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THE LAPLACIAN SPECTRUM OF CORONA OF TWO GRAPHS

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Abstract. Let G_1,G_2 be two connected graphs. Denote the corona and the edge corona of G_1, G_2 by $G_1 \circ G_2$ and $G = G_1 \diamond G_2$, respectively. In this paper, we compute the Laplacian spectrum of the corona $G \circ H$ of two arbitrary graphs G and H and the edge corona of a connected regular graph G_1 and an arbitrary graph G_2.

1. Introduction

Throughout this paper, we consider only simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of G denoted by $A(G)$ is defined as $A(G) = (a_{ij})$, where $(a_{ij}) = 1$ if vertices i and j are adjacent in G and 0 otherwise. The Laplacian matrix of G, denoted by $L(G)$ is defined as $D(G)-A(G)$, where $D(G)$ is the diagonal degree matrix of G. The Laplacian spectrum of $L(G)$ is defined as $L(G) = (\mu_1(G), \mu_2(G), \ldots, \mu_n(G))$ where $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ are the eigenvalues of $L(G)$. There is extensively literature available on works related to Laplacian spectrum of a graph. See [4, 5, 6] and the references therein to know more.

The following two definitions come frow [2, 9]. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end-vertices of the ith edge of G_1 to every vertex in the i-th copy of G_2.

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Note that the corona $G_1 \circ G_2$ has $n_1(n_2+1)$ vertices and $m_1 + n_1(m_2+n_2)$ edges, and that the edge corona $G_1 \diamond G_2$ has $n_1 + m_1 n_2$ vertices and $m_1 + m_1(m_2 + 2n_2)$ edges.

There have been some results on the corona and the edge corona of two graphs. Barik et al. [1] provided complete information about the adjacency spectrum of $G_1 \circ G_2$ for a connected graph G_1 and a regular graph G_2, and complete information about the Laplacian spectrum of $G_1 \circ G_2$ using the spectrum (and the Laplacian spectrum, respectively) of G_1 and G_2. In 2010, Hou and Shiu [7] considered the adjacency spectrum of $G_1 \diamond G_2$ for a connected regular graph G_1 and a regular graph G_2 and the Laplacian spectrum of $G_1 \diamond G_2$ for a connected regular graph G_1 and a graph G_2. Recently, McLeman and McNicholas [3], by introducing a new invariant called the coronal of a graph, also discussed the spectrum of $G_1 \circ G_2$.

Motivated by these researches, we discuss the Laplacian spectrum of $G_1 \circ G_2$ and $G_1 \diamond G_2$. We also consider the spectrum of $G_1 \diamond G_2$ when G_1 is regular. This paper is organized as follows. In Section 3, we introduce a new invariant, the L-coronal of a Laplacian matrix, and use it to compute the characteristic polynomial of the Laplacian matrix of $G_1 \circ G_2$. Using this result, we give a complete description of the Laplacian eigenvalues of $G_1 \circ G_2$ when G_1 is an arbitrary graph and G_2 is also an arbitrary graph. In Section 4, we give the characteristic polynomials of the Laplacian matrix of $G_1 \diamond G_2$ for a regular graph G_1 and any graph G_2. Using these results, we give a complete description of Laplacian eigenvalues of $G_1 \diamond G_2$ when G_1 is an r_1-regular graph and G_2 is an arbitrary graph.

2. Preliminaries

The Kronecker product of matrices $A = (a_{ij})$ and B, denoted by $A \otimes B$, is defined to be the partition matrix $(a_{ij}B)$. See [8]. In cases where each multiplication makes sense, we have $M_1 M_2 \otimes M_3 M_4 = (M_1 \otimes M_3)(M_2 \otimes M_4)$.

This implies that for nonsingular matrix M and N, $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$, Recall also that for square matrices M and N of order k and s, respectively. $det(M \otimes N) = (detM)^k (detN)^s$

If M_4 is invertible, then

$$det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = det(M_4) det(M_1 - M_2 M_4^{-1} M_3).$$

Let j_n be the column vector of size n with all entries equal to one, 0_n a column zero vector of size n, and I_n the identity matrix of order n. Let e_i be the i-th unit column vector of size n_1 for $i = 1, 2, \cdots, n_1$. For a vertex u of a graph G, let $d_G(u)$ be the degree of vertex u in G. For vertices u and v in a graph, $u \sim v$ means that v is adjacent to u.

For graphs G_1 with n_1 vertices and G_2 with n_2 vertices we have

$$L(G_1) = (\mu_1, \mu_2, \cdots, \mu_{n_1}), L(G_2) = (\delta_1, \delta_2, \cdots, \delta_{n_2}). \tag{2.1}$$

3. The Laplacian spectrum of corona

In [3], McLeman and McNicholas introduced a new invariant, the coronal of a graph G of order n. It is defined to be the sum of the entries of the matrix $(\lambda I_n - A)^{-1}$, where I_n and A are the identity matrix of order n and the adjacency matrix of G, respectively. Now we shall generalize this concept to Laplacian matrix of graph G.

For a graph G on n vertices, the Laplacian characteristic polynomial of G is $f_G(\lambda) = det(\lambda I_n - L(G))$.

Definition 3.1. Let G be a graph on n vertices, with the Laplacian matrix $L(G)$. Note that, viewed as a matrix over the field of rational functions $C(\lambda)$, the characteristic matrix $((\lambda-1)I_n - L(G))$ is invertible since its determinant $det((\lambda-1)I_n - L(G)) = f_G(\lambda-1) \neq 0$, The L-coronal $\chi_H(\lambda) \in C(\lambda)$ of G is defined to be the sum of the entries of the matrix. Note this can be calculated as $\chi_G(\lambda) = j_n^T((\lambda-1)I_n - L(G))^{-1} j_n$.

Our main theorem is that, beyond the spectra of G and H, only the coronal of H is needed to compute the spectrum of $G = G \circ H$.

Let G_1 be a graph with $V(G_1) = \{1, 2, \cdots, n_1\}$ and G_2 be a graph with n_2 vertices. Let $G = G_1 \circ G_2$. Then

$$A(G) = \begin{pmatrix} A(G_1) & j_{n_2}^T \otimes I_{n_1} \\ (j_{n_2}^T \otimes I_{n_1})^T & A(G_2) \otimes I_{n_1} \end{pmatrix}$$

$$L(G) = \begin{pmatrix} L(G_1) + n_2 I_{n_1} & -j_{n_2}^T \otimes I_{n_1} \\ -(j_{n_2}^T \otimes I_{n_1})^T & (I_{n_2} + L(G_2) \otimes I_{n_1} \end{pmatrix}$$

Theorem 3.1. *Let G_1 be a graph with n_1 vertices, G_2 be a graph with n_2 vertices. Let $\chi_{G_2}(\lambda)$ be the L-coronal of G_2. Then*

$$f_{G_1 \circ G_2}(\lambda) = (f_{G_2}(\lambda - 1))^{n_1} f_{G_1}(\lambda - n_2 - \chi_{G_2}(\lambda)).$$

In particular, the Laplacian characteristic polynomial of $G_1 \circ G_2$ is completely determined by the Laplacian characteristic polynomials $f_{G_1}(\lambda)$ and $f_{G_2}(\lambda)$ and the L-coronal of $\chi_{G_2}(\lambda)$.

Proof. It is easily seen that

$$\begin{aligned} f_{G_1 \circ G_2}(\lambda) &= det(\lambda I_{n_1(n_2+1)} - L(G_1 \circ G_2)) \\ &= det \begin{pmatrix} \lambda I_{n_1} - L(G_1) - n_2 I_{n_1} & j_{n_2}^T \otimes I_{n_1} \\ (j_{n_2}^T \otimes I_{n_1})^T & \lambda I_{n_1 n_2} - (I_{n_2} + L(G_2)) \otimes I_{n_1} \end{pmatrix} \\ &= det \begin{pmatrix} \lambda I_{n_1} - L(G_1) - n_2 I_{n_1} & j_{n_2}^T \otimes I_{n_1} \\ (j_{n_2}^T \otimes I_{n_1})^T & (\lambda I_{n_2} - (I_{n_2} + L(G_2)) \otimes I_{n_1} \end{pmatrix} \\ &= det((\lambda I_{n_2} - (I_{n_2} + L(G_2))) \otimes I_{n_1}) \times det(\lambda I_{n_1} - L(G_1) - n_2 I_{n_1} \\ &\quad - (j_{n_2}^T \otimes I_{n_1})((\lambda I_{n_2} - (I_{n_2} + L(G_2))) \otimes I_{n_1})^{-1} (j_{n_2}^T \otimes I_{n_1})^T \end{aligned}$$

$$\begin{aligned}
&= det((\lambda-1)I_{n_2}-L(G_2))^{n_1}\times det(\lambda I_{n_1}-L(G_1)-n_2I_{n_1}\\
&\quad-(j_{n_2}^T(\lambda I_{n_2}-I_{n_2}-L(G_2))^{-1})j_{n_2})\otimes I_{n_1})\\
&= det((\lambda-1)I_{n_2}-L(G_2))^{n_1}det((\lambda-\chi_{G_2}(\lambda)-n_2)I_{n_1}-L(G_1))\\
&= (f_{G_2}(\lambda-1))^{n_1}f_{G_1}(\lambda-n_2-\chi_{G_2}(\lambda)),
\end{aligned}$$

as desired. □

The following Theorem 3.2, first addressed in [1], is an immediate consequence of Theorem 3.1. We remark that here our method is straight-forward and different from that of Theorem 2.4.

Theorem 3.2. *Let G_1 be any graph with $n_1$vertices, m_1 edges and G_2 be any graph with n_2 vertices, m_2 edges. Suppose that $L(G_1)=(\mu_1,\mu_2,\dots,\mu_{n_1})$ and $L(G_2)=(\delta_1,\delta_2,\dots,\delta_{n_2})$. Then the Laplacian spectrum of $G_1\circ G_2$ is given by*

(i) *The eigenvalue δ_j+1 with multiplicity n_1 for every eigenvalue $\delta_j (j=2,\dots,n_2)$ of $L(G_2)$,*

(ii) *Two multiplicity-one eigenvalues $\frac{(\mu_i+n_2+1)\pm\sqrt{(n_2+1+\mu_i)^2-4\mu_i}}{2}$ for each eigenvalue $\mu_i (i=1,2,\dots,n_1)$ of $L(G_1)$.*

Proof. Since the sum of all entries on every row of Laplacian matrix is zero, we have $L(G_2)j_{n_2}=0j_{n_2}$, and then $((\lambda-1)I_{n_2}-L(G_2))j_{n_2}=(\lambda-1)j_{n_2}$. Thus

$$\chi_{G_2}(\lambda)=j_{n_2}^T((\lambda-1)I_{n_2}-L(G_2))^{-1}j_{n_2}=\frac{j_{n_2}^Tj_{n_2}}{\lambda-1}=\frac{n_2}{\lambda-1}$$

The only pole of $\chi_{G_2}(\lambda)$ is $\lambda=1$. By Theorem 3.1, δ_j+1 is an eigenvalue of $L(G_1\circ G_2)$ with multiplicity of n_1 for $j=2,\dots,n_2$. The remaining $2n_1$ eigenvalues are obtained by solving $\lambda-n_2-\frac{n_2}{\lambda-1}=\mu_i$, the theorem is proved. □

Now, we give an alternative proof of the above theorem by an idea from [10].

Theorem 3.3. *Let G_1 be a graph with n_1 vertices and G_2 be a graph with n_2 vertices, and their Laplacian spectrum are as in (2.1). Let*

$$\lambda_i,\overline{\lambda_i}=\frac{(\mu_i+n_2+1)\pm\sqrt{(n_2+1+\mu_i)^2-4\mu_i}}{2}$$

for $i=1,2,\dots,n_1$. Then $L(G)$ is

$$\begin{pmatrix}\delta_1+1, & \delta_2+1, & \dots & \delta_{n_2-1}+1, & \lambda_1, & \overline{\lambda_1}, & \dots & \lambda_{n_1}, & \overline{\lambda_{n_1}}\\ n_1, & n_1, & \dots & n_1, & 1, & 1, & \dots & 1, & 1\end{pmatrix},$$

where entries in the first row are the eigenvalues with the multiplicities written below.

Proof. Let $Z_1,Z_2,\dots,Z_{n_2}$ be the orthogonal of $L(G_2)$ corresponding to the eigenvalue $0=\delta_1,\delta_2,\dots,\delta_{n_2}$, respectively. Then for $i=1,2,\dots,n_1$ and for $k=2,\dots,n_2$, we have

$$L(G)\begin{pmatrix} 0_{n_1} \\ Z_k \otimes e_i \end{pmatrix} = \begin{pmatrix} 0_{n_1} \\ (I_{n_2}+L(G_2))\otimes I_{n_1})(Z_k\otimes e_i) \end{pmatrix}$$
$$= \begin{pmatrix} 0_{n_1} \\ (I_{n_2}+L(G_2))Z_k\otimes I_{n_1}e_i \end{pmatrix}$$
$$= (\delta_k+1)\begin{pmatrix} 0_{n_1} \\ Z_k\otimes e_i \end{pmatrix}$$

and thus we obtain $n_1(n_2-1)$ eigenvalues and corresponding eigenvectors of $L(G)$.

Let $X_1, X_2, \ldots, X_{n_1}$ be the orthogonal eigenvectors of $L(G_1)$ corresponding to the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n_1}$, respectively. Let $F_i = (X_i^T, \frac{X_i^T}{1-\lambda_i}, \ldots, \frac{X_i^T}{1-\lambda_i})$ for $i = 1, 2, \ldots, n_1$. For any vertex u in the s-th copy of G_2, $F_i(u) = \frac{X_i(s)}{1-\lambda_i}$, and thus

$$d_G(u)F_i(u) - \sum_{v\sim u} F_i(v) = (d_{G_2}(u)+1)F_i(u) - X_i(s) - d_{G_2}(u)\frac{X_i(s)}{1-\lambda_i} = \lambda_i F_i(u).$$

For any vertex u in G_1, we have $d_{G_1}(u)X_i(u) - \sum_{v\sim u, v\in V(G_1)} X_i(v) = \mu_i X_i(u)$, thus

$$\begin{aligned} d_G(u)F_i(u) - \sum_{v\sim u} F_i(v) &= (d_{G_1}(u)+n_2)X_i(u) - \sum_{v\sim u, v\in V(G_1)} X_i(v) - \frac{n_2}{1-\lambda_i}X_i(v) \\ &= (d_{G_1}(u)+n_2)F_i(u) - \sum_{v\sim u, v\in V(G_1)} F_i(v) - \sum_{v\sim u, v\notin V(G_1)} F_i(v) \\ &= (d_{G_1}(u)+n_2-\frac{n_2}{1-\lambda_i})X_i(u) + (\mu_i - d_{G_1}(u))X_i(u) \\ &= \lambda_i X_i(u) = \lambda_i F_i(u) \end{aligned}$$

It follows that $d_{G_1}(u)F_i(u) - \sum_{v\sim u} F_i(v) = \lambda_i F_i(u)$ for every vertex u in G. Similarly, if $\overline{F_i} = (X_i^T, \frac{X_i^T}{1-\overline{\lambda_i}}, \ldots, \frac{X_i^T}{1-\overline{\lambda_i}})$, then $d_G(u)\overline{F_i(u)} - \sum_{v\sim u} F_i(u) = \mu_i F_i(u)$ for every vertex u in G. Therefore we obtain $2n_1$ eigenvalues $\lambda_i, \overline{\lambda_i}$ and corresponding eigenvectors F_i and $\overline{F_i}$ of $L(G)$ for $i = 1, 2, \ldots, n_1$.

Now we have obtained $n_1(n_2+1)$ eigenvalues and corresponding eigenvectors of $L(G)$ and it is easy to see that these eigenvectors of $L(G)$ are linearly independent. Hence, the proof is completed. □

4. The spectrum and Laplacian spectrum of $G_1 \diamond G_2$

Let G_1 be a r_1- regular graph with n_1 vertices, m_1 edges and G_2 be any graph with n_2 vertices, m_2 edges. Also let $L(G_1)$ and $L(G_2)$ be the Laplacian matrices of the graphs G_1 and G_2, respectively. Then the Laplacian matrix of $G = G_1 \diamond G_2$ is

$$L(G) = \begin{pmatrix} L(G_1)+r_1n_2I_{n_1} & -R(G_1)\otimes j_{n_2} \\ -(R(G_1)\otimes j_{n_2})^T & I_{n_1}\otimes(2I_{n_2}+L(G_2) \end{pmatrix}$$

where $R(G_1) = (r_{ij})$ is the vertex-edge incident matrix with entry $r_{ij} = 1$ if the vertex i is incident the edge e_j and 0 otherwise.

Theorem 4.1. *Let G_1 be an r_1- regular graph with n_1 vertices, m_1 edges and G_2 be any graph with n_2 vertices, m_2 edges. Also let $\chi_{G_2}(\lambda)$ be the L-coronal of G_2. Then*

$$f_G(\lambda) = [f_{G_2}(\lambda - 2)]^{m_1} f_{G_1}(\frac{\lambda - r_1n_2 - 2r_1\chi_{G_2}(\lambda - 1)}{1 - \chi_{G_2}(\lambda - 1)})[1 - \chi_{G_2}(\lambda - 1)]^{n_1}.$$

In particular, the Laplacian characteristic polynomial of $G_1 \diamond G_2$ is completely determined by the characteristic polynomials $f_{G_1}(\lambda)$ and $f_{G_2}(\lambda)$ and the L-coronal of $\chi_{G_2}(\lambda)$.

Proof. Note that, viewed as a matrix over the field of rational functions $C(\lambda)$, the following equalities make sense. If λ is not a pole of $\chi_{G_2}(\lambda - 1)$, then

$$\begin{aligned} f_{G_1 \diamond G_2}(\lambda) &= det(\lambda I_{n_1+m_1n_2} - L(G_1 \diamond G_2)) \\ &= det\begin{pmatrix} \lambda I_{n_1} - (L(G_1) + r_1n_2I_{n_1}) & R(G_1) \otimes j_{n_2} \\ (R(G_1) \otimes j_{n_2})^T & \lambda I_{m_1n_2} - (I_{m_1} \otimes (2I_{n_2} + L(G_2)) \end{pmatrix} \\ &= det\begin{pmatrix} (\lambda - r_1n_2)I_{n_1} - L(G_1) & R(G_1) \otimes j_{n_2} \\ (R(G_1) \otimes j_{n_2})^T & I_{m_1} \otimes ((\lambda - 2)I_{n_2} - L(G_2)) \end{pmatrix} \\ &= det(I_{m_1} \otimes ((\lambda - 2)I_{n_2} - L(G_2)) \times detB, \end{aligned}$$

where $B = (\lambda I_{n_1+m_1n_2} - L(G_1)/(I_{m_1} \otimes ((\lambda - 2)I_{n_2} - L(G_2))$ is the Schur complement with respect to $I_{m_1} \otimes ((\lambda-2)I_{n_2} - L(G_2))$. Using many elementary results of Kronecker product of matrices, one has $det(I_{m_1} \otimes ((\lambda - 2)I_{n_2} - L(G_2)) = ((\lambda - 2)I_{n_2} - L(G_2))^{m_1}$ and

$$\begin{aligned} detB &= det[(\lambda I_{n_1+m_1n_2} - L(G_1)/(I_{m_1} \otimes ((\lambda - 2)I_{n_2} - L(G_2))] \\ &= det\{(\lambda - r_1n_2)I_{n_1} - L(G_1) - (R(G_1) \otimes j_{n_2}) \\ &\quad [(I_{m_1}^{-1} \otimes ((\lambda - 2)I_{n_2} - L(G_2)]^{-1}(R(G_1) \otimes j_{n_2})^T\} \\ &= det\{(\lambda - r_1n_2)I_{n_1} - L(G_1) - (R(G_1)I_{m_1}R(G_1)^T) \\ &\quad \otimes (j_{n_2}((\lambda - 2)I_{n_2} - L(G_2))^{-1}j_{n_2}^T)\} \\ &= det\{(\lambda - r_1n_2)I_{n_1} - L(G_1) - (2r_1I_{n_1} - L(G_1)) \otimes \chi_{G_2}(\lambda - 1)\} \\ &= det\{(\lambda - r_1n_2 - 2r_1\chi_{G_2}(\lambda - 1))I_{n_1} - (1 - \chi_{G_2}(\lambda - 1))L(G_1)\} \\ &= f_{G_1}(\frac{\lambda - r_1n_2 - 2r_1\chi_{G_2}(\lambda - 1)}{1 - \chi_{G_2}(\lambda - 1)})[1 - \chi_{G_2}(\lambda - 1)]^{n_1}, \end{aligned}$$

where $R(G_1)R(G_1)^T = 2r_1I_{n_1} - L(G_1)$. Hence, the Laplacian characteristic polynomial of G is $f_G(\lambda) = [f_{G_2}(\lambda - 2)]^{m_1} f_{G_1}(\frac{\lambda - r_1n_2 - 2r_1\chi_{G_2}(\lambda-1)}{1-\chi_{G_2}(\lambda-1)})[1 - \chi_{G_2}(\lambda - 1)]^{n_1}$. □

The following Theorem 4.2, first addressed in [9], is an immediate consequence of Theorem 4.1. We remark that here our method is straight-forward and different from that of Theorem 2.4.

Theorem 4.2. *Let G_1 be an r_1- regular graph with $n_1$vertices, m_1 edges and G_2 be any graph with n_2 vertices, m_2 edges. Suppose that $L(G_1) = (\mu_1, \mu_2, \cdots, \mu_{n_1})$, $L(G_2) = (\delta_1, \delta_2, \cdots, \delta_{n_2})$. Then the Laplacian spectrum of $G = G_1 \diamond G_2$ is given by*

(i) *The eigenvalue $\delta_j + 2$ with multiplicity m_1 for every non-maximum eigenvalue $\delta_j (j = 2, \ldots, n_2)$ of $L(G_2)$,*
(ii) *Two multiplicity-one eigenvalues $\frac{r_1n_2+\mu_i+2\pm\sqrt{(r_1n_2+\mu_i+2)^2-4(n_i+2)\mu_i}}{2}$ for each eigenvalue $\mu_i (i = 1, 2, \ldots, n_1)$ of $L(G_1)$ and*
(iii) *The eigenvalue 2 with multiplicity $m_1 - n_1$(if possible).*

Proof. Since the sum of all entries on every row of Laplacian matrix is zero, we have $L(G_2)j_{n_2} = 0j_{n_2}$, and then $\chi_{G_2}(\lambda) = \frac{n_2}{\lambda-1}$. The only pole of $\chi_{G_2}(\lambda)$ is $\lambda = 1$, which is equivalent to the minimial Laplacian eigenvalue $\lambda - 2 = 0$ of G_2.

Suppose that λ is not the only pole of $\chi_{G_2}(\lambda)$. By theorem 4.1, one has:

(i) The $m_1(n_2-1)$ eigenvalues are δ_j+2 with multiplicity m_1 for every non-minnimal eigenvalue δ_j $(j = 2, \ldots, n_2)$ of $L(G_1)$

(ii) The $2n_1$ eigenvalues are obtained by solving $\lambda - r_1n_2 - 2r_1\frac{n_2}{\lambda-2} = \mu_i(1 - \frac{n_2}{\lambda-2})$ for each eigenvalue $\mu_i(i = 1, \ldots, n_1)$.

(iii) Now we obtain $m_1(n_2 - 1) + 2n_1$ Laplacian eigenvalues of G. The other $n_1 + m_1n_2 - m_1(n_2 - 1) - 2n_1 = m_1 - n_1$ Laplacian eigenvalues of G must come from the only pole $\lambda = 2$ of $\chi_{G_2}(\lambda - 1)$. This completes the proof of the theorem. □

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