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DIFFERENCE BETWEEN TWO RIEMANN-STIELTJES INTEGRAL MEANS

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Abstract. In this paper, several bounds for the difference between two Rieman-Stieltjes integral means under various assumptions are proved.

1. Introduction

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows.

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In 2001, Matić and Pečarić [7] have proved the following estimates of the difference of two integral means.

Theorem 1.2. *Given a function $f : [a,b] \to \mathbb{R}$ satisfying the Lipschitz condition with constant $M > 0$, and $a \le c < d \le b$, then the following two-point Ostrowski inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(s)\,ds \right| \le M \cdot \frac{(c-a)^2 + (b-d)^2}{2(c-a+b-d)},$$

holds.

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Another result was proved by Barnett et al. [4], as follows.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} |f'(t)|.$$

Then for $a \le c < d \le b$, we have the inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| \\
&\le \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{(b-a)-(d-c)}\right)^2\right] \cdot [(b-a)-(d-c)] \cdot \|f'\|_\infty \\
&\le \frac{1}{2}[(b-a)-(d-c)] \cdot \|f'\|_\infty .
\end{aligned} \tag{1.1}$$

The constant $1/4$ in the first inequality and $1/2$ in the second inequality are the best possible.

After that, several authors obtained interesting bounds for the difference of two integral means under various assumptions. For other results, the reader may refer to [1]–[10] and the references therein.

The aim of this paper and for the first time several bounds for the difference between two Stieltjes–integral means are obtained. Namely, bounds for

$$\mathcal{D}(f,u;a,b;c,d) := \mathcal{S}(f,u;a,b) - \mathcal{S}(f,u;c,d), \quad a \le c < d \le b \tag{1.2}$$

where

$$\mathcal{S}(f,u;a,b) := \frac{1}{u(b)-u(a)}\int_a^b f(x)\,du(x) \tag{1.3}$$

and

$$\mathcal{S}(f,u;c,d) := \frac{1}{u(d)-u(c)}\int_c^d f(t)\,du(t) \tag{1.4}$$

such that the integrand f is assumed to be of r–H–Hölder type mapping on $[a,b]$ and the integrator u is to be of bounded variation, Lipschitzian and monotonic mappings; respectively on $[a,b]$ are given.

2. The Result

The following result holds.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be of r-H–Hölder type mapping on $[a,b]$, where, $H > 0$ and $r \in (0,1]$ are given, and $u : [a,b] \to \mathbb{R}$ is a mapping of bounded variation* *on* $[a,b]$*. Then we have the inequality*

$$|\mathcal{D}(f,u;a,b;c,d)| \leq \frac{H(b-c)^r}{[u(b)-u(a)]\cdot[u(d)-u(c)]} \cdot \bigvee_c^d(u) \cdot \bigvee_a^b(u),$$

for all $a \leq c < d \leq b$ *such that* $x \in [a,b]$ *and* $t \in [c,d]$*.*

Proof. First we observe that, using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \int_c^d [f(x)-f(t)]\,du(t)\,du(x) \\
&= \int_a^b \left(\int_c^d [f(x)-f(t)]\,du(t)\right) du(x) \\
&= \int_a^b \left(f(x)[u(d)-u(c)] - \int_c^d f(t)\,du(t)\right) du(x) \\
&= [u(d)-u(c)] \int_a^b f(x)\,du(x) - [u(b)-u(a)] \int_c^d f(t)\,du(t)
\end{aligned}$$

for all $a \leq c < d \leq b$ such that $x \in [a,b]$ and $t \in [c,d]$.

It is well-known that for a continuous function $p:[a,b] \to \mathbb{R}$ and a function $\nu:[a,b] \to \mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \leq \sup_{t\in[a,b]} |p(t)| \bigvee_a^b(\nu). \tag{2.1}$$

Therefore, as u is of bounded variation on $[a,b]$ (and then on $[c,d]$), we have

$$\begin{aligned}
|\mathcal{D}(f,u;a,b;c,d)| &= \frac{\left|\int_a^b \int_c^d [f(x)-f(t)]\,du(t)\,du(x)\right|}{[u(b)-u(a)]\cdot[u(d)-u(c)]} \\
&= \frac{\left|\int_a^b \left(\int_c^d [f(x)-f(t)]\,du(t)\right) du(x)\right|}{[u(b)-u(a)]\cdot[u(d)-u(c)]} \\
&\leq \sup_{x\in[a,b]} \left|\int_c^d [f(x)-f(t)]\,du(t)\right| \cdot \frac{\bigvee_a^b(u)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}.
\end{aligned} \tag{2.2}$$

Now, apply (2.1) again on the right hand side of the above inequality, we get

$$\left|\int_c^d [f(x)-f(t)]\,du(t)\right| \leq \sup_{t\in[c,d]} |f(x)-f(t)| \cdot \bigvee_c^d(u).$$

Since f is of r-Holder type on $[c,d]$, then

$$\begin{aligned}
\sup_{t\in[c,d]}|f(x)-f(t)| &\le H\sup_{t\in[c,d]}|x-t|^r \\
&= H\begin{cases}\max\left\{(c-x)^r,(d-x)^r\right\}, & \text{if } \ x<c\\ \max\left\{(x-c)^r,(d-x)^r\right\}, & \text{if } \ c<x<d\\ \max\left\{(x-c)^r,(x-d)^r\right\}, & \text{if } \ d<x\end{cases}\\
&= H\begin{cases}\left[\max\left\{(c-x),(d-x)\right\}\right]^r, & \text{if } \ x<c\\ \left[\max\left\{(x-c),(d-x)\right\}\right]^r, & \text{if } \ c<x<d\\ \left[\max\left\{(x-c),(x-d)\right\}\right]^r, & \text{if } \ d<x\end{cases}\\
&= H\begin{cases}(d-x)^r, & \text{if } \ x<c\\ \left[\frac{d-c}{2}+\left|x-\frac{c+d}{2}\right|\right]^r, & \text{if } \ c<x<d\\ (x-c)^r, & \text{if } \ d<x\end{cases}
\end{aligned}$$

therefore by (2.2), we have

$$\begin{aligned}
&|\mathcal{D}(f,u;a,b;c,d)|\\
&\le \frac{H}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\sup_{a\le x\le b}\left|\int_c^d[f(x)-f(t)]\,du(t)\right|\cdot\bigvee_a^b(u)\\
&\le H\sup_{x\in[a,b]}\left(\begin{cases}(d-x)^r, & \text{if } \ x<c\\ \left[\frac{d-c}{2}+\left|x-\frac{c+d}{2}\right|\right]^r, & \text{if } \ c<x<d\\ (x-c)^r, & \text{if } \ d<x\end{cases}\right)\cdot\frac{\bigvee_c^d(u)\cdot\bigvee_a^b(u)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\\
&\le \frac{H\bigvee_c^d(u)\cdot\bigvee_a^b(u)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\cdot\begin{cases}(d-a)^r, & \text{if } \ x<c\\ (d-c)^r, & \text{if } \ c<x<d\\ (b-c)^r, & \text{if } \ d<x\end{cases}\\
&\le \frac{H\bigvee_c^d(u)\cdot\bigvee_a^b(u)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\max\left\{(d-a)^r,(d-c)^r,(b-c)^r\right\}\\
&\le \frac{H\bigvee_c^d(u)\cdot\bigvee_a^b(u)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\max\left\{(d-a)^r,(b-c)^r\right\}
\end{aligned}$$

$$= \frac{H \bigvee_c^d (u) \cdot \bigvee_a^b (u)}{[u(b) - u(a)] \cdot [u(d) - u(c)]} [\max\{(d-a), (b-c)\}]^r$$
$$= \frac{H \bigvee_c^d (u) \cdot \bigvee_a^b (u)}{[u(b) - u(a)] \cdot [u(d) - u(c)]} \left[\frac{(b-a)+(d-c)}{2} + \frac{1}{2}|(c+d)-(a+b)|\right]^r$$
$$= \frac{H (b-c)^r}{[u(b) - u(a)] \cdot [u(d) - u(c)]} \cdot \bigvee_c^d (u) \cdot \bigvee_a^b (u)$$

since $a \le c < d \le b$, then $a+b > d+c$ and therefore $|(c+d)-(a+b)| = a+b-c-d$ it follows that

$$\frac{(b-a)+(d-c)}{2} + \frac{1}{2}|(c+d)-(a+b)| = b-c,$$

which completes the proof. □

Corollary 2.1. *Let u be as in Theorem 2.1 and $f : [a,b] \to \mathbb{R}$ be a K–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$|\mathcal{D}(f,u;a,b;c,d)| \le \frac{K(b-c)}{[u(b) - u(a)] \cdot [u(d) - u(c)]} \cdot \bigvee_c^d (u) \cdot \bigvee_a^b (u),$$

for all $a \le c < d \le b$ such that $x \in [a,b]$ and $t \in [c,d]$.

Corollary 2.2. *Let f be as in Theorem 2.1. Let $u \in C^{(1)}[a,b]$. Then we have the inequality*

$$|\mathcal{D}(f,u;a,b;c,d)| \le \frac{H(b-c)^r}{[u(b) - u(a)] \cdot [u(d) - u(c)]} \cdot \|u'\|_{1,[a,b]} \cdot \|u'\|_{1,[c,d]}$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\|u'\|_{1,[\alpha,\beta]} := \int_\alpha^\beta |u'(t)|\,dt$.

Corollary 2.3. *Let f be as in Theorem 2.1. Let $u : [a,b] \to \mathbb{R}$ be a L-Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

$$|\mathcal{D}(f,u;a,b;c,d)| \le \frac{HL^2(b-a) \cdot (d-c) \cdot (b-c)^r}{[u(b) - u(a)] \cdot [u(d) - u(c)]}.$$

Corollary 2.4. *Let f be as in Theorem 2.1. Let $u : [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$|\mathcal{D}(f,u;a,b;c,d)| \le \frac{H \cdot |u(b) - u(a)| \cdot |u(d) - u(c)|}{[u(b) - u(a)] \cdot [u(d) - u(c)]} \cdot (b-c)^r.$$

Remark 2.1. Let us assume that $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$, then $u(z) = \int_a^z g(s)\,ds$ is differentiable almost everywhere. Using the properties of the Stieltjes integral, we have

$$\int_a^b f(x)\,du(x) = \int_a^b f(x)\,g(x)\,dx, \qquad \int_c^d f(t)\,du(t) = \int_c^d f(t)\,g(t)\,dt$$

and

$$\bigvee_a^b (u) = \int_a^b |u'(s)|\, ds = \int_a^b |g(s)|\, ds, \qquad \bigvee_c^d (u) = \int_c^d |u'(s)|\, ds = \int_c^d |g(s)|\, ds.$$

Therefore, the weighted version of $\mathcal{D}(f, u; a, b; c, d)$ is

$$\mathcal{WD}(f, u; a, b; c, d) := \int_c^d g(t)\, dt \cdot \int_a^b f(x)\, du(x) - \int_a^b g(x)\, dx \cdot \int_c^d f(t)\, du(t).$$

A general weighted version of the Stieltjes-integral mean, may be deduced as follows

$$\overline{\mathcal{WD}}(f, u; a, b; c, d) := \frac{\int_a^b f(x)\, g(x)\, dx}{\int_a^b g(x)\, dx} - \frac{\int_c^d f(t)\, g(t)\, dt}{\int_c^d g(t)\, dt},$$

for all $a \le c < d \le b$, provided that $g(s) \ge 0$, for almost every $s \in [a, b]$ and $\int_a^b g(x)\, dx \ne 0$ and $\int_c^d g(t)\, dt \ne 0$.

Therefore, we can state the following result.

Corollary 2.5. *Let $f : [a, b] \to \mathbb{R}$ be of r-H–Hölder type mapping on $[a, b]$, where, $H > 0$ and $r \in (0, 1]$ are given, and $g : [a, b] \to \mathbb{R}$ is a continuous mapping on $[a, b]$. Then we have the inequality*

$$|\mathcal{WD}(f, u; a, b; c, d)| \le H(b - c)^r \cdot \int_c^d |g(t)|\, dt \cdot \int_a^b |g(x)|\, dx,$$

for all $a \le c < d \le b$ such that $x \in [a, b]$ and $t \in [c, d]$. Moreover, we have

$$\left|\overline{\mathcal{WD}}(f, u; a, b; c, d)\right| \le H(b - c)^r,$$

We can deduce the following result.

Corollary 2.6. *If $g(s) = 1$, for all $s \in [a, b]$ then we get the following two-point Ostrowski inequality for mapping f defined on $[a, b]$ which is of r-Hölder type*

$$\left|\frac{1}{b - a}\int_a^b f(x)\, dx - \frac{1}{d - c}\int_c^d f(t)\, dt\right| \le H(b - c)^r.$$

Moreover, if $c = a$ and $a \le d \le b$, then we have

$$\left|\frac{1}{b - a}\int_a^b f(x)\, dx - \frac{1}{d - a}\int_a^d f(t)\, dt\right| \le H(b - a)^r.$$

Also, if $b = d$ and $a \le c \le d$, then we have

$$\left|\frac{1}{d - a}\int_a^d f(x)\, dx - \frac{1}{d - c}\int_c^d f(t)\, dt\right| \le H(d - c)^r.$$

Bound for the difference between two Stieltjes integral means for L-Lipschitz integrator is incorporated in the following result.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be of r-H–Hölder type mapping on $[a,b]$, where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ be a L–Lipschitzian mapping of on $[a,b]$. Then we have the inequality*

$$\left|\mathcal{D}\left(f,u;a,b;c,d\right)\right|\leq HL^{2}\left[\frac{\left(d-a\right)^{r+2}-\left(c-a\right)^{r+2}+\left(b-c\right)^{r+2}-\left(b-d\right)^{r+2}}{\left(r+1\right)\left(r+2\right)\left[u\left(b\right)-u\left(a\right)\right]\cdot\left[u\left(d\right)-u\left(c\right)\right]}\right],$$

for all $a\leq c<d\leq b$ such that $x\in[a,b]$ and $t\in[c,d]$.

Proof. It is well-known that for a Riemann integrable function $p:[a,b]\to\mathbb{R}$ and L–Lipschitzian function $\nu:[a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int_a^b p\left(t\right)d\nu\left(t\right)\right|\leq L\int_a^b\left|p\left(t\right)\right|dt.$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned}\left|\mathcal{D}\left(f,u;a,b;c,d\right)\right|&=\frac{\left|\int_a^b\int_c^d\left[f\left(x\right)-f\left(t\right)\right]du\left(t\right)du\left(x\right)\right|}{\left[u\left(b\right)-u\left(a\right)\right]\cdot\left[u\left(d\right)-u\left(c\right)\right]}\\&=\frac{\left|\int_a^b\left(\int_c^d\left[f\left(x\right)-f\left(t\right)\right]du\left(t\right)\right)du\left(x\right)\right|}{\left[u\left(b\right)-u\left(a\right)\right]\cdot\left[u\left(d\right)-u\left(c\right)\right]}\\&\leq L\frac{\int_a^b\left|\int_c^d\left[f\left(x\right)-f\left(t\right)\right]du\left(t\right)\right|dx}{\left[u\left(b\right)-u\left(a\right)\right]\cdot\left[u\left(d\right)-u\left(c\right)\right]}\end{aligned}\tag{2.3}$$

Now, apply (2.3) again on the right hand side of the above inequality, we get

$$\left|\int_c^d\left[f\left(x\right)-f\left(t\right)\right]du\left(t\right)\right|\leq L\int_c^d\left|f\left(x\right)-f\left(t\right)\right|dt$$

and since f is of r-Holder type on $[c,d]$, then

$$\begin{aligned}\left|\int_c^d\left[f\left(x\right)-f\left(t\right)\right]du\left(t\right)\right|&\leq L\int_c^d\left|f\left(x\right)-f\left(t\right)\right|dt\\&\leq L\int_c^d\left|x-t\right|^{r}dt\end{aligned}$$

$$\leq L\begin{cases}\int_c^d (t-x)^r\,dt, & \text{if} \quad x<c\\ \left[\int_c^x (x-t)^r\,dt+\int_x^d (t-x)^r\,dt\right], & \text{if} \quad c<x<d\\ \int_c^d (x-t)^r\,dt, & \text{if} \quad d<x\end{cases}$$

$$\leq L\begin{cases}\frac{(d-x)^{r+1}-(c-x)^{r+1}}{1+r}, & \text{if} \quad x<c\\ \frac{(x-c)^{r+1}+(d-x)^{r+1}}{1+r}, & \text{if} \quad c<x<d\\ \frac{(x-c)^{r+1}-(x-d)^{r+1}}{1+r}, & \text{if} \quad d<x\end{cases}$$

which gives by (2.3) that,

$$\begin{aligned}
&|\mathcal{D}(f,u;a,b;c,d)|\\
&\leq L\frac{\int_a^b\left|\int_c^d [f(x)-f(t)]\,du(t)\right|dx}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\\
&\leq \frac{L^2}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\int_a^b\left(\begin{cases}\frac{(d-x)^{r+1}-(c-x)^{r+1}}{1+r}, & \text{if} \quad x<c\\ \frac{(x-c)^{r+1}+(d-x)^{r+1}}{1+r}, & \text{if} \quad c<x<d\\ \frac{(x-c)^{r+1}-(x-d)^{r+1}}{1+r}, & \text{if} \quad ,d<x\end{cases}\right)dx\\
&= \frac{L^2}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\left(\int_a^c\left[\frac{(d-x)^{r+1}-(c-x)^{r+1}}{1+r}\right]dx\right.\\
&\quad\left.+\int_c^d\left[\frac{(x-c)^{r+1}+(d-x)^{r+1}}{1+r}\right]dx+\int_d^b\left[\frac{(x-c)^{r+1}-(x-d)^{r+1}}{1+r}\right]dx\right)\\
&= \frac{L^2}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\left[\frac{(d-a)^{r+2}-(d-c)^{r+2}-(c-a)^{r+2}}{(r+1)(r+2)}\right.\\
&\quad\left.+\frac{2(d-c)^{r+2}}{(r+1)(r+2)}+\frac{(b-c)^{r+2}-(d-c)^{r+2}-(b-d)^{r+2}}{(r+1)(r+2)}\right]\\
&= L^2\left[\frac{(d-a)^{r+2}-(c-a)^{r+2}+(b-c)^{r+2}-(b-d)^{r+2}}{(r+1)(r+2)[u(b)-u(a)]\cdot[u(d)-u(c)]}\right]
\end{aligned}$$

which is required. □

Remark 2.2. Let us assume that $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$, then $u(z)=\int_a^z g(s)\,ds$ is differentiable almost everywhere. Therefore, g is L-Lipschitzian with the constant $L = \|g\|_\infty$. Using the properties of the Stieltjes integral, we have

$$\int_a^b f(x)\,du(x) = \int_a^b f(x)\,g(x)\,dx, \qquad \int_c^d f(t)\,du(t) = \int_c^d f(t)\,g(t)\,dt$$

Therefore, we can state the following result.

Corollary 2.7. *Let $f : [a,b] \to \mathbb{R}$ be of r-H–Hölder type mapping on $[a,b]$, where, $H > 0$ and $r \in (0,1]$ are given, and $g : [a,b] \to \mathbb{R}$ is a continuous mapping on $[a,b]$. Then we have the inequality*

$$|\mathcal{WD}(f,u;a,b;c,d)| \le H\,\|g\|_\infty^2 \left[\frac{(d-a)^{r+2} - (c-a)^{r+2} + (b-c)^{r+2} - (b-d)^{r+2}}{(r+1)(r+2)}\right]$$

for all $a \le c < d \le b$ such that $x \in [a,b]$ and $t \in [c,d]$. Moreover, we have

$$\left|\overline{\mathcal{WD}}(f,u;a,b;c,d)\right| \le H\,\|g\|_\infty^2 \left[\frac{(d-a)^{r+2} - (c-a)^{r+2} + (b-c)^{r+2} - (b-d)^{r+2}}{(r+1)(r+2)\cdot\left(\int_a^b g(x)\,dx\right)\cdot\left(\int_c^d g(t)\,dt\right)}\right],$$

We may deduce the following result.

Corollary 2.8. *If $g(s) = 1$, for all $s \in [a,b]$ then we get the following two-point Ostrowski inequality for mapping f defined on $[a,b]$ which is of r-Hölder type*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{d-c}\int_c^d f(t)\,dt\right| \le H\,\|g\|_\infty^2 \left[\frac{(d-a)^{r+2} - (c-a)^{r+2} + (b-c)^{r+2} - (b-d)^{r+2}}{(r+1)(r+2)\cdot(b-a)\cdot(d-c)}\right].$$

Moreover, if $c = a$ and $a \le d \le b$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{d-a}\int_a^d f(t)\,dt\right| \le H\,\|g\|_\infty^2 \left[\frac{(d-a)^{r+2} + (b-a)^{r+2} - (b-d)^{r+2}}{(r+1)(r+2)\cdot(b-a)\cdot(d-a)}\right].$$

Also, if $b = d$ and $a \le c \le d$, then we have

$$\left|\frac{1}{d-a}\int_a^d f(x)\,dx - \frac{1}{d-c}\int_c^d f(t)\,dt\right| \le H\,\|g\|_\infty^2 \left[\frac{(d-a)^{r+2} - (c-a)^{r+2} + (d-c)^{r+2}}{(r+1)(r+2)\cdot(d-a)\cdot(d-c)}\right].$$

Finally, we point out the difference between two Stieltjes integrals mean for monotonic integrator.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be of r-H–Hölder type mapping on $[a,b]$, where, $H > 0$ and $r \in (0,1]$ are given, and $u : [a,b] \to \mathbb{R}$ be a monotonic non-decreasing mapping of on $[a,b]$. Then we have the inequality*

$$\begin{aligned} &|\mathcal{D}(f,u;a,b;c,d)| \\ &\quad \le H\left[(d-a)^r \cdot \frac{u(c)-u(a)}{u(b)-u(a)} + (d-c)^r \cdot \frac{u(d)-u(c)}{u(b)-u(a)} + (b-c)^r \cdot \frac{u(b)-u(d)}{u(b)-u(a)}\right], \end{aligned}$$

for all $a \le c < d \le b$ such that $x \in [a,b]$ and $t \in [c,d]$.

Proof. It is well-known that for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\, d\nu(t)\right| \le \int_a^b |p(t)|\, d\nu(t). \tag{2.4}$$

Therefore, as u is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned} |\mathcal{D}(f,u;a,b;c,d)| &= \frac{\left|\int_a^b \left(\int_c^d [f(x)-f(t)]\, du(t)\right) du(x)\right|}{[u(b)-u(a)] \cdot [u(d)-u(c)]} \\ &\le \frac{\int_a^b \left|\int_c^d [f(x)-f(t)]\, du(t)\right| du(x)}{[u(b)-u(a)] \cdot [u(d)-u(c)]} \end{aligned} \tag{2.5}$$

Now, apply (2.4) again on the right hand side of the above inequality, we get

$$\left|\int_c^d [f(x)-f(t)]\, du(t)\right| \le \int_c^d |f(x)-f(t)|\, du(t)$$

and since f is of r-Holder type on $[c,d]$, then

$$\begin{aligned} \left|\int_c^d [f(x)-f(t)]\, du(t)\right| &\le \int_c^d |f(x)-f(t)|\, du(t) \\ &\le \int_c^d |x-t|^r\, du(t) \\ &= \begin{cases} \int_c^d (t-x)^r\, du(t), & \text{if } x < c \\ \left[\int_c^x (x-t)^r\, du(t) + \int_x^d (t-x)^r\, du(t)\right], & \text{if } c < x < d \\ \int_c^d (x-t)^r\, du(t), & \text{if } d < x \end{cases} \end{aligned}$$

$$= \begin{cases} (d-x)^r u(d) - (c-x)^r u(c) - r\int_c^d (t-x)^{r-1} u(t)\,dt, & \text{if} \quad x < c \\ \\ (d-x)^r u(d) - (x-c)^r u(c) \\ +r\left[\int_c^x (x-t)^{r-1} u(t)\,dt - \int_x^d (t-x)^{r-1} u(t)\,dt\right], & \text{if} \quad c < x < d \\ \\ (x-d)^r u(d) - (x-c)^r u(c) + r\int_c^d (x-t)^{r-1} u(t)\,dt, & \text{if} \quad d < x \end{cases}$$

Since u is monotonic nondecreasing on $[a,b]$ and then on $[c,d]$, hence

$$\int_c^d (t-x)^{r-1} u(t)\,dt \geq u(c)\int_c^d (t-x)^{r-1}\,dt = \frac{1}{r}u(c)\left[(d-x)^r - (c-x)^r\right],$$
$$\int_c^x (x-t)^{r-1} u(t)\,dt \leq u(x)\int_c^x (x-t)^{r-1}\,dt = \frac{1}{r}(x-c)^r u(x),$$
$$\int_x^d (t-x)^{r-1} u(t)\,dt \geq u(x)\int_x^d (t-x)^{r-1}\,dt = \frac{1}{r}(d-x)^r u(x),$$

and

$$\int_c^d (x-t)^{r-1} u(t)\,dt \leq u(d)\int_c^d (x-t)^{r-1}\,dt = \frac{1}{r}u(d)\left[(x-c)^r - (x-d)^r\right]$$

from which it follows that

$$\begin{cases} (d-x)^r u(d) - (c-x)^r u(c) - r\int_c^d (t-x)^{r-1} u(t)\,dt, & \text{if} \quad x < c \\ \\ (d-x)^r u(d) - (x-c)^r u(c) \\ +r\left[\int_c^x (x-t)^{r-1} u(t)\,dt - \int_x^d (t-x)^{r-1} u(t)\,dt\right], & \text{if} \quad c < x < d \\ \\ (x-d)^r u(d) - (x-c)^r u(c) + r\int_c^d (x-t)^{r-1} u(t)\,dt, & \text{if} \quad d < x \end{cases}$$
$$\leq \begin{cases} (d-x)^r u(d) - (c-x)^r u(c) - u(c)\left[(d-x)^r - (c-x)^r\right], & \text{if} \quad x < c \\ \\ (d-x)^r u(d) - (x-c)^r u(c) + (x-c)^r u(x) - (d-x)^r u(x), & \text{if} \quad c < x < d \\ \\ (x-d)^r u(d) - (x-c)^r u(c) + u(d)\left[(x-c)^r - (x-d)^r\right], & \text{if} \quad d < x \end{cases}$$
$$= \begin{cases} (d-x)^r\left[u(d) - u(c)\right], & \text{if} \quad x < c \\ \\ (d-x)^r\left[u(d) - u(x)\right] + (x-c)^r\left[u(x) - u(c)\right], & \text{if} \quad c < x < d \\ \\ (x-c)^r\left[u(d) - u(c)\right], & \text{if} \quad d < x \end{cases}$$

which, by (2.5), gives that,

$$
\begin{aligned}
&\frac{\int_a^b \left|\int_c^d [f(x)-f(t)]\,du(t)\right| du(x)}{[u(b)-u(a)]\cdot[u(d)-u(c)]} \\
&\le H\int_a^b \left(\begin{cases} (d-x)^r \frac{[u(d)-u(c)]}{[u(b)-u(a)]\cdot[u(d)-u(c)]}, & \text{if } x<c \\ \frac{(d-x)^r[u(d)-u(x)]+(x-c)^r[u(x)-u(c)]}{[u(b)-u(a)]\cdot[u(d)-u(c)]}, & \text{if } c<x<d \\ \frac{(x-c)^r[u(d)-u(c)]}{[u(b)-u(a)]\cdot[u(d)-u(c)]}, & \text{if } d<x \end{cases}\right) du(x) \\
&= H\left\{\frac{\int_a^c (d-x)^r\,du(x)}{[u(b)-u(a)]} + \frac{\int_c^d (d-x)^r[u(d)-u(x)]\,du(x)}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\right. \\
&\quad\left. + \frac{\int_c^d (x-c)^r[u(x)-u(c)]\,du(x)}{[u(b)-u(a)]\cdot[u(d)-u(c)]} + \frac{\int_d^b (x-c)^r[u(d)-u(c)]\,du(x)}{[u(b)-u(a)]}\right\}
\end{aligned}
$$

(2.6)

$$
\begin{aligned}
&= H\left\{\frac{1}{[u(b)-u(a)]}\cdot\left[(d-c)^r u(c)-(d-a)^r u(a)+r\int_a^c (d-x)^{r-1}u(x)\,dx\right]\right. \\
&\quad - \frac{(d-c)^r u(c)}{[u(b)-u(a)]} + \frac{(d-c)^r u(d)}{[u(b)-u(a)]} \\
&\quad + \frac{r}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\int_c^d (d-x)^{r-1}[u(d)-u(x)]\,u(x)\,dx \\
&\quad - \frac{r}{[u(b)-u(a)]\cdot[u(d)-u(c)]}\int_c^d (x-c)^{r-1}[u(x)-u(c)]\,u(x)\,dx \\
&\quad \left. + \frac{1}{[u(b)-u(a)]}\cdot\left[(b-c)^r u(b)-(d-c)^r u(d)-r\int_d^b (x-c)^{r-1}u(x)\,dx\right]\right\}.
\end{aligned}
$$

Since u is monotonic nondecreasing on $[a,b]$, hence

$$\int_a^c (d-x)^{r-1}u(t)\,dt \le u(c)\int_a^c (d-x)^{r-1}\,dt = \frac{1}{r}u(c)\left[(d-a)^r-(d-c)^r\right]. \tag{2.7}$$

Now, for the term

$$
\begin{aligned}
&\int_c^d (d-x)^{r-1}[u(d)-u(x)]\,u(x)\,dx \\
&\qquad = \int_c^d (d-x)^{r-1}u(d)\,u(x)\,dx - \int_c^d (d-x)^{r-1}u^2(x)\,dx
\end{aligned} \tag{2.8}
$$

and

$$\int_c^d (d-x)^{r-1}u(d)\,u(x)\,dx \le u^2(d)\int_c^d (d-x)^{r-1}\,dx = \frac{1}{r}u^2(d)(d-c)^r,$$

since $u(c) \leq u(d)$, then we have

$$\int_c^d (d-x)^{r-1} u^2(x)\, dx \geq u^2(d) \int_c^d (d-x)^{r-1}\, dx \geq \frac{1}{r} u^2(c)(d-c)^r, \tag{2.9}$$

which implies by (2.8)–(2.9), that

$$\int_c^d (d-x)^{r-1} [u(d) - u(x)]\, u(x)\, dx \leq \frac{1}{r}(d-c)^r \left[u^2(d) - u^2(c)\right]. \tag{2.10}$$

Similarly, for the term

$$\begin{aligned} &\int_c^d (x-c)^{r-1} [u(c) - u(x)]\, u(x)\, dx \\ &\quad = \int_c^d (x-c)^{r-1} u(c)\, u(x)\, dx - \int_c^d (x-c)^{r-1} u^2(x)\, dx \end{aligned} \tag{2.11}$$

since $u(c) \leq u(d)$, then we have

$$\begin{aligned} \int_c^d (x-c)^{r-1} u(c)\, u(x)\, dx &\leq u(c)\, u(d) \int_c^d (x-c)^{r-1}\, dx \\ &\leq \frac{1}{r} u^2(d)(d-c) \end{aligned} \tag{2.12}$$

and

$$\int_c^d (x-c)^{r-1} u^2(x)\, dx \geq \frac{1}{r} u^2(c)(d-c)^r \tag{2.13}$$

which implies by (2.11)–(2.13), that

$$\int_c^d (x-c)^{r-1} [u(x) - u(c)]\, u(x)\, dx \geq \frac{1}{r}(d-c)^r \left[u^2(d) - u^2(c)\right] \tag{2.14}$$

and finally, we have

$$\int_d^b (x-c)^{r-1} u(x)\, dx \geq \frac{1}{r}\left[(b-c)^r - (d-c)^r\right] u(d) \tag{2.15}$$

and if we substitute (2.7), (2.10), (2.14) and (2.15) in (2.6), we get

$$\int_a^b \left| \int_c^d [f(x) - f(t)]\, du(t) \right| du(x)$$
$$\leq H \left\{ \frac{1}{[u(b)-u(a)]} \cdot \left[(d-c)^r u(c) - (d-a)^r u(a) + r \int_a^c (d-x)^{r-1} u(x)\, dx \right] \right.$$
$$- \frac{(d-c)^r u(c)}{[u(b)-u(a)]} + \frac{(d-c)^r u(d)}{[u(b)-u(a)]}$$
$$+ \frac{r}{[u(b)-u(a)] \cdot [u(d)-u(c)]} \int_c^d (d-x)^{r-1} [u(d) - u(x)]\, u(x)\, dx$$
$$- \frac{r}{[u(b)-u(a)] \cdot [u(d)-u(c)]} \int_c^d (x-c)^{r-1} [u(x) - u(c)]\, u(x)\, dx$$
$$\left. + \frac{1}{[u(b)-u(a)]} \cdot \left[(b-c)^r u(b) - (d-c)^r u(d) - r \int_d^b (x-c)^{r-1} u(x)\, dx \right] \right\}.$$
$$\leq H \left\{ \frac{1}{[u(b)-u(a)]} \cdot [(d-c)^r u(c) - (d-a)^r u(a) + u(c) [(d-a)^r - (d-c)^r]] \right.$$
$$- \frac{(d-c)^r u(c)}{[u(b)-u(a)]} + \frac{(d-c)^r u(d)}{[u(b)-u(a)]}$$
$$\left. + \frac{1}{[u(b)-u(a)]} \cdot [(b-c)^r u(b) - (d-c)^r u(d) - [(b-c)^r - (d-c)^r] u(d)] \right\}$$
$$= H \left[(d-a)^r \cdot \frac{u(c)-u(a)}{u(b)-u(a)} + (d-c)^r \cdot \frac{u(d)-u(c)}{u(b)-u(a)} + (b-c)^r \cdot \frac{u(b)-u(d)}{u(b)-u(a)} \right].$$

which proves this theorem. □

Corollary 2.9. *If in Theorem 2.3, we choose $u(s) = s$, $s \in [a,b]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{1}{d-c} \int_c^d f(t)\, dt \right|$$
$$\leq \frac{H}{(b-a)} \left[(d-a)^r (c-a) + (d-c)^{r+1} + (b-c)^r (b-d) \right].$$

Moreover, if $c = a$ and $a \leq d \leq b$, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{1}{d-a} \int_a^d f(t)\, dt \right| \leq \frac{H}{(b-a)} \left[(d-a)^{r+1} + (b-a)^r (b-d) \right].$$

Also, if $b = d$ and $a \leq c \leq d$, then we have

$$\left| \frac{1}{d-a} \int_a^d f(x)\, dx - \frac{1}{d-c} \int_c^d f(t)\, dt \right| \leq \frac{H}{(d-a)} \left[(d-a)^r (c-a) + (d-c)^{r+1} \right].$$

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