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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE HIGHER ORDER PARTIAL DERIVATIVES ARE CO-ORDINATED s-CONVEX

MUHAMMAD AMER LATIF

Abstract. In this paper we point out some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense. Our established results generalize the Hermite-Hadamard type inequalities established for co-ordinated s-convex functions and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex proved in recent literature.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1.1}$$

holds for all x, $y \in I$ and $\lambda \in [0, 1]$. The inequality (1.1) holds in reverse direction if f is concave.

The most famous inequality concerning the class of convex functions, is the Hermite-Hadamard's inequality.

This double inequality is stated as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.2) are in reversed order if f a concave function.

The inequalities (1.2) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a

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variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.2), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [8, 14, 19, 29, 32, 33] and the references therein.

In the paper [15], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined follows.

A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda) y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0,\infty), \lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [9], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.1. [9] *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$ and $a, b \in [0,\infty)$, $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

For more about properties and Hermite-Hadamard type inequalities of s-convex functions in the second sense we refer the interested readers to [7, 9, 12, 15, 20].

Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex functions on Δ, known as co-ordinated convex functions, was introduced by S. S. Dragomir [10] as follows.

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition for co-ordinated convex functions may be stated as follow.

Definition 1.1. [21] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the following inequality holds for all $t, r \in [0,1]$ and $(x,u), (y,w) \in \Delta$

$$\begin{aligned} f(tx + (1-t)y, ru + (1-r)w) \le\ & trf(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) \\ & + (1-t)(1-r)f(y,w). \end{aligned}$$

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true [10].

The following Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [10].

Theorem 1.2. [10] *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx \right. \\
&\quad \left. +\frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right] \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{1.4}
$$

The above inequalities are sharp.

The concept of s-convex functions on the co-ordinates in the second sense was introduced by Alomari and Darus in [3] as a generalization of the usual co-ordinated convexity.

Definition 1.2. [3] Consider the bidimensional interval $\Delta = [a,b]\times[c,d]$ in $[0,\infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is s-convex in the second sense on Δ if $f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \leq \lambda^s f(x,y)+(1-\lambda)^s f(z,w)$, holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$ with some fixed $s\in(0,1]$.

A function $f : \Delta \subseteq [0,\infty)^2 \to \mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y : [a,b]\to\mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d]\to\mathbb{R}$, $f_x(v) = f(x,v)$, are s-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s\in(0,1]$, i.e., the partial mappings f_y and f_x are s-convex in the second sense with some fixed $s\in(0,1]$.

A formal definition of co-ordinated s-convex function in second sense may be stated as follows.

Definition 1.3. A function $f : \Delta \subseteq [0,\infty)^2 \to \mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if

$$
\begin{aligned}
f(tx+(1-t)y, ru+(1-r)w) \leq{}& t^s r^s f(x,u)+t^s(1-r)^s f(x,w) \\
&+r^s(1-t)^s f(y,u)+(1-t)^s(1-r)^s f(y,w)
\end{aligned} \tag{1.5}
$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,u),(x,w),(y,w)\in\Delta$, for some fixed $s\in(0,1]$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.5) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$ with some fixed $s\in(0,1]$.

Furthermore, Alomari and Darus [5] introduced a new class of s-convex functions on the co-ordinates on the rectangle from the plane as follows.

Definition 1.4. [5] Consider the bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is s-convex in the second sense on Δ if there exist $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$ such that

$$f(\lambda x+(1-\lambda)\,z,\lambda y+(1-\lambda)\,w)\le \lambda^{s_1}f(x,y)+(1-\lambda)^{s_2}f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$. This class of functions is denoted by $MWO^2_{s_1,s_2}$.

A function $f:\Delta\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are s_1-convex and s_2-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$, respectively, i.e., the partial mappings f_y and f_x are s_1-convex and s_2-convex in the second sense, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$.

The definition 1.3 can be generalized as follows.

Definition 1.5. A function $f:\Delta=:[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if

$$\begin{aligned}f(tx+(1-t)y,ru+(1-r)w)&\le t^{s_1}r^{s_2}f(x,u)+t^{s_1}(1-r)^{s_2}f(x,w)\\&\quad+r^{s_2}(1-t)^{s_1}f(y,u)+(1-t)^{s_1}(1-r)^{s_2}f(y,w)\end{aligned}\tag{1.6}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,u),(x,w),(y,w)\in\Delta$, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.6) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$.

In [5], Alomari et al. also proved a variant of inequalities given above by (1.4) for s-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.3. [5] *Suppose $f:\Delta\subseteq[0,\infty)^2\to[0,\infty)$ is s-convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities*

$$\begin{aligned}\frac{4^{s_1-1}+4^{s_2-1}}{2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{2^{s_1-2}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\&\quad+\frac{2^{s_2-2}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\end{aligned}\tag{1.7}$$

$$\leq \frac{1}{2(s_1+1)}\left(\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right)$$
$$\leq \frac{1}{2}\left(\frac{1}{(s_1+1)^2}+\frac{1}{(s_2+1)^2}\right)[f(a,c)+f(b,c)+f(a,d)+f(b,d)].$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1, 3, 4, 5, 6], [10], [13], [21]-[24], [25]-[28] and [31]. Alomari et al. [1, 3, 4, 5, 6], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions and co-ordinated log-convex functions. Dragomir [10], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [13], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated to the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [12]-[14], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, differentiable functions whose higher order partial derivatives are co-ordinated convex, product of two co-ordinated convex mappings and for co-ordinated h-convex mappings. Özdemir et. al [25]-[28], proved Hadamard's type inequalities for co-ordinated convex functions, co-ordinated s-convex functions and co-ordinated m-convex and (α, m)-convex functions.

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities for differentiable functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense on the rectangle from the plane $\mathbb{R}^2$ which generalize the Hermite-Hadamard type inequalities proved for co-ordinated s-convex functions in the second sense and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex on the rectangle from the plane $\mathbb{R}^2$ (see [24]).

2. Main Results

In this section we establish new Hermite-Hadamard type inequalities for double integrals of functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense.

To make the presentation easier and compact to understand, we make some symbolic representations as follows

$$A' = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right]$$
$$+\frac{1}{2}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^l}{2(l+1)!}\left[\frac{\partial^l f(a,c)}{\partial y^l}+\frac{\partial^l f(b,c)}{\partial y^l}\right]$$
$$+\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\left[\frac{\partial^k f(a,c)}{\partial x^k}+\frac{\partial^k f(a,d)}{\partial x^k}\right]$$

$$
\begin{aligned}
&-\frac{1}{b-a}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^{l}}{2(l+1)!}\int_{a}^{b}\frac{\partial^{l}f(x,c)}{\partial y^{l}}dx\\
&-\frac{1}{d-c}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^{k}}{2(k+1)!}\int_{c}^{d}\frac{\partial^{k}f(a,y)}{\partial x^{k}}dy\\
&-\sum_{k=2}^{n-1}\sum_{l=2}^{m-1}\frac{(k-1)(l-1)(b-a)^{k}(d-c)^{l}}{4(k+1)!(l+1)!}\frac{\partial^{k+l}f(a,c)}{\partial x^{k}y^{l}},
\end{aligned}
$$

and

$$
\begin{aligned}
B_{(n,m)} &= \left|\tfrac{\partial^{n+m}f(a,c)}{\partial t^{n}\partial r^{m}}\right|, & C_{(n,m)} &= \left|\tfrac{\partial^{n+m}f(a,d)}{\partial t^{n}\partial r^{m}}\right|, & D_{(n,m)} &= \left|\tfrac{\partial^{n+m}f(b,c)}{\partial t^{n}\partial r^{m}}\right|,\\
E_{(n,m)} &= \left|\tfrac{\partial^{n+m}f(b,d)}{\partial t^{n}\partial r^{m}}\right|, & F_{(n,m)} &= \left|\tfrac{\partial^{n+m}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t^{n}\partial r^{m}}\right|, & G_{(n,m)} &= \left|\tfrac{\partial^{n+m}f\left(a,\frac{c+d}{2}\right)}{\partial t^{n}\partial r^{m}}\right|,\\
H_{(n,m)} &= \left|\tfrac{\partial^{n+m}f\left(\frac{a+b}{2},c\right)}{\partial t^{n}\partial r^{m}}\right|, & J_{(n,m)} &= \left|\tfrac{\partial^{n+m}f\left(\frac{a+b}{2},d\right)}{\partial t^{n}\partial r^{m}}\right|, & I_{(n,m)} &= \left|\tfrac{\partial^{n+m}f\left(b,\frac{c+d}{2}\right)}{\partial t^{n}\partial r^{m}}\right|,
\end{aligned}
$$

where the sums above take 0, when $m=n=1$ and $m=n=2$ and hence

$$
A^{'}=A=\frac{1}{2}\left[\frac{1}{b-a}\int_{a}^{b}[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_{c}^{d}[f(a,y)+f(b,y)]\,dy\right].
$$

In what follows Δ° is the interior of $\Delta=[a,b]\times[c,d]$ and $L(\Delta)$ is the space of integrable functions over Δ.

The following two results will be very useful in the sequel of the paper

Theorem 2.1. [18] *Let $f:\Delta\to\mathbb{R}$ be a continuous mapping such that the partial derivatives $\frac{\partial^{k+l}f(.,.)}{\partial x^{k}\partial y^{l}}$, $k=0,1,\ldots,n-1$, $l=0,1,\ldots,m-1$ exist on Δ° and are continuous on Δ, then*

$$
\begin{aligned}
\int_{a}^{b}\int_{c}^{d}f(t,r)\,drdt &= (-1)^{m+n}\int_{a}^{b}\int_{c}^{d}K_{n}(x,t)\,S_{m}(y,r)\,\frac{\partial^{n+m}f(t,r)}{\partial t^{n}\partial r^{m}}drdt\\
&+\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_{k}(x)\,Y_{l}(y)\,\frac{\partial^{k+l}f(x,y)}{\partial x^{k}\partial y^{l}}+(-1)^{m}\sum_{k=0}^{n-1}X_{k}(x)\int_{c}^{d}S_{m}(y,r)\,\frac{\partial^{k+m}f(x,r)}{\partial x^{k}\partial r^{m}}dr\\
&+(-1)^{n}\sum_{l=0}^{m-1}Y_{l}(y)\int_{a}^{b}K_{n}(x,t)\,\frac{\partial^{n+l}f(t,y)}{\partial t^{n}\partial y^{l}}dt,
\end{aligned}
$$

where, for $(x,y)\in\Delta$, we have

$$
\left\{
\begin{array}{l}
K_{n}(x,t):=\left\{\begin{array}{l}\frac{(t-a)^{n}}{n!},\,t\in[a,x]\\ \frac{(t-b)^{n}}{n!},\,t\in(x,b]\end{array}\right.\\[2ex]
S_{m}(y,r):=\left\{\begin{array}{l}\frac{(r-c)^{m}}{m!},\,r\in[c,y]\\ \frac{(r-d)^{m}}{m!},\,r\in(y,d]\end{array}\right.
\end{array}
\right.
\quad\text{and}\quad
\left\{
\begin{array}{l}
X_{k}(x):=\frac{(b-x)^{k+1}+(-1)^{k}(x-a)^{k+1}}{(k+1)!}\\[2ex]
Y_{l}(y):=\frac{(d-y)^{l+1}+(-1)^{l}(y-c)^{l+1}}{(l+1)!}
\end{array}
\right..
$$

Lemma 2.1. [24] *Let $f:\Delta\to\mathbb{R}$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial x^n\partial y^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial x^n\partial y^m}\in L(\Delta)$ for $m,n\geq 1$, then*

$$\begin{aligned}(2.1)\quad &\frac{(b-a)^n(d-c)^m}{4n!m!}\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\\ &\times\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}dtdr+A'\\ &=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx.\end{aligned}$$

Now we prove our main results.

Theorem 2.2. *Let $f:\Delta\subseteq[0,\infty)^2\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial r^m}\right|$ is s-convex on the co-ordinates on Δ in the second sense, for m, $n\in\mathbb{N}$, m, $n\geq 2$, then we have the following inequality*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|\\ (2.2)\quad &\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left[LB_{(n,m)}+MC_{(n,m)}+ND_{(n,m)}+RE_{(n,m)}\right],\end{aligned}$$

where $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$,

$$\begin{aligned}L&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right]\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right],\\ M&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right][mB(m,s_2+1)-2B(m+1,s_2+1)],\\ N&=\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right][nB(n,s_1+1)-2B(n+1,s_1+1)],\\ R&=[nB(n,s_1+1)-2B(n+1,s_1+1)][mB(m,s_2+1)-2B(m+1,s_2+1)],\end{aligned}$$

and $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Euler Beta function.

Proof. Suppose $m,n\geq 2$. By Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|\\ (2.3)\quad &\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\\ &\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|dtdr.\end{aligned}$$

By s-convexity of $\left|\frac{\partial^{m+n}f}{\partial t^n \partial s^m}\right|$ on the co-ordinates on Δ, we get that

$$\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|dtdr$$

(2.4)

$$\begin{aligned}&\leq B_{(n,m)}\int_0^1\int_0^1 t^{n+s_1-1}r^{m+s_2-1}(n-2t)(m-2r)\,drdt\\&\quad+C_{(n,m)}\int_0^1\int_0^1 t^{n+s_1-1}r^{m-1}(1-r)^{s_2}(n-2t)(m-2r)\,drdt\\&\quad+E_{(n,m)}\int_0^1\int_0^1 t^{n-1}(1-t)^{s_1}(n-2t)r^{m-1}(1-r)^{s_2}(m-2r)\,drdt\\&\quad+D_{(n,m)}\int_0^1\int_0^1 t^{n-1}r^{m+s_2-1}(1-t)^{s_1}(n-2t)(m-2r)\,drdt.\end{aligned}$$

Since

$$\begin{aligned}(2.5)\qquad&\int_0^1\int_0^1 t^{n+s_1-1}r^{m+s_2-1}(n-2t)(m-2r)\,drdt\\&=\int_0^1 t^{n+s_1-1}(n-2t)\,dt\int_0^1 r^{m+s_2-1}(m-2r)\,dr\\&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right]\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right].\end{aligned}$$

Analogously,

$$\begin{aligned}(2.6)\qquad&\int_0^1\int_0^1 t^{n+s_1-1}r^{m-1}(1-r)^{s_2}(n-2t)(m-2r)\,drdt\\&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right][mB(m,s_2+1)-2B(m+1,s_2+1)],\end{aligned}$$

$$\begin{aligned}(2.7)\qquad&\int_0^1\int_0^1 t^{n-1}r^{m+s_2-1}(1-t)^{s_1}(n-2t)(m-2r)\,drdt\\&=\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right][nB(n,s_1+1)-2B(n+1,s_1+1)]\end{aligned}$$

and

$$\int_0^1\int_0^1 t^{n-1}(1-t)^{s_1}(n-2t)r^{m-1}(1-r)^{s_2}(m-2r)\,drdt$$

(2.8) $$=[nB(n,s_1+1)-2B(n+1,s_1+1)][mB(m,s_2+1)-2B(m+1,s_2+1)].$$

From (2.4)-(2.8) in (2.3), we get the required inequality. This completes the proof of the theorem. □

Theorem 2.3. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial r^m}\right|^q$, $q\geq 1$, is s-convex on the co-ordinates on Δ, $m,n\in\mathbb{N}$, $m,n\geq 2$, then*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right| \\
&\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left(\frac{(n-1)(m-1)}{(n+1)(m+1)}\right)^{1-1/q} \\
&\quad\times\sqrt[q]{LB^q_{(n,m)}+MD^q_{(n,m)}+NC^q_{(n,m)}+RE^q_{(n,m)}},
\end{aligned}\tag{2.9}$$

where $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$ and L, M, N, R and $B(x,y)$ are as defined in Theorem 2.2.

Proof. The case $q=1$ is the Theorem 2.2. Suppose $q>1$, then by Lemma 2.1 and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right| \\
&\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left\{\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\,drdt\right\}^{1-1/q} \\
&\quad\times\left\{\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\right. \\
&\quad\left.\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|^q dtdr\right\}^{1/q}.
\end{aligned}\tag{2.10}$$

By the similar arguments used to obtain (2.2) and the fact

$$\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\,drdt=\frac{(n-1)(m-1)}{(n+1)(m+1)},$$

we get (2.9). This completes the proof of the theorem. □

Theorem 2.4. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exist on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q\geq 1$, is s-convex on the co-ordinates on Δ, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$, m, $n\in\mathbb{N}$, $m,n\geq 1$.*

Then

$$\left|-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^{k}\right]\left[1+(-1)^{l}\right]}{2^{k+l+2}}\frac{(b-a)^{k}(d-c)^{l}}{(k+1)!(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^{k}\partial y^{l}}\right.$$
$$+\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}f(t,r)\,drdt$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^{k}\right](b-a)^{k}}{2^{k+1}(k+1)!}\int_{c}^{d}Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^{k}\partial r^{m}}dr$$
$$\left.+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^{l}\right](d-c)^{l}}{2^{l+1}(l+1)!}\int_{a}^{b}P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^{n}\partial y^{l}}dt\right|$$

(2.11)

$$\leq\frac{1}{4n!m!}\left(\frac{4}{(n+1)(m+1)}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{2}\right)^{n}\left(\frac{d-c}{2}\right)^{m}$$
$$\times\left[\left(B_{(n,m)}^{q}+C_{(n,m)}^{q}+D_{(n,m)}^{q}+E_{(n,m)}^{q}\right)B(n+1,s_{1}+1)\,B(m+1,s_{2}+1)\right.$$
$$+\frac{2\left(G_{(n,m)}^{q}+I_{(n,m)}^{q}\right)B(n+1,s_{1}+1)}{m+s_{2}+1}+\frac{2\left(H_{(n,m)}^{q}+J_{(n,m)}^{q}\right)B(m+1,s_{2}+1)}{n+s_{1}+1}$$
$$\left.+\frac{4F_{(n,m)}^{q}}{(n+s_{1}+1)(m+s_{2}+1)}\right]^{\frac{1}{q}},$$

where

$$P(t):=\begin{cases}(t-a)^{n},t\in\left[a,\frac{a+b}{2}\right]\\(t-b)^{n},t\in\left(\frac{a+b}{2},b\right]\end{cases}\quad\text{and}\quad Q(r):=\begin{cases}(r-c)^{m},r\in\left[c,\frac{c+d}{2}\right]\\(r-d)^{m},r\in\left(\frac{c+d}{2},d\right]\end{cases}.$$

Proof. By letting $x\mapsto\frac{a+b}{2}$ and $y\mapsto\frac{c+d}{2}$ in Theorem 2.1 and using the properties of the absolute value, we obtain

$$\left|-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^{k}\right]\left[1+(-1)^{l}\right]}{2^{k+l+2}}\frac{(b-a)^{k}(d-c)^{l}}{(k+1)!(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^{k}\partial y^{l}}\right.\tag{2.12}$$
$$+\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}f(t,r)\,drdt$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^{k}\right](b-a)^{k}}{2^{k+1}(k+1)!}\int_{c}^{d}Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^{k}\partial r^{m}}dr$$

$$+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg|$$

$$\leq \frac{1}{(b-a)(d-c)\,m!n!}\int_a^b\int_c^d |P(t)|\,|Q(r)|\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt.$$

By the power mean inequality for double integrals, we have

$$\int_a^b\int_c^d |P(t)|\,|Q(r)|\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt$$

(2.13)

$$\leq\left(\int_a^b\int_c^d |P(t)|\,|Q(r)|\,drdt\right)^{1-\frac{1}{q}}\left(\int_a^b\int_c^d |P(t)|\,|Q(r)|\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt\right)^{\frac{1}{q}}$$

$$=\left(\int_a^b\int_c^d |P(t)|\,|Q(r)|\,drdt\right)^{1-\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt\right.$$

$$+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}(b-t)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt$$

$$+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d(t-a)^n(d-r)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt$$

$$\left.+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d(b-t)^n(d-r)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt\right]^{\frac{1}{q}}.$$

Now we calculate each integral in (2.13). Since $t=\left(\frac{\frac{a+b}{2}-t}{\frac{a+b}{2}-a}\right)a+\left(\frac{t-a}{\frac{a+b}{2}-a}\right)\frac{a+b}{2}$ and $r=\left(\frac{\frac{c+d}{2}-r}{\frac{c+d}{2}-c}\right)c+\left(\frac{r-c}{\frac{c+d}{2}-c}\right)\frac{c+d}{2}$. By the co-ordinated s-convexity of $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, we have

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt\leq\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2} \tag{2.14}$$

$$\times\left[B^q_{(n,m)}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left(\frac{a+b}{2}-t\right)^{s_1}\left(\frac{c+d}{2}-r\right)^{s_2}drdt\right.$$

$$+G^q_{(n,m)}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n\left(\frac{a+b}{2}-t\right)^{s_1}(r-c)^{s_2+m}drdt$$

$$+H^q_{(n,m)}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}\left(\frac{c+d}{2}-r\right)^{s_2}(r-c)^m drdt$$

$$\left.+F^q_{(n,m)}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}(r-c)^{s_2+m}drdt\right].$$

Now by the change of variables $u=t-a$, $v=r-c$ and then by the change of variables $x=\frac{2u}{b-a}$, $y=\frac{2v}{d-c}$, we get that

(2.15)
$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\times\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left(\frac{a+b}{2}-t\right)^{s_1}\left(\frac{c+d}{2}-r\right)^{s_2}drdt\\
&=\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_0^{\frac{b-a}{2}}u^n\left(\frac{b-a}{2}-u\right)^{s_1}du\int_0^{\frac{d-c}{2}}v^m\left(\frac{d-c}{2}-v\right)^{s_2}dv\\
&=\int_0^{\frac{b-a}{2}}u^n\left(1-\frac{2u}{b-a}\right)^{s_1}du\int_0^{\frac{d-c}{2}}v^m\left(1-\frac{2v}{d-c}\right)^{s_2}dv\\
&=\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\int_0^1x^n(1-x)^{s_1}dx\int_0^1y^m(1-y)^{s_2}dy\\
&=\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(n+1,s_1+1)\,B(m+1,s_2+1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n\left(\frac{a+b}{2}-t\right)^{s_1}(r-c)^{s_2+m}drdt\\
&=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(n+1,s_1+1)}{m+s_2+1},
\end{aligned}\tag{2.16}$$

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}\left(\frac{c+d}{2}-r\right)^{s_2}(r-c)^mdrdt\\
&=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(m+1,s_2+1)}{n+s_1+1}
\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}(r-c)^{s_2+m}drdt\\
&=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}}{(n+s_1+1)(m+s_2+1)}.
\end{aligned}\tag{2.18}$$

Using (2.15)-(2.18) in (2.14), we obtain

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^qdrdt\le\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\\
&\times\left[B^q_{(n,m)}B(n+1,s_1+1)\,B(m+1,s_2+1)+\frac{G^q_{(n,m)}B(n+1,s_1+1)}{m+s_2+1}\right.
\end{aligned}\tag{2.19}$$

$$+\frac{H^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + \frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}\Bigg].$$

Analogously,

$$\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} (b-t)^n (r-c)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \le \left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1} \times \left[\frac{H^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + D^q_{(n,m)} B(n+1,s_1+1) B(m+1,s_2+1) + \frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)} + \frac{I^q_{(n,m)} B(n+1,s_1+1)}{m+s_2+1}\right], \tag{2.20}$$

$$\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} (t-a)^n (d-r)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \le \left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1} \left[\frac{G^q_{(n,m)} B(n+1,s_1+1)}{m+s_2+1} + C^q_{(n,m)} B(n+1,s_1+1) B(m+1,s_2+1) + \frac{J^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + \frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}\right] \tag{2.21}$$

and

$$\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} (b-t)^n (d-r)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \le \left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1} \left[\frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)} + \frac{I^q_{(n,m)} B(n+1,s_1+1)}{m+s_2+1} + \frac{J^q_{(n,m)} B(m+1,s_2+1)}{n+s_1+1} + E^q_{(n,m)} B(n+1,s_1+1) B(m+1,s_2+1)\right]. \tag{2.22}$$

It is not difficult to observe that

$$\int_a^b\int_c^d |P(t)|\,|Q(r)|\,drdt = \frac{4}{(n+1)(m+1)}\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}. \tag{2.23}$$

From (2.12)-(2.23), we get the desired inequality. The proof of the Theorem for $q=1$ is the same. This completes the proof. □

Some results can be deduced from the inequalities (2.9) and (2.12) as follows. Letting $s_1 = s_2 = 1$ in Theorem 2.3 gives the following corollary.

Corollary 2.1. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial r^m}\right|^q$, $q\geq 1$, is convex on the co-ordinates on Δ, $m,n\in\mathbb{N}$, $m,n\geq 2$, then*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right| \\
&\leq\frac{(b-a)^n(d-c)^m(n-1)^{1-1/q}(m-1)^{1-1/q}}{4(n+1)!(m+1)!(n+2)^{1/q}(m+2)^{1/q}}\Big[\left(m^2-2\right)\left(n^2-2\right)B^q_{(n,m)} \\
&\quad+m\left(n^2-2\right)C^q_{(n,m)}+n\left(m^2-2\right)D^q_{(n,m)}+nmE^q_{(n,m)}\Big]^{\frac{1}{q}}.
\end{aligned}\tag{2.24}$$

Corollary 2.2. *Under the assumptions of Corollary 2.1 with $m=n=2$, we have*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right| \\
&\leq\frac{(b-a)^2(d-c)^2}{9\cdot 2^{\frac{2}{q}+4}}\sqrt[q]{\left|\frac{\partial^4 f(a,c)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(b,c)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(a,d)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(b,d)}{\partial t^2\partial r^2}\right|^q}.
\end{aligned}$$

The following corollary is a special case of Theorem 2.4 for $s_1=s_2=1$.

Corollary 2.3. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exist on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q\geq 1$, is convex on the co-ordinates on Δ, m, $n\in\mathbb{N}$, $m,n\geq 1$. Then*

$$\begin{aligned}
&\Bigg|-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l} \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt \\
&+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr \\
&+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg|
\end{aligned}\tag{2.25}$$

$$\leq \frac{(b-a)^n (d-c)^m}{2^{m+n+\frac{2}{q}} (n+1)! (m+1)!} \left[\frac{B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}}{(n+2)(m+2)} \right.$$
$$+ \frac{2(m+1)\left(G^q_{(n,m)} + I^q_{(n,m)}\right)}{(n+2)(m+2)} + \frac{2(n+1)\left(H^q_{(n,m)} + J^q_{(n,m)}\right)}{(n+2)(m+2)}$$
$$\left. + \frac{4(n+1)(m+1) F^q_{(n,m)}}{(n+2)(m+2)} \right]^{\frac{1}{q}},$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 2.4.

The following corollary is a special case of Theorem 2.4 for $s_1 = s_2 = 1$ and $m = n = 1$, which gives tighter estimate than those from [23, Theorem 4, page 8].

Corollary 2.4. *Under the assumptions of Corollary 2.3 with $m = n = 1$, we have*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.$$
$$\left. - \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, r\right) dr - \frac{1}{2(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt \right|$$
$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{2}{q}}} \left[\frac{B^q_{(1,1)} + C^q_{(1,1)} + D^q_{(1,1)} + E^q_{(1,1)}}{9} \right. \tag{2.26}$$
$$\left. + \frac{4\left(G^q_{(1,1)} + I^q_{(1,1)}\right)}{9} + \frac{4\left(H^q_{(1,1)} + J^q_{(1,1)}\right)}{9} + \frac{8F^q_{(1,1)}}{9} \right]^{\frac{1}{q}},$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 2.4.

It is easy to see that, when $\left| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right|^q$, $q \geq 1$, is convex on the co-ordinates on Δ, m, $n \in \mathbb{N}$, $m, n \geq 1$, then

$$2\left(G^q_{(n,m)} + I^q_{(n,m)}\right) \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)},$$

$$2\left(H^q_{(n,m)} + J^q_{(n,m)}\right) \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}$$

and

$$4F^q_{(n,m)} \leq B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}.$$

Substituting these inequalities in Corollary 2.3, we get the following corollary which is [24, Theorem 2.3, page 12].

Corollary 2.5. *Let $f : \Delta \subset [0,\infty) \times [0,\infty) \to [0,\infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial r^m}$ exist on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial r^m} \in L(\Delta)$. If $\left| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right|^q$, $q \geq 1$, is* *convex on the co-ordinates on* Δ, m, $n \in \mathbb{N}$, $m, n \geq 1$. *Then*

$$\begin{aligned}(2.27)\quad &\Bigg| -\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}} \frac{(b-a)^k (d-c)^l}{(k+1)!\,(l+1)!} \frac{\partial^{k+l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt \\ &+ \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, r\right)}{\partial x^k \partial r^m} dr \\ &+ \frac{(-1)^{n+1}}{(b-a)\, n!} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \Bigg| \\ &\leq \frac{(b-a)^n (d-c)^m}{2^{m+n+\frac{2}{q}}(n+1)!\,(m+1)!} \sqrt[q]{B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}},\end{aligned}$$

where $P(t)$ *and* $Q(r)$ *are as defined in Theorem 2.4.*

A different approach leads us to the following result.

Theorem 2.5. *Let* $f : \Delta \subset [0,\infty) \times [0,\infty) \to [0,\infty)$, $a < b$, $c < d$, *be a continuous mapping such that* $\frac{\partial^{m+n} f}{\partial t^n \partial r^m}$ *exist on* Δ° *and* $\frac{\partial^{m+n} f}{\partial t^n \partial r^m} \in L(\Delta)$. *If* $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^q$, $q \geq 1$, *is* s*-convex on the co-ordinates on* Δ, $s_1, s_2 \in (0,1]$ *with* $s = \frac{s_1+s_2}{2}$, m, $n \in \mathbb{N}$, $m, n \geq 1$. *Then*

$$\begin{aligned}&\Bigg| -\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}} \frac{(b-a)^k (d-c)^l}{(k+1)!\,(l+1)!} \frac{\partial^{k+l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt \\ &+ \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, r\right)}{\partial x^k \partial r^m} dr \\ &+ \frac{(-1)^{n+1}}{(b-a)\, n!} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \Bigg| \\ (2.28)\quad &\leq \frac{1}{4n!m!} \left(\frac{1}{(n+1)(m+1)}\right)^{1-\frac{1}{q}} \left(\frac{b-a}{2}\right)^n \left(\frac{d-c}{2}\right)^m \times\end{aligned}$$

$$\times\left\{\left[B^q_{(n,m)}B\left(n+1,s_1+1\right)B\left(m+1,s_2+1\right)+\frac{G^q_{(n,m)}B\left(n+1,s_1+1\right)}{m+s_2+1}\right.\right.$$

$$\left.+\frac{H^q_{(n,m)}B\left(m+1,s_2+1\right)}{n+s_1+1}+\frac{F^q_{(n,m)}}{\left(n+s_1+1\right)\left(m+s_2+1\right)}\right]^{\frac{1}{q}}$$

$$+\left[\frac{H^q_{(n,m)}B\left(m+1,s_2+1\right)}{n+s_1+1}+D^q_{(n,m)}B\left(n+1,s_1+1\right)B\left(m+1,s_2+1\right)\right.$$

$$\left.+\frac{F^q_{(n,m)}}{\left(n+s_1+1\right)\left(m+s_2+1\right)}+\frac{I^q_{(n,m)}B\left(n+1,s_1+1\right)}{m+s_2+1}\right]^{\frac{1}{q}}$$

$$+\left[\frac{G^q_{(n,m)}B\left(n+1,s_1+1\right)}{m+s_2+1}+C^q_{(n,m)}B\left(n+1,s_1+1\right)B\left(m+1,s_2+1\right)\right.$$

$$\left.+\frac{J^q_{(n,m)}B\left(m+1,s_2+1\right)}{n+s_1+1}+\frac{F^q_{(n,m)}}{\left(n+s_1+1\right)\left(m+s_2+1\right)}\right]^{\frac{1}{q}}$$

$$+\left[\frac{F^q_{(n,m)}}{\left(n+s_1+1\right)\left(m+s_2+1\right)}+\frac{I^q_{(n,m)}B\left(n+1,s_1+1\right)}{m+s_2+1}+\frac{J^q_{(n,m)}B\left(m+1,s_2+1\right)}{n+s_1+1}\right.$$

$$\left.\left.+E^q_{(n,m)}B\left(n+1,s_1+1\right)B\left(m+1,s_2+1\right)\right]^{\frac{1}{q}}\right\},$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 2.4.

Proof. By letting $x\mapsto\frac{a+b}{2}$ and $y\mapsto\frac{c+d}{2}$ in Theorem 2.1, using the properties of the absolute value, we obtain

$$\left|-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}\right.$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt$$

$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr$$

$$\left.+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\right|$$

$$\begin{aligned}
(2.29)\qquad &\leq \frac{1}{(b-a)(d-c)\,m!n!}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt\right.\\
&+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}(b-t)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt\\
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}(t-a)^n(d-r)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt\\
&\left.+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(b-t)^n(d-r)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt\right].
\end{aligned}$$

Using the power-mean inequality for each integral on the right-side of (2.29) and by the similar arguments as in proving Theorem 2.4, we get (2.28). □

Corollary 2.6. *If the conditions of Theorem 2.5 are satisfied and if $m=n=1$ and $s_1=s_2=1$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt+\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.-\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},r\right)dr-\frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\right|\\
&\leq\left(\frac{1}{4}\right)^{2-\frac{1}{q}}\left(\frac{b-a}{2}\right)\left(\frac{d-c}{2}\right)\left\{\left[\frac{1}{36}B^q_{(1,1)}+\frac{1}{18}G^q_{(1,1)}+\frac{1}{18}H^q_{(1,1)}+\frac{1}{9}F^q_{(1,1)}\right]^{\frac{1}{q}}\right.\\
&+\left[\frac{1}{18}H^q_{(1,1)}+\frac{1}{36}D^q_{(1,1)}+\frac{1}{9}F^q_{(1,1)}+\frac{1}{18}I^q_{(1,1)}\right]^{\frac{1}{q}}\\
&+\left[\frac{1}{18}G^q_{(1,1)}+\frac{1}{36}C^q_{(1,1)}+\frac{1}{18}J^q_{(1,1)}+\frac{1}{9}F^q_{(1,1)}\right]^{\frac{1}{q}}\\
&\left.+\left[\frac{1}{9}F^q_{(1,1)}+\frac{1}{18}I^q_{(1,1)}+\frac{1}{18}J^q_{(1,1)}+\frac{1}{36}E^q_{(1,1)}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we use the Hölder's inequality instead of the power-mean inequality we get the following result.

Theorem 2.6. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exist on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^p$, $p>1$, is s-convex on the co-ordinates on Δ, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$, m, $n\in\mathbb{N}$, $m,n\geq 1$.*

Then for $P(t)$ *and* $Q(r)$ *defined as in Theorem 2.4 and* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\begin{aligned}(2.30)\quad &\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt \\ &-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l} \\ &+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr \\ &+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg| \\ &\le\frac{(b-a)^n(d-c)^m}{2^{n+m}n!m!\left[(np+1)(mp+1)\right]^{\frac{1}{p}}}\left[\frac{1}{2}\left(\frac{1}{(s_1+1)^2}+\frac{1}{(s_2+1)^2}\right)\right]^{\frac{1}{q}} \\ &\times\left[B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. The inequality (2.30) follows from the Hölder's inequality and (1.7). □

Corollary 2.7. *Under the assumptions of Theorem 2.6, if* $m=n=1$ *and* $s_1=s_2=1$*, then for* $\frac{1}{p}+\frac{1}{q}=1$ *we have the inequality*

$$\begin{aligned}&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &-\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},r\right)dr-\frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\Bigg| \\ &\le\frac{(b-a)(d-c)}{2^{2+\frac{2}{q}}(p+1)^{\frac{2}{p}}}\sqrt[q]{\left|\frac{\partial^2 f(a,c)}{\partial t\partial r}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial r}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial r}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial r}\right|^q}.\end{aligned}$$

Our last result is for the s-concave functions can be stated as follows.

Theorem 2.7. *Let* $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$*,* $a<b$*,* $c<d$*, be a continuous mapping such that* $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ *exist on* Δ° *and* $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$*. If* $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^p$*,* $p>1$*, is s-concave on the co-ordinates on* Δ*,* $s_1,s_2\in(0,1]$ *with* $s=\frac{s_1+s_2}{2}$*,* m*,* $n\in\mathbb{N}$*,* $m,n\ge 1$*. Then for* $P(t)$ *and* $Q(r)$ *defined as in Theorem 2.4 and* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt\right.$$
$$-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr$$
$$\left.+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\right|$$
$$\leq\frac{(b-a)^n(d-c)^m}{2^{n+m}n!m!\left[(np+1)(mp+1)\right]^{\frac{1}{p}}}\left[\frac{4^{s_1+1}+4^{s_2+1}}{2}\right]^{\frac{1}{q}}\left|\frac{\partial^{n+m}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t^n\partial r^m}\right|. \tag{2.31}$$

Proof. The inequality (2.31) follows from the Hölder's inequality and the inequality (1.7) with inequalities in reversed direction. □

Corollary 2.8. *If the conditions of Theorem 2.7 are satisfied and if $m=n=1$ and $s_1=s_2=1$, then for $\frac{1}{p}+\frac{1}{q}=1$ we have the inequality*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.-\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},r\right)dr-\frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\right|$$
$$\leq\frac{(b-a)(d-c)}{2^{2-\frac{4}{q}}(p+1)^{\frac{2}{p}}}\left|\frac{\partial^2 f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t\partial r}\right|.$$

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School of Computational and Applied Mathematics,
University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg,
South Africa
E-mail address: m_amer_latif@hotmail.com