

$$f(v_i) = \begin{cases} i-1, & 1 \leq i \leq m+2n-3 \text{ and } i \text{ is odd} \\ i+1, & m+2n-4 \leq i \leq 2m+2n-3 \text{ and } i \text{ is odd} \\ i-1, & 2 \leq i \leq m+n-2 \text{ and } i \text{ is even} \\ i+1, & m+n-1 \leq i \leq 2m+2n-4 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 2i-1, & 1 \leq i \leq m+n-2 \\ 2i+1, & m+n-1 \leq i \leq m+2n-3 \\ 2i+3, & m+2n-2 \leq i \leq 2m+2n-4 \end{cases}$$
$$f^*(v_{2m+2n-4}v_1) = 2m + 2n - 3 \text{ and}$$
$$f^*(v_{2n-2}v_{2m+2n-3}) = 2m + 4n - 3.$$

Case (vi). m is even and $n = m$.

$$f(v_i) = \begin{cases} i-1, & 1 \leq i \leq m-1 \text{ and } i \text{ is odd} \\ i+1, & m \leq i \leq 2m-3 \text{ and } i \text{ is odd} \\ 6m-i-3, & 2m-2 \leq i \leq 4m-3 \text{ and } i \text{ is odd} \\ i-1, & 2 \leq i \leq 2m-2 \text{ and } i \text{ is even} \\ 6m-i-3, & 2m-1 \leq i \leq 3m-2 \text{ and } i \text{ is even} \\ 6m-i-5, & 3m-1 \leq i \leq 4m-4 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 2i-1, & 1 \leq i \leq m-1 \\ 2i+1, & m \leq i \leq 2m-3 \\ 6m-5, & i = 2m-2 \\ 12m-2i-7, & 2m-1 \leq i \leq 3m-2 \\ 12m-2i-9, & 3m-1 \leq i \leq 4m-4 \end{cases}$$
$$f^*(v_{4m-4}v_1) = 2m - 1 \text{ and}$$
$$f^*(v_{2m-2}v_{4m-3}) = 4m - 3.$$

Hence, f is an odd sum labeling. Hence, $S(C_m@C_n)$ is an odd sum graph. □

Proposition 2.7. *The Grid $S(P_m \times P_n)$ is an odd sum graph.*

Proof. Let $u_{i,j}, 1 \leq i \leq m$ and $1 \leq j \leq n$ be the vertices of the grid $P_m \times P_n$. Let $v_{i,j}$ be the vertex divides the edge $u_{i,j}u_{i,j+1}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n-1$ and $w_{i,j}$ be the vertex divides the edge $u_{i,j}u_{i+1,j}$ for each $1 \leq i \leq m-1$ and $1 \leq j \leq n$.

We define $f : V(S(P_m \times P_n)) \to \{0, 1, 2, \ldots, 4mn - 2m - 2n\}$ as follows:

$$f(u_{i,j}) = \begin{cases} 2(i-1)(2n-1) + 2j - 2, & 1 \le i \le m \text{ and } i \text{ is odd}, 1 \le j \le n \\ 2(i-2)(2n-1) + 6n - 2j - 2, & 1 \le i \le m \text{ and } i \text{ is even}, 1 \le j \le n \end{cases}$$

$$f(v_{i,j}) = \begin{cases} 2j - 1, & i = 1, 1 \le j \le n-1 \\ (2i-3)(2n-1) + 4j - 2, & 3 \le i \le m \text{ and } i \text{ is odd}, 1 \le j \le n-1 \\ (2i-1)(2n-1) - 4j, & 2 \le i \le m \text{ and } i \text{ is even}, 1 \le j \le n-1 \end{cases}$$

$$f(w_{i,j}) = \begin{cases} (2i+1)(2n-1) - 4j + 2, & 1 \le i \le m-1 \text{ and } i \text{ is odd}, 1 \le j \le n \\ (2i-1)(2n-1) + 4(j-1), & 1 \le i \le m-1 \text{ and } i \text{ is even}, 1 \le j \le n. \end{cases}$$

The induced edge label is obtained as follows:

$$f^*(u_{i,j}v_{i,j}) = \begin{cases} 4j - 3, & i = 1 \text{ and } 1 \le j \le n-1 \\ (4i-5)(2n-1) + 6j - 4, & 3 \le i \le m \text{ and } i \text{ is odd}, \\ & 1 \le j \le n-1 \\ (4i-5)(2n-1) + 6n - 6j - 2, & 2 \le i \le m \text{ and } i \text{ is even}, \\ & 1 \le j \le n-1 \end{cases}$$

$$f^*(v_{i,j}u_{i,j+1}) = \begin{cases} 4j - 1, & i = 1 \text{ and } 1 \le j \le n-1 \\ (4i-5)(2n-1) + 6j - 2, & 3 \le i \le m \text{ and } i \text{ is odd}, \\ & 1 \le j \le n-1 \\ (4i-5)(2n-1) + 6n - 6j - 4, & 2 \le i \le m \text{ and } i \text{ is even}, \\ & 1 \le j \le n-1 \end{cases}$$

$$f^*(u_{i,j}w_{i,j}) = \begin{cases} (4i-1)(2n-1) - 2j, & 1 \le i \le m-1 \text{ and } i \text{ is odd}, \\ & 1 \le j \le n \\ (4i-5)(2n-1) + 6n + 2j - 6, & 1 \le i \le m-1 \text{ and } i \text{ is even}, \\ & 1 \le j \le n \end{cases}$$

and

$$f^*(w_{i,j}u_{i+1,j}) = \begin{cases} (4i-1)(2n-1) + 6n - 6j, & 1 \le i \le m-1 \text{ and } i \text{ is odd}, \\ & 1 \le j \le n \\ (4i-1)(2n-1) + 6j - 6, & 1 \le i \le m-1 \text{ and } i \text{ is even}, \\ & 1 \le j \le n \end{cases}$$

Hence, f is an odd sum labeling. The odd sum labeling is shown in the following Figure 3. □

Proposition 2.8. *Let G be a graph obtained by duplicating a vertex in P_n. Then $S(G)$ is an odd sum graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n.

Case (i). v_1' or v_n' is the duplicating vertex of v_1 (or v_n) in P_n. Let u_i, $1 \le i \le n-1$ be the subdividing vertices of the edge v_iv_{i+1} and u_1' be the subdividing vertex of the edge $v_1'v_2$.

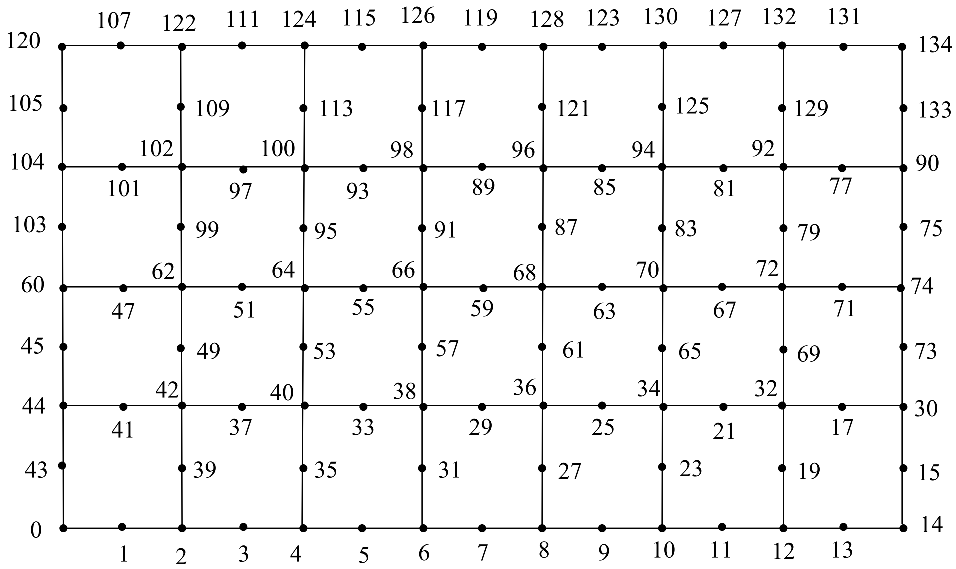

FIGURE 3. $S(P_5 \times P_8)$.

Subcase (i). n is odd.

The vertex labeling $f : V(S(G)) \to \{0, 1, 2, \dots, 2n\}$ is defined as follows:

$$f(v_1') = 0, f(u_1') = 1, f(v_1) = 4, f(v_2) = 2, f(v_3) = 6$$

$$f(v_i) = \begin{cases} 2i-2, & 4 \le i \le n \text{ and } i \text{ is odd} \\ 2i+2, & 4 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(u_1) = 3, f(u_2) = 7, f(u_3) = 5, f(u_4) = 9 \text{ and}$$

$$f(u_i) = \begin{cases} 2i+3, & 5 \le i \le n-1 \text{ and } i \text{ is odd} \\ 2i-1, & 5 \le i \le n-1 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling f^* is obtained as follows:

$$f^*(v_1'u_1') = 1, f^*(u_1'v_2) = 3$$

$$f^*(v_iu_i) = \begin{cases} 2i+5, & 1 \le i \le 3 \\ 4i+3, & i = 4 \\ 4i+1, & 5 \le i \le n-1 \end{cases} \quad \text{and}$$

$$f^*(u_iv_{i+1}) = \begin{cases} 5, & i = 1 \\ 2i+3, & 2 \le i \le 4 \\ 4i+7, & 5 \le i \le n-1 \text{ and } i \text{ is odd} \\ 4i-1, & 5 \le i \le n-1 \text{ and } i \text{ is even.} \end{cases}$$

Thus, f is an odd sum labeling.

Subcase (ii). n is even.

The vertex labeling $f : V(S(G)) \rightarrow \{0, 1, 2, \dots, 2n\}$ is defined as follows:

$$f(v_1') = 0, f(u_1') = 1, f(v_1) = 3, f(v_2) = 2,$$

$$f(v_i) = \begin{cases} 2i+2, & 3 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i-2, & 3 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(u_1) = 4 \text{ and}$$

$$f(u_i) = \begin{cases} 2i+3, & 2 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 2i-1, & 2 \leq i \leq n-1 \text{ and } i \text{ is odd.} \end{cases}$$

The induced edge labeling f^* is obtained as follows:

$$f^*(v_1'u_1') = 1,$$

$$f^*(u_1'v_2) = 3,$$

$$f^*(v_iu_i) = \begin{cases} 7, & i = 1 \\ 2i+5, & 2 \leq i \leq n-1 \end{cases}$$

$$\text{and } f^*(u_iv_{i+1}) = \begin{cases} 5, & i = 1 \\ 4i+7, & 2 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 4i-1, & 3 \leq i \leq n-1 \text{ and } i \text{ is odd.} \end{cases}$$

Thus, f is an odd sum labeling.

Case (ii). v_i' is the duplicating vertex of v_i in P_n, $2 \leq i \leq n-1$.

Let $u_i, 1 \leq i \leq n-1$ be the subdividing vertex of the edge v_iv_{i+1} and u_i', u_i'' be the subdividing vertices of the edges $v_{i-1}v_i'$ and $v_i'v_{i+1}$ respectively. The vertex labeling $f : V(S(G)) \rightarrow \{0, 1, 2, \dots, 2n+2\}$ is defined as follows:

$$f(v_j) = \begin{cases} 2j-1, & 1 \leq j \leq i-1 \\ 2j+1, & i \leq j \leq n \end{cases}$$

$$f(u_j) = \begin{cases} 2j-2, & 1 \leq j \leq i-2 \\ 2j+4, & i-1 \leq j \leq n-1 \end{cases}$$

$$f(v_i') = 2i-1, f(u_i') = 2i-4, \text{ and } f(u_i'') = 2i-2.$$

The induced edge labeling f^* is obtained as follows:

$$f^*(v_ju_j) = \begin{cases} 4j-3, & 1 \leq j \leq i-2 \\ 4i-1, & j = i-1 \\ 4j+5, & 1 \leq j \leq n-1 \end{cases}$$

$$f^*(u_jv_{j+1}) = \begin{cases} 4j-1, & 1 \leq j \leq i-2 \\ 4j+7, & i-1 \leq j \leq n-1 \end{cases}$$

$$f^*(v_{i-1}u_i') = 4i-7, f^*(u_i'v_i') = 4i-5, f^*(v_i'u_i'') = 4i-3, \text{ and } f^*(u_i''v_{i+1}) = 4i+1.$$

Thus, f is an odd sum labeling. □

Proposition 2.9. *Let G be a graph obtained by duplicating a vertex in C_n. Then $S(G)$ is an odd sum graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n and let v_2' be the duplicating vertex of v_2 in C_n. Let $u_i, 1 \leq i \leq n-1$ and let u_n be the subdividing vertices of the edges v_iv_{i+1} and v_nv_1 respectively and u_2', u_2'' be the subdividing vertices of the edges v_1v_2' and $v_2'v_3$ respectively.

Case (i). n is odd and $n \geq 7$.

The vertex labeling $f : V(S(G)) \to \{0, 1, 2, \ldots, 2n+4\}$ is defined as follows:

$$f(v_1) = 2$$

$$f(v_i) = \begin{cases} 2i+2, & 2 \leq i \leq \frac{n-1}{2} \\ 2i+4, & \frac{n+1}{2} \leq i \leq n \end{cases}$$

$$f(u_i) = \begin{cases} 2i+1, & 1 \leq i \leq 2 \\ 2i+3, & 3 \leq i \leq n \end{cases}$$

$$f(v_2') = 0, f(u_2') = 1, \text{and } f(u_2'') = 7.$$

The induced edge labeling f^* is obtained as follows:

$$f^*(v_iu_i) = \begin{cases} 5, & i = 1 \\ 11, & i = 2 \\ 4i+5, & 3 \leq i \leq \frac{n-1}{2} \\ 4i+7, & \frac{n+1}{2} \leq i \leq n \end{cases}$$

$$f^*(u_iv_{i+1}) = \begin{cases} 4i+5, & 1 \leq i \leq 2 \\ 4i+7, & 3 \leq i \leq \frac{n-3}{2} \\ 4i+9, & \frac{n-1}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(u_nv_1) = 2n+5, f^*(v_1u_2') = 3, f^*(u_2'v_2') = 1, f^*(v_2'u_2'') = 7$$
$$\text{and } f^*(u_2''v_3) = 15.$$

Thus, f is an odd sum labeling.

Case (ii). n is even and $n \geq 6$.

The vertex labeling $f : V(S(G)) \to \{0, 1, 2, \ldots, 2n+4\}$ is defined as follows:

$$f(v_i) = \begin{cases} 2i+1, & 1 \leq i \leq \frac{n}{2} \\ 2i+3, & \frac{n}{2}+1 \leq i \leq n \end{cases}$$

$$f(u_i) = 2i+4, \quad 1 \leq i \leq n$$

$$f(v_2') = 1, f(u_2') = 2 \text{ and } f(u_2'') = 0.$$

The induced edge labeling f^* is obtained as follows:

$$f^*(v_iu_i) = \begin{cases} 4i+5, & 1 \leq i \leq \frac{n}{2} \\ 4i+7, & \frac{n}{2}+1 \leq i \leq n \end{cases}$$

$$f^*(u_iv_{i+1}) = \begin{cases} 4i+7, & 1 \leq i \leq \frac{n}{2}-1 \\ 4i+9, & \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(u_nv_1) = 2n+7, f^*(v_1u_2') = 5, f^*(u_2'v_2') = 3, f^*(v_2'u_2'') = 1$$
$$\text{and } f^*(u_2''v_3) = 7.$$

Thus, f is an odd sum labeling.

The odd sum labeling for G when $n = 3, 4, 5$ are given as follows in the Figure 4.

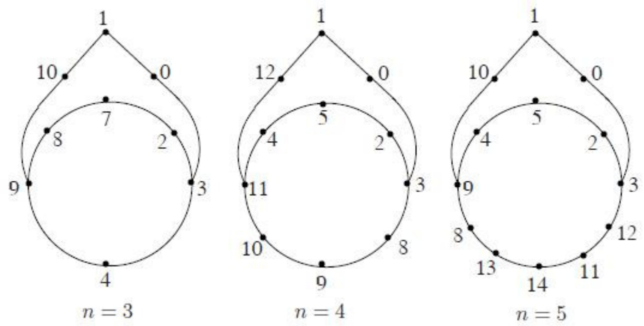

FIGURE 4

□

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[1]DEPARTMENT OF MATHEMATICS,
MEPCO SCHLENK ENGINEERING COLLEGE,
SIVAKASI-626 005, TAMILNADU, INDIA.
E-mail address: psarockiaraj@gmail.com

[2]DEPARTMENT OF MATHEMATICS,
KAMARAJ COLLEGE OF ENGINEERING AND TECHNOLOGY,
VIRUDHUNAGAR- 626 001, TAMILNADU, INDIA.
E-mail address: mahajai1979@gmail.com

[3]DEPARTMENT OF MATHEMATICS,
M.D.T. HINDU COLLEGE,
TIRUNELVELI-627 010, TAMILNADU, INDIA.
E-mail address: vasuhe2010@gmail.com