Kragujevac Journal of Mathematics
Volume 38(1) (2014), Pages 83–94.

SOME RESULTS FOR ROMAN DOMINATION NUMBER ON CARDINAL PRODUCT OF PATHS AND CYCLES

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Abstract. For a graph $G = (V, E)$, *a Roman dominating function* (RDF) is a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF equals $w(f) = \sum_{v\in V} f(v) = |V_1| + 2|V_2|$ where $V_i = \{v \in V : f(v) = i\}$, $i \in \{1, 2\}$. An RDF for which $w(f)$ achieves its minimum is called *a γ_R-function* and its weight, denoted by $\gamma_R(G)$, is called *the Roman domination number*.
In this paper we determine a lower and the upper bounds for $\gamma_R(P_m \times P_n)$ as well as the exact value of $\lim\limits_{m,n\to\infty} \dfrac{\gamma_R(P_m \times P_n)}{mn}$ where $P_m \times P_n$ stands for the cardinal product of two paths. We also present some results concerning the cardinal product of two cycles $C_m \times C_n$ as well as the exact value of $\lim\limits_{m,n\to\infty} \dfrac{\gamma_R(C_m \times C_n)}{mn}$.

1. Basic definitions and historical background

Let $G = (V, E)$ be a graph of order n. If $G' = (V', E')$ is also a graph such that $V' \subseteq V$ and $E' \subseteq E$, then G' is said to be *a subgraph* of graph G. In case every pair of vertices in V' which are adjacent in G, if and only if, are also adjacent in G', the subgraph G' is called *an induced subgraph.* A subgraph induced by the set of vertices V' is usually denoted by $G' = G[V']$. We can define *the open neighborhood of* $S \subseteq V$ to be the set $N(S) = \bigcup_{v\in S} N(v)$, where $N(v) = \{u \in V : uv \in E\}$ represents *the open neighborhood of vertex* $v \in V$. We also define *the closed neighborhood of* S, denoted by $N[S]$, as a union of sets S and $N(S)$.

A dominating function on G is any function $f\colon V \to \{0, 1\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 1$. Such a function obviously induces the ordered partition (V_0, V_1) of V such

Key words and phrases. Roman dominating function, Roman domination number γ_R, Cardinal product of paths, Cardinal product of cycles.
2010 *Mathematics Subject Classification.* Primary: 05C69. Secondary: 05C38.
Received: March 11, 2014.

that each vertex in V_0 is adjacent to at least one vertex in V_1. Therefore, the set V_1 is called *a dominating set*.

A dominating set $D \subseteq V$ is *perfect* if all vertices not in D are dominated only by one vertex from D.

There is a bijection between the set of all functions $f\colon V \to \{0,1\}$ and the set of all ordered partitions (V_0, V_1). Thus we are allowed to write $f = (V_0, V_1)$. *The weight of* f equals $w(f) = \sum_{v\in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| = |V_1|$. Obviously, the most interesting dominating functions are those of minimum weight.

Since we established a 1-1 correspondence between the set of all functions and the set of all ordered partitions they induce, this optimization problem can be interpreted as finding a dominating set V_1 of minimum cardinality. Such a set is called *a γ-set* of G and its weight, denoted by $\gamma(G)$, is called *the domination number* of G.

Domination on graphs is well studied, but in 1999 an article "Defend the Roman Empire" written by Ian Stewart motivated numerous mathematicians to expand their understanding of this topic. In that article the author suggested a new variant of domination known as *Roman domination* thanks to its historical background.

In the 4th century AD the Roman Empire was under the rule of Constantine the Great. During that time the Empire suffered from numerous barbaric attacks, so Constantine had to arrange Roman legions in a way all strategically important places were protected. Not only did this arrangement have to be successful in defending the Empire, it also had to be easy to maintain.

If at least one Roman legion was stationed at a certain location, that location was considered to be secured. Unsecured locations, on the other hand, had no legions stationed at them, but they had to be adjacent to at least one secured location. If an unsecured location was under attack, sending a legion from its secured neighbor would not be effective if doing so makes that location unsecured. Therefore, Constantine decreed that at least two legions must be stationed at a location before one of them is sent to help its attacked neighbor. In order to reduce costs of maintaining an army, Constantine had to use as few legions as possible, but still secure the whole Empire.

Representing locations of the Empire as graph vertices and roads of the Empire as graph edges, the problem of defending the Roman Empire transforms to the problem of protecting (or dominating) a graph. However, this type of domination is slightly different from the one previously described because it uses another type of dominating function called a Roman dominating function.

For a graph $G = (V, E)$, *a Roman dominating function* (RDF) is a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Since this function also induces the ordered partition of V, $V_i = \{v \in V : f(v) = i\}$, $i \in \{0, 1, 2\}$, we are allowed to write $f = (V_0, V_1, V_2)$. The weight of an RDF equals $w(f) = \sum_{v\in V} f(v) = 0 \cdot |V_0| + 1 \cdot |V_1| + 2 \cdot |V_2| = |V_1| + 2|V_2|$.

The vertex partition (V_0, V_1, V_2) gives us another definition of an RDF. A function f is called an RDF if $V_0 \subseteq N(V_2)$, i.e. if the set V_2 is a dominating set of $G[V_0 \cup V_2]$.

This definition implies that the same graph can be protected under several different RDFs, but the most important RDFs are those for which $w(f)$ achieves its minimum. Such minimum weight is called *the Roman domination number* of G and we denote it by $\gamma_R(G)$. An RDF which satisfies $w(f) = \gamma_R(G)$ is called *a γ_R-function.* It is obvious that $\gamma_R(G) \leq |V_1| + 2|V_2|$ for any RDF $f = (V_0, V_1, V_2)$.

The main result describing a connection between domination and Roman domination number of an arbitrary graph G is $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. Graphs for which $\gamma_R(G) = 2\gamma(G)$ are called *Roman graphs.* It is obvious that $|V_1| = 0$ for every minimum weight RDF of any Roman graph.

Despite the fact that Roman domination has been studied for jast a little over a decade, we already know the exact values of Roman domination numbers for many classes of graphs ([1], [2]). However, the complexity of some graphs allows the author(s) to establish only an upper bound. An example of such a graph is the grid graph, i.e. the Cartesian product of two paths $P_m \Box P_n$. An upper bound for $\gamma_R(P_m \Box P_n)$ is

$$\gamma_R(P_m \Box P_n) \leq 2\left(\left\lceil \frac{mn}{5} \right\rceil + \left\lceil \frac{m}{5} \right\rceil + \left\lceil \frac{n}{5} \right\rceil\right)$$

and it was determined by Cockayne et al (for complete proof see [2]).

Motivated by this result we used the same idea to study the cardinal (tensor, direct, Kronecker) product of two paths $P_m \times P_n$. For arbitrary graphs G and H, the cardinal product of G and H is the graph $G \times H$ which satisfies the following

- its vertex set is $V(G \times H) = V(G) \times V(H)$;
- two vertices $(g, h), (g', h') \in V(G \times H)$ are adjacent if and only if g is adjacent to g' in G and h is adjacent to h' in H.

The cardinal product of two paths $P_m \times P_n$ has two connected components. If the vertices of P_m and P_n are denoted by $\{1, 2, 3, \dots, m\}$ and $\{1, 2, 3, \dots, n\}$, respectively, then the component of $P_m \times P_n$ containing the vertex $(1, 1)$ will be denoted by K_1, and the other component by K_2. If at least one of the parameters m or n is even, components K_1 and K_2 are isomorphic. Otherwise, the component K_1 has one vertex more than the component K_2. Further in the text we will mostly be using K_1 (and K_2) not as a label of a connected component, but as a label of its vertex set. The meaning of K_1 and K_2 will be clear from the context.

2. Upper and lower bound for $\gamma_R(P_m \times P_n)$

Theorem 2.1. *For every two paths P_m and P_n, $m, n \geq 2$*

$$\gamma_R(P_m \times P_n) \leq 4\left\lceil \frac{mn}{10} \right\rceil + 8\left\lceil \frac{m}{10} \right\rceil + 8\left\lceil \frac{n}{10} \right\rceil.$$

Proof. Let us assume that at least one of the parameters m or n is even. This allows us to observe only the component K_1. Since K_1 and K_2 in this case are isomorphic, $|K_1| = \frac{mn}{2}$. Let us construct an RDF $f = (V_0, V_1, V_2)$ on the component K_1. First let us color the vertices of K_1 with three colors - yellow, red and black following the pattern shown in Fig. 1, and then label them by 0,1 and 2, respectively. Doing so, we partition the vertex set of K_1 into V_0, V_1 and V_2, where V_i represents the subset of K_1 labeled by i, $i \in \{0, 1, 2\}$. Fig. 1 also shows vertices of the component K_2, but in order to draw the graph as simple as possible they were intentionally left uncolored and isolated. $\square$

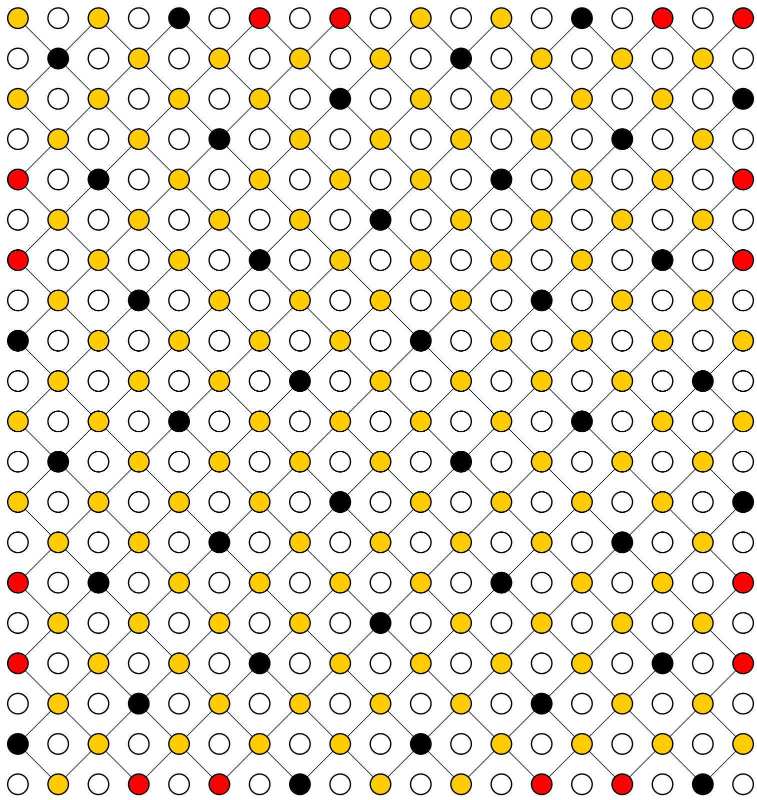

FIGURE 1. A pattern of coloring the vertices of the component K_1 of $P_{20} \times P_{19}$.

Note that all yellow vertices are adjacent to at least one black vertex, i.e. $V_0 \subseteq N[V_2]$. This is enough to conclude that the function f under which the vertex partition (V_0, V_1, V_2) was made is an RDF. Let us calculate the upper bound of it's weight.

Black vertices placed on boundaries dominate themselves and two of their neighbors, while inner black vertices dominate themselves and four of their neighbors. Therefore, the vertex set of the induced subgraph $G[V_0 \cup V_2]$ can be partitioned into disjoint subsets of cardinality 5 (one inner black vertex and four yellow vertices it dominates) or 3 (one boundary black vertex and two yellow vertices it dominates). Since $N[u] \cap N[v] = \emptyset$ for all $u, v \in V_2$, we conclude that the set V_2 is a dominating set of $G[V_0 \cup V_2]$ which implies that $|V_2|$ in the component K_1 achieved its minimum. Furthermore, note that the red vertices of K_1, i.e. vertices not in $G[V_0 \cup V_2]$, appear only on boundaries where every two of them are either preceded or followed by a boundary black vertex. These conclusions enable us to determine the following upper bound for $|V_2|$

$$|V_2| \leq \left\lceil \frac{\frac{mn}{2}}{5} \right\rceil = \left\lceil \frac{mn}{10} \right\rceil.$$

Let R_1 be the set of vertices placed in the first row, R_m the set of vertices placed in the last row, S_1 the set of vertices placed in the first column and S_n the set of vertices placed in the last column of $P_m \times P_n$. Let $(i, j) \in V_2 \cap K_1$ be an inner black vertex such that $4 \leq i \leq m - 3$ and $4 \leq j \leq n - 3$. An example of such vertex is marked in Fig. 1. Now let us observe the subgraph induced by the set of vertices $S = \{(k, l) \in K_1 : i - 3 \leq k \leq i + 3, j - 3 \leq l \leq j + 3\}$, emphasized in Fig. 1 with a square. Let U be the set of all unprotected (undominated) vertices of the S-induced subgraph

$$\begin{aligned} U \;=\; & \{(i-3, j-3), (i-1, j-3), (i+3, j-3), (i+3, j-1), \\ & (i+3, j+3), (i+1, j+3), (i-3, j+3), (i-3, j+1)\}. \end{aligned}$$

Note that all vertices in U appear only on boundaries of the S-induced subgraph and that they are either preceded or followed by a boundary black vertex. In order to protect them, they must be labeled by 1, i.e. colored red.

Exactly the same conclusion holds for boundary vertices of the component K_1. One can easily show that red vertices in sets $(R_1 \cup S_n) \cap K_1$ and $(S_1 \cup R_m) \cap K_1$ appear only in positions $i + 2$ and $i + 4$ (or $i - 6$ and $i - 8$) for $(R_1 \cup S_n) \cap K_1$, and $i - 2$ and $i - 4$ (or $i + 6$ and $i + 8$) for $(S_1 \cup R_m) \cap K_1$, with i in both cases marking the position of a boundary black vertex. Now the set V_1 can be represented as $V_1 = V_1 \cap \big((R_1 \cup S_n \cup S_1 \cup R_m) \cap K_1\big)$ and an upper bound of its cardinality is

$$|V_1| \leq 2\left\lceil \frac{m}{10} \right\rceil + 2\left\lceil \frac{m}{10} \right\rceil + 2\left\lceil \frac{n}{10} \right\rceil + 2\left\lceil \frac{n}{10} \right\rceil = 4\left\lceil \frac{m}{10} \right\rceil + 4\left\lceil \frac{n}{10} \right\rceil.$$

Therefore, $\gamma_R(P_m \times P_n) = 2\gamma_R(K_1) \leq 2\big(|V_1| + 2|V_2|\big) \leq 4\left\lceil \frac{mn}{10} \right\rceil + 8\left\lceil \frac{m}{10} \right\rceil + 8\left\lceil \frac{n}{10} \right\rceil$.

Remark 2.1. In case both parameters m and n are odd we obtain the same final result, although components K_1 and K_2 are not isomorphic. However, red vertices will again appear only on the boundaries R_1, R_m, S_1 and S_n. Furthermore, they appear in exactly the same positions with regard to the positions of black vertices as we described in the proof of Theorem (2.1). The only difference is that the set $R_1 \cap K_2$ always starts with a red vertex as shown in Fig. 2, but the following sequence of colors is the same as in the set $R_1 \cap K_1$, and since we give only upper bound that particular vertex doesn't change our result.

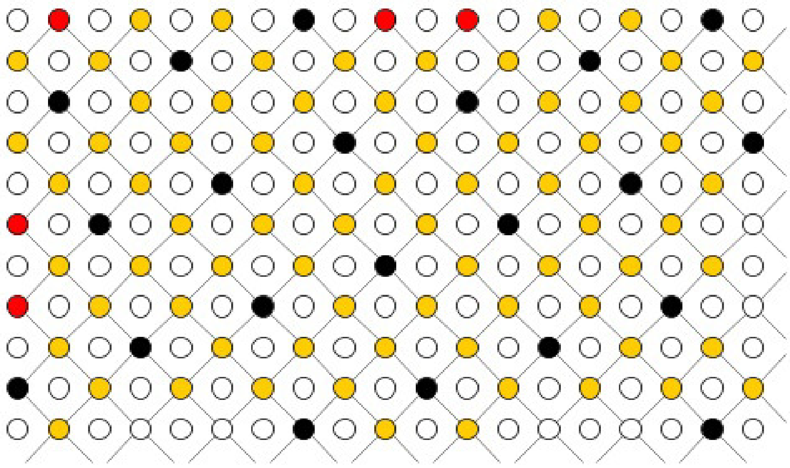

FIGURE 2. A pattern of coloring the vertices of the component K_2 in case both m and n are odd.

Theorem 2.2. *For every two paths P_m and P_n, $m, n \geq 2$ we have that*

$$\gamma_R(P_m \times P_n) > 4\left\lfloor \frac{mn-m-n+1}{10} \right\rfloor + 4\left\lfloor \frac{m-1}{10} \right\rfloor + 4\left\lfloor \frac{n-1}{10} \right\rfloor.$$

Proof. The assumption that at least one of the parameters m or n is even allows us to make all observations only on the component K_1, as we did in the proof of the previous theorem. Yellow, red and black color are again used as substitutes for vertex labels 0, 1 and 2, respectively. Now let us observe the graph $H = G[K_1 \setminus (R_m \cup S_n)]$, a color preserving subgraph induced by all vertices of K_1 with the exception of those vertices placed in the last row and the last column. The vertices of H can be represented as the set $\big(V(P_m \times P_n) \setminus (R_m \cup S_n)\big) \setminus K_2$. Therefore,

$$|V(H)| \geq \frac{|V(P_m \times P_n)| - m - n + 1}{2} = \frac{mn - m - n + 1}{2}.$$

Note that the function $f = (V_0, V_1, V_2)$ which was an RDF on K_1 isn't an RDF on H because the construction of H leaves some of its boundary yellow vertices undominated. To be more precise, a yellow vertex $u \in H$ is undominated if and only if $u \in N[v]$ for some black $v \in (R_m \cup S_n) \cap K_1$. Let us this disregard undominated vertices of H. By the same methods as in the proof of Theorem (2.1) it follows that $\gamma_R(K_1) > 2|V_2 \cap V(H)| + |V_1 \cap V(H)|$. Because the best case is that each "black" vertex r-dominates 5 vertices, and that "red" vertices are on the bound of graph, it follows

$$|V_2 \cap V(H)| \geq \left\lfloor \frac{\frac{mn-m-n+1}{2}}{5} \right\rfloor = \left\lfloor \frac{mn-m-n+1}{10} \right\rfloor$$

and $|V_1 \cap V(H)| \geq 2\left\lfloor \frac{m-1}{10} \right\rfloor + 2\left\lfloor \frac{n-1}{10} \right\rfloor$.

Now the following is obvious

$$\gamma_R(P_m \times P_n) = 2\gamma_R(K_1) > 4\left(\left\lfloor \frac{mn-m-n+1}{10} \right\rfloor + \left\lfloor \frac{m-1}{10} \right\rfloor + \left\lfloor \frac{n-1}{10} \right\rfloor\right).$$

□

3. Determining $\lim_{m,n\to\infty} \frac{\gamma_R(P_m \times P_n)}{mn}$

First let us observe a similar result $\lim_{m,n\to\infty} \frac{\gamma(P_m \Box P_n)}{mn} = \frac{1}{5}$, presented by Klobučar in [4], which we use as a motivation to determine $\lim_{m,n\to\infty} \frac{\gamma_R(P_m \times P_n)}{mn}$.

Theorem 3.1. *For every two paths P_m and P_n, $m,n \geq 2$ we have that* $\lim_{m,n\to\infty} \frac{\gamma_R(P_m \times P_n)}{mn} = \frac{2}{5}$.

Proof. In Theorem (2.1) and Theorem (2.2) we have determined the upper and the lower bound for $\gamma_R(P_m \times P_n)$. Furthermore, we know that the floor and ceiling functions satisfy the following

$$\frac{k}{10} - 1 < \left\lfloor \frac{k}{10} \right\rfloor \leq \frac{k}{10} \leq \left\lceil \frac{k}{10} \right\rceil < \frac{k}{10} + 1, \qquad k \in \mathbb{N}. \tag{3.1}$$

Now Theorem (2.1), Theorem (2.2) and inequality (3.1) imply

$$\frac{2mn}{5} - \frac{62}{5} < \gamma_R(P_m \times P_n) < \frac{2mn}{5} + \frac{4m}{5} + \frac{4n}{5} + 20. \tag{3.2}$$

If we divide (3.2) by mn, we obtain

$$\frac{2}{5} - \frac{62}{5mn} < \frac{\gamma_R(P_m \times P_n)}{mn} < \frac{2}{5} + \frac{4}{5n} + \frac{4}{5m} + \frac{20}{mn}. \tag{3.3}$$

For $m,n \to \infty$, the left and the right hand side of (3.3) tend to $\frac{2}{5}$. Applying *the sandwich rule* gives us the desired result $\lim_{m,n\to\infty} \frac{\gamma_R(P_m \times P_n)}{mn} = \frac{2}{5}$. □

4. Some results about $\gamma_R(C_m \times C_n)$

Back in 1968. D. J. Miller had shown that the cardinal and Cartesian product of two graphs G and H are mutually nonisomorphic, except in case when both G and H are odd cycles of the same size [8]. This obviously implies that for odd m

$$\gamma_R(C_m \Box C_m) = \gamma_R(C_m \times C_m).$$

More general result concerning domination number of Cartesian product of two cycles $C_m \Box C_n$ was given by Klavžar and Seifter in [3]. They had shown that in case when $m,n \equiv 0 \pmod 5$ we have that $\gamma(C_m \Box C_n) = \frac{mn}{5}$, which can easily be verified by observing one example of such graph and concluding the following:

- each dominating vertex dominates all four of its neighbors,
- each dominated vertex is dominated only by one dominating vertex.

Corollary 4.1. *For $m,n \equiv 0 \pmod 5$ it holds that* $\gamma(C_m \times C_n) = \frac{mn}{5}$.

Proof. Each vertex of $C_m \times C_n$, $m, n \equiv 0 \pmod 5$ satisfies the above listed corresponding condition which makes that the dominating set is perfect and therefore minimal. □

Theorem 4.1. *In case when* $m, n \equiv 0 \pmod{10}$, $C_m \times C_n$ *is a Roman graph, i. e.*

$$\gamma_R(C_m \times C_n) = 2\gamma(C_m \times C_n) = \frac{2mn}{5}.$$

Proof. Similarly as in previous corollary, black vertices make perfect dominating set

$$\begin{aligned} D \;=\; & \{(i_1, j_1), i_1 \equiv 1(\mathrm{mod}\,5), j_1 \equiv 0(\mathrm{mod}\,5), (i_2, j_2), i_2, j_2 \equiv 2(\mathrm{mod}\,5), \\ & (i_3, j_3), i_3 \equiv 3(\mathrm{mod}\,5), j_3 \equiv 4(\mathrm{mod}\,5), (i_4, j_4), i_4 \equiv 4(\mathrm{mod}\,5), j_4 \equiv 1(\mathrm{mod}\,5), \\ & (i_5, j_5), i_5 \equiv 0(\mathrm{mod}\,5), j_5 \equiv 3(\mathrm{mod}\,5)\} \end{aligned}$$

and giving them weight 2 instead of 1 we obtain minimal Roman dominating set as one can see in Fig. 3. □

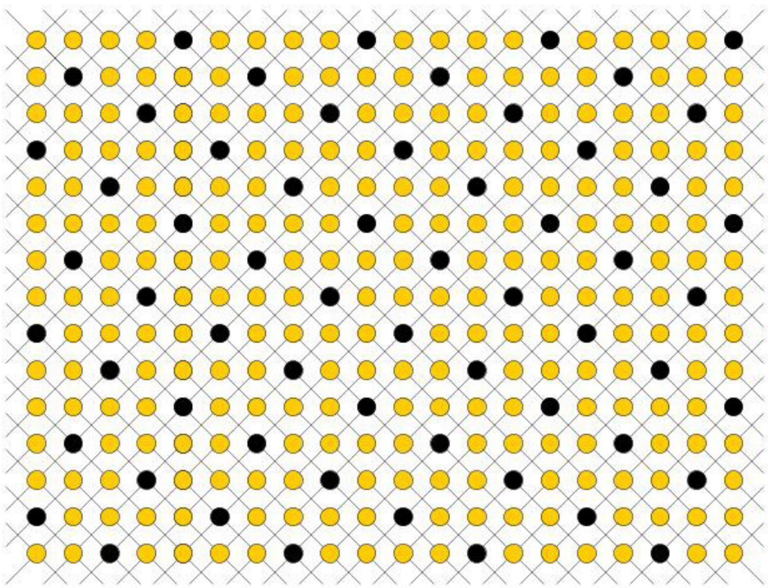

FIGURE 3. $C_{15} \times C_{20}$ drawn on torus. The set V_2 is represented by black and the set V_0 by yellow vertices.

Theorem 4.2. *For* $m \equiv 0 \pmod 6$ *and* $n \equiv 0 \pmod 4$ *we have that* $\gamma_R(C_m \times C_n) \leq \frac{mn}{2}$.

Proof. Vertex set of $C_m \times C_n$ can be partitioned into $\frac{mn}{24}$ disjoint blocks as one can see in Fig. 4. If we denote the vertex set of one of such blocks by $B = \{(i, j) : i \in$

$\{1,2,3,4,5,6\}, j \in \{1,2,3,4\}\}$, the Roman dominating function used for domination of $C_m \times C_n$ induces the following partition of B

$$V_2 = \{(2,k),(3,k),(5,k+1),(6,k+1) : k \in \{1,2,3,4\}\}$$
$$V_1 = \{(2,k+2),(3,k+2),(5,k+3),(6,k+3)\}$$
$$V_0 = B \setminus (V_1 \cup V_2)$$

with all addition performed modulo 4. It is obvious that $\gamma_R(B) = \gamma_R(C_6 \times C_4) = 12$ from which directly follows that $\gamma_R(C_m \times C_n) \leq \frac{mn}{24} \cdot 12 = \frac{mn}{2}$. □

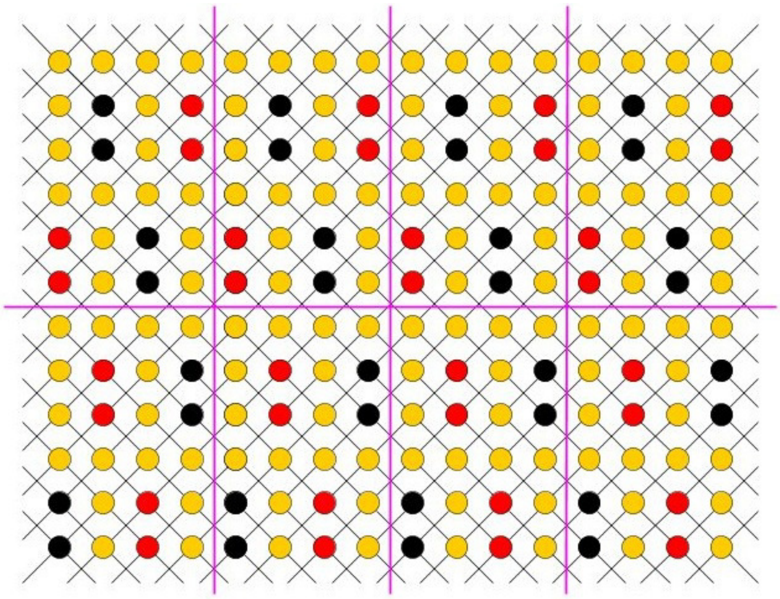

FIGURE 4. $C_{12} \times C_{16}$ drawn on torus and divided on 8 blocks. The set V_2 is represented by black, the set V_1 by red and the set V_0 by yellow vertices.

Theorem 4.3. *For odd $m \geq 11$*

$$\gamma_R(C_m \times C_m) \leq \begin{cases} \frac{2m(m-1)}{5} + m, & m \equiv 1 \pmod{10}; \\ \frac{2m(m-3)}{5} + 3m, & m \equiv 3 \pmod{10}; \\ \frac{2m(m-k)}{5} + 4m, & m \equiv k \pmod{10},\ k \in \{7,9\}. \end{cases}$$

$$\gamma_R(C_m \times C_m) = \frac{2m^2}{5}, m \equiv 5 \pmod{10}$$

Proof. a) $m \equiv 1 (\mathrm{mod}\, 10)$

Let us construct an RDF f on $C_m \times C_m$. The easiest and the most logical way to do that is first to determine the sets V_2 and V_0, and finally the set V_1.

If the vertices of $C_m \times C_m$ are denoted by (i,j), $i,j \in \{1,2,\ldots,m\}$, the first vertex we add into the set V_2 is denoted by $(2,2)$. Once we have chosen this initial vertex, determining its followers is easy. Starting from the vertex $(2,2)$, the rule is to move along $C_m \times C_m$ three rows down and one column to the right and add to V_2 all vertices we come across. Since these vertices are characterized as dominating, at the same time their neighbors are used for construction of the set V_0.

Note that after we have added first m vertices into the set V_2, the regularity of $C_m \times C_m$ will return us to our first initial vertex $(2,2)$. Thus we must repeat the rule explained above, but this time with another initial vertex chosen by one of the following two criteria:

– if an arbitrary column of $C_m \times C_m$ contains odd number of vertices we added into V_2 and if among them we detect some vertex (i,j), but not the vertex $(i+1,j)$, then the latter is chosen to be the next initial vertex;
– if an arbitrary column of $C_m \times C_m$ contains even number of vertices we added into V_2 and if among them we detect both vertices (i,j) and $(i+1,j)$, then the vertex $(i+2,j+4)$ is chosen to be the next initial vertex.

When we have exhausted all possible initial vertices, i.e. when both sets V_2 and V_0 are fully determined, the remaining m vertices of $C_m \times C_m$ (one vertex at each column) are used for construction of the set V_1.
The described construction of the sets V_2 and V_1 enables us to easily determine their cardinality $|V_2| = m \cdot \frac{m-1}{10} \cdot 2$, $|V_1| = m$ from which follows that $\gamma_R(C_m \times C_m) \leq 2|V_2| + |V_1| = \frac{2m(m-1)}{5} + m$. Fig. 5 a) showing vertex partition of $C_{11} \times C_{11}$ illustrates this result. Note that the red vertices of $C_{11} \times C_{11}$, i.e. vertices added into V_1, are additionally emphasized with purple square.

Remark 4.1. Vertex partition of $C_m \times C_m$ in the following three cases when $m \equiv k \pmod{10}$, $k \in \{3,7,9\}$ is obtained by applying the same logic as described in case a). However, exhausting all possible initial vertices in each of the following cases leaves us with exactly k unsorted vertices in each column of $C_m \times C_m$. These mk unsorted vertices (emphasized with purple rectangles) must be colored by following the pattern shown in Fig. 5.

b) $m \equiv 3 (\mathrm{mod}\, 10)$

$$|V_2| = m \cdot \frac{m-3}{10} \cdot 2 + m, \quad |V_1| = m$$

$$\gamma_R(C_m \times C_m) \leq \frac{2m(m-3)}{5} + 3m$$

c) $m \equiv 7 (\mathrm{mod}\, 10)$

$$|V_2| = m \cdot \frac{m-7}{10} \cdot 2 + 2m$$

$$\gamma_R(C_m \times C_m) \leq \frac{2m(m-7)}{5} + 4m$$

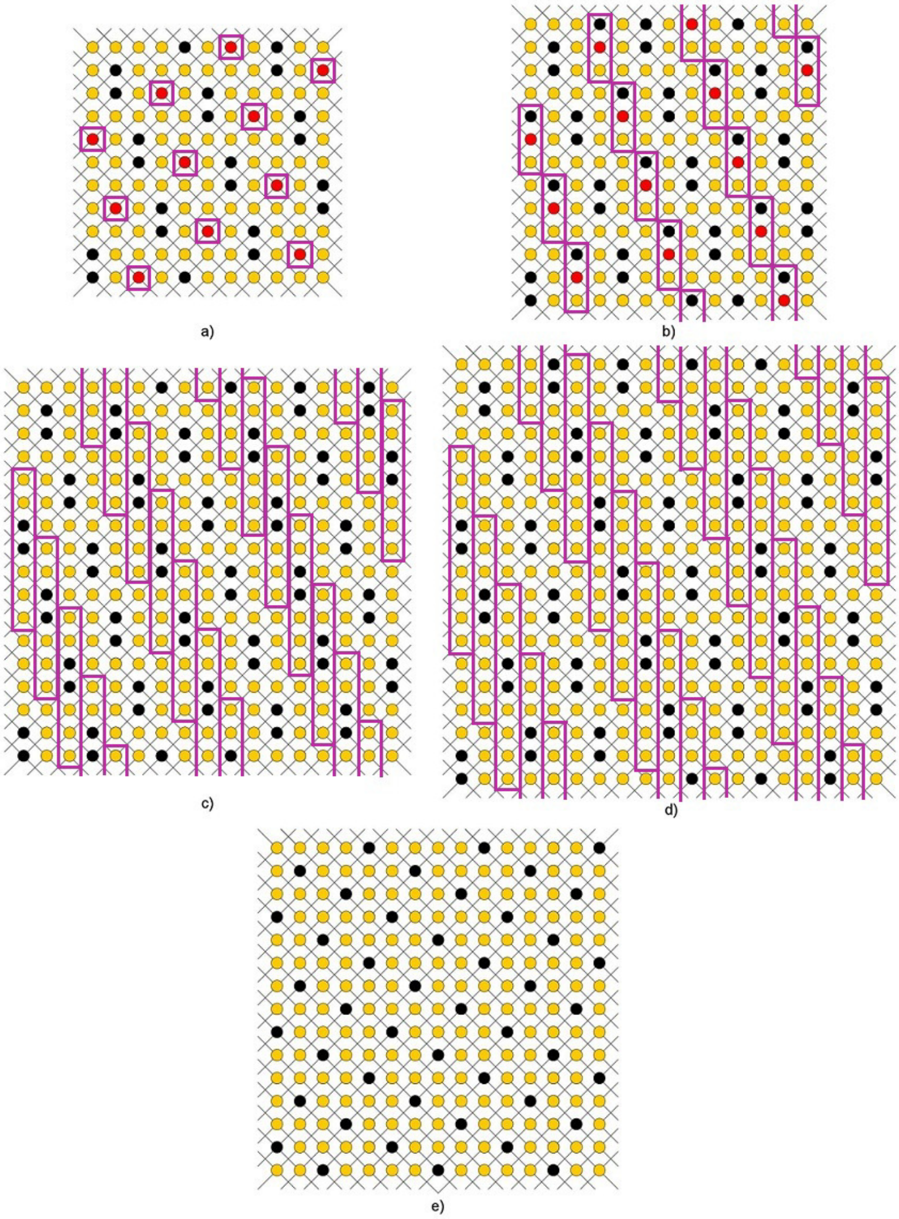

FIGURE 5. a) $C_{11} \times C_{11}$; b) $C_{13} \times C_{13}$; c) $C_{17} \times C_{17}$; d) $C_{19} \times C_{19}$; e) $C_{15} \times C_{15}$ all drawn on torus.

d) $m \equiv 9 (\mathrm{mod}\, 10)$

$$|V_2| = m \cdot \frac{m-9}{10} \cdot 2 + 2m$$

$$\gamma_R(C_m \times C_m) \le \frac{2m(m-9)}{5} + 4m$$

e) $m \equiv 5 (\mathrm{mod}\, 10)$

Follows directly from Theorem 4.1. □

Proposition 4.1. $\lim_{m,n\to\infty} \frac{\gamma_R(C_m\times C_n)}{mn} = \frac{2}{5}$

Proof. Since $\lim_{m,n} \frac{\gamma_R(P_m\times P_n)}{mn} = \frac{2}{5}$, the claim follows from the relations $\frac{2}{5} \leq \frac{\gamma_R(C_m\times C_n)}{mn} \leq \frac{\gamma_R(P_m\times P_n)}{mn} \leq \frac{2}{5}$. The central inequality is true since $P_m \times P_n$ is a spanning subgraph of $C_m \times C_n$. □

Acknowledgment: This work has been financed by Croatian Science Foundation within the Eurocores Programme Eurogiga (project GReGAS) of the European Science Foundation.

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