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THE χ^{2I} CONVERGENT SEQUENCE SPACES DEFINED BY A MODULI

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Abstract. In this paper we introduce the sequence spaces χ_F^{2I} and Λ_F^{2I} for the sequence of moduli $F = (f_{mn})$ and study some of the general properties of these spaces.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Başarır and Solankan [2], Tripathy [17], Türkmenoğlu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1, \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

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$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t),$$
$$\mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively.

Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [21, 22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Başar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Başar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35–39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$ was studied extensively by Robison and Hamilton. We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! \, |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ be the set of all finitely non-zero sequences.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \, (1 \leq p < \infty)$ subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektaş and Altın [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for small u or at 0 if for each $k \in \mathbb{N}$, there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an modulus function M is said to satisfy the Δ_2- condition if and only if

$$\lim_{u \to 0+} \sup \frac{M(2u)}{M(u)} < \infty$$

Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b].$$

An Orlicz function M can always be represented in the following integral form

$$M(u) = \int_0^u \eta(t) \, dt,$$

where η, the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$ for $t > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel η associated with the Orlicz function M and let

$$\mu(s) = \sup\{t : \eta(t) \leq s\}.$$

Then μ possesses the same properties as the function η. Suppose now

$$\Phi = \int_0^x \mu(s)\, ds.$$

Then, Φ is an Orlicz function. The functions M and Φ are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let M and Φ are mutually complementary Orlicz functions. Then, we have

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{(Young's inequality)} \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some} \quad \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions

(i) X'; the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{for each x} \in \text{X}\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each x} \in \text{X}\right\}$;

(iv) $X^\gamma = \left\{a = (a_{mn}) : \sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{for each x} \in \text{X}\right\}$;

(v) Let X be an FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^\delta = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{for each x} \in \text{X}\right\}$;

where $X^\alpha, X^\beta, X^\gamma$ are called $\alpha-(or\,K\ddot{o}the-Toeplitz)$dual of $X, \beta-(or\,generalized-K\ddot{o}the-Toeplitz)\,dual\,of\,X, \gamma-\,dual\,of\,X, \delta-\,dual\,of\,X$ respectively. X^α is defined by Gupta and Kamptan [20]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kızmaz.

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and scalar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay in [42] and in the case $0 < p < 1$ by Altay and Başar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many other [52, 53]. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

The notion of the statistical convergence was introduced by H. Fast. Later on it was studied by J.A.Fridy from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, $\check{S}$alát and Wilezyński. Later on it was studied by $\check{S}$alát, Tripathy and Ziman and Demirci, Das, Kostyrko, Wilczynski, and Malik, Mursaleen and Alotaibi, Mursaleen, Mohiuddine and Edely, Mursaleen and Mohiuddine, Sahiner, Gurdal, Saltan and Gunawan and Kumar, V.A.Khan, Suthep Suantai and Khalid Ebadullah. Here we give some preliminaries about the notion of $I-$ convergence.

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \bigcup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\mathcal{L}(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \mathcal{L}(I)$, for $A, B \in \mathcal{L}(I)$ we have $A \bigcap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I), A \subseteq B \Rightarrow B \in \mathcal{L}(I)$.

An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\mathcal{L}(I)$ corr esponding to I i.e.

$$\mathcal{L}(I) = \{K \subseteq N : K^c \in I\},$$

where $K^c = N - K$.

The idea of modulus was structured in 1953 by Nakano (see [12]). A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if:

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(iii) f is increasing, and
(iv) f is continuous from the right at zero.

Ruckle, used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch. The approach of constructing a new sequence space by means of the modulus function has recently been employed by Candan [54] and some others. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H.Galing, G.Köthe and W.H.Ruckle.

After then E.Kolk gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

2. Definitions and preliminaries

Definition 2.1. Let λ be a sequence space. Then λ is called

(i) Solid (or normal) if $(\alpha_{mn} x_{mn}) \in \lambda$ whenever $(x_{mn}) \in \lambda$ for all sequences (α_{mn}) of scalars with $|\alpha_{mn}| \leq 1$.
(ii) Monotone if provided λ contains the canonical preimages of all its step spaces.
(iii) Perfect if $\lambda = \lambda^{\alpha\alpha}$ [20].

Definition 2.2. A sequence space E is said to be convergence free if $(y_{mn}) \in E$ whenever $(x_{mn}) \in E$ and $x_{mn} = 0$ implies $(y_{mn}) = 0$.

Definition 2.3. A sequence space E is said to be a sequence algebra if $(x_{mn} y_{mn}) \in E$ whenever $(x_{mn}) \in E, (y_{mn}) \in E$.

Definition 2.4. A sequence space E is said to be symmetric if $\left(x_{\pi(mn)}\right) \in E$ whenever $(x_{mn}) \in E$ where $\pi(mn)$ is a permutation on $\mathbb{N} \times \mathbb{N}$.

Definition 2.5. A sequence $(x_{mn}) \in w^2$ is said to be χ^{2I} then we write

$$I - \lim_{m,n \to \infty} ((m+n)! \, |x_{mn}|)^{1/m+n} = 0.$$

Definition 2.6. A map h defined on a domain $D \subset X$ i.e $h : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|h(x) - h(y)| \leq K |x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $h \in (D, K)$.

Definition 2.7. A convergence filed of $I-$ convergence is a set

$$F(I) = \{x = (x_{mn}) \in \Lambda^2 : \text{there exists } \mathrm{I} - \lim \mathrm{x} \in \mathbb{R}\}.$$

The convergence filed $F(I)$ is a closed linear subspace of Λ^2 with respect to the supremum metric $F(I) = \Lambda^2 \bigcap c^{2I}$.

Define a function $h : F(I) \to \mathbb{R}$ such that $h(x) = I - \lim x$, for all $x \in F(I)$, then the function $h : F(I) \to \mathbb{R}$ is a Lipschitz function.

Throughout the article Λ^2, χ^{2I} and η_F^{2I} represent the Prinsheims double analytic, Prinsheims double $I-$ gai and Prinsheims double analytic $I-$ gai sequence spaces respectively.

In this article we introduce the following classes of double sequence of spaces:

$$\chi_F^{2I} = \left\{ x = (x_{mn}) \in w^2 : I - \lim f_{mn} \left((m+n)! \left| x_{mn} \right| \right)^{1/m+n} = 0 \right\} \in I,$$

$$\Lambda_F^{2I} = \left\{ x = (x_{mn}) \in w^2 : \sup_{mn} f_{mn} \left(\left| x_{mn} \right|^{1/m+n} \right) < \infty \right\} \in I.$$

We also denote $\eta_F^{2I} = \chi_F^{2I} \bigcap \Lambda_F^2$.

3. Main results

Theorem 3.1. *The classes of sequences χ_F^{2I} and η_F^{2I} of moduli $F = (f_{mn})$, are linear spaces.*

Proof. We shall prove the result for the space χ_F^{2I}. The proof for the other spaces will follow similarly. Let $(x_{mn}), (y_{mn}) \in \chi_F^{2I}$ and let α, β be the scalars. Then

$$I - \lim_{m,n \to \infty} f_{mn} \left((m+n)! \left| x_{mn} \right| \right)^{1/m+n} = 0,$$

$$I - \lim_{m,n \to \infty} f_{mn} \left((m+n)! \left| y_{mn} \right| \right)^{1/m+n} = 0.$$

That is, for a given $\epsilon > 0$, we have

$$N_1 = \left\{ m, n \in \mathbb{N} : f_{mn} \left((m+n)! \left| x_{mn} \right| \right)^{1/m+n} > \frac{\epsilon}{2} \right\} \in I, \tag{3.1}$$

$$N_2 = \left\{ m, n \in \mathbb{N} : f_{mn} \left((m+n)! \left| y_{mn} \right| \right)^{1/m+n} > \frac{\epsilon}{2} \right\} \in I, \tag{3.2}$$

Since f_{mn} is a modulus function, we have

$$\begin{aligned} & f_{mn} \left((m+n)! \left| \alpha x_{mn} + \beta y_{mn} \right| \right)^{1/m+n} \leq \\ & \quad \leq f_{mn} \left(|\alpha| \left((m+n)! \left| x_{mn} \right| \right)^{1/m+n} \right) + f_{mn} \left(|\beta| \left((m+n)! \left| y_{mn} \right| \right)^{1/m+n} \right) \\ & \quad \leq f_{mn} \left(\left((m+n)! \left| x_{mn} \right| \right)^{1/m+n} \right) + f_{mn} \left(\left((m+n)! \left| y_{mn} \right| \right)^{1/m+n} \right). \end{aligned}$$

Now, by (3.1) and (3.2), $\left\{ m, n \in \mathbb{N} : f_{mn} \left((m+n)! \left| \alpha x_{mn} + \beta y_{mn} \right| \right)^{1/m+n} > \epsilon \right\} \subset N_1 \bigcup N_2$. Therefore $(\alpha x_{mn} + \beta y_{mn}) \in \chi_F^{2I}$. This completes the proof. □

Proposition 3.1. *The spaces χ_F^{2I} and η_F^{2I} are solid and monotone.*

Proof. We shall prove the result for χ_F^{2I}. Let $x_{mn} \in \chi_F^{2I}$. Then

$$I - \lim_{mn\to\infty} f_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n} = 0 \tag{3.3}$$

Let (α_{mn}) be a sequence of scalars with $|\alpha_{mn}|^{1/m+n} \leq 1$ for all $m,n \in \mathbb{N}$. Therefore the equation from (3.3) and the following inequality $f_{mn}\left((m+n)!\,|\alpha_{mn}x_{mn}|\right)^{1/m+n} \leq |\alpha_{mn}|^{1/m+n} f_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n} \leq f_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n}$ for all $m,n \in \mathbb{N}$. Since χ_F^{2I} be sequence space. If χ_F^{2I} is solid then χ_F^{2I} is monotone. Hence the space χ_F^{2I} is monotone. Similarly the result is true for η_F^{2I}. □

Proposition 3.2. *The space χ_F^{2I} is sequence algebra.*

Proof. Let $(x_{mn}),(y_{mn}) \in \chi_F^{2I}$. Then

$$I - \lim_{mn\to\infty} f_{mn}\left((m+n)!\,|x_{mn}|\right)^{1/m+n} = 0$$
$$I - \lim_{mn\to\infty} f_{mn}\left((m+n)!\,|y_{mn}|\right)^{1/m+n} = 0$$

Then we have

$$I - \lim_{mn\to\infty} f_{mn}\left((m+n)!\,|x_{mn}y_{mn}|\right)^{1/m+n} = 0.$$

Thus $(x_{mn}y_{mn}) \in \chi_F^{2I}$ is a sequence algebra. □

Proposition 3.3. *The space χ_F^{2I} is not convergence free in general.*

Proof. Here we give a counter example.

Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0,\infty)$. Consider the sequence (x_{mn}) and (y_{mn}) defined by

$$x_{mn} = \tfrac{1}{(m+n)!(mn)^{m+n}} \text{ and } y_{mn} = (m+n)!\,(mn)^{m+n} \text{ for all } m,n \in \mathbb{N}.$$

Then $(x_{mn}) \in \chi_F^{2I}$, but $(y_{mn}) \notin \chi_F^{2I}$. Hence the space χ_F^{2I} is not convergence free. □

Proposition 3.4. *If I is not maximal and $I \neq I_f$, then the space χ_F^{2I} is not symmetric.*

Proof. Let $N \in I$ be infinite and $f(x) = x$ for all $x \in [0,\infty)$. If

$$(x_{mn}) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } m,n \in N; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $(x_{mn}) \in \chi_F^{2I}$. Let $K \subset \mathbb{N}\times\mathbb{N}$ be such that $K \notin I$ and $\mathbb{N}\times\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N}\times\mathbb{N} - K \to \mathbb{N}\times\mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N}\times\mathbb{N}$ defined by

$$\pi(mn) = \begin{cases} \phi(m,n), & \text{for } m,n \in K; \\ \psi(m,n), & \text{otherwise.} \end{cases}$$

is a permutation on $\mathbb{N}\times\mathbb{N}$, but $x_{\pi(m,n)} \notin \chi_F^{2I}$. Hence χ_F^{2I} is not symmetric. □

Theorem 3.2. *Let $F = (f_{mn})$ be the sequence of moduli. Then $\chi_F^{2I} \subset \Lambda_F^{2I}$ and the inclusions are proper*

Proof. Let $(x_{mn}) \in \chi_F^{2I}$, i.e. $I - \lim_{mn\to\infty} f_{mn}\left((m+n)!\left|x_{mn}y_{mn}\right|\right)^{1/m+n} = 0$.
We have

$$f_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} \leq \frac{1}{2} f_{mn}\left((m+n)!\left|x_{mn}-0\right|\right)^{1/m+n} + \frac{1}{2} f_{mn}\left((m+n)!\left|0\right|\right)^{1/m+n}.$$

Therefore

$$\sup_{mn}\left(f_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\right) \leq \sup_{mn}\left(\frac{1}{2} f_{mn}\left((m+n)!\left|x_{mn}-0\right|\right)^{1/m+n}\right) + \sup_{mn}\left(\frac{1}{2} f_{mn}\left((m+n)!\left|0\right|\right)^{1/m+n}\right).$$

We get $(x_{mn}) \in \Lambda_F^2$. □

Theorem 3.3. *The function $h : \Lambda^{2I}_{\chi_F^2} \to \mathbb{R}$ is the Lipschitz function, where $\Lambda_F^{2I} = \chi_F^{2I} \bigcap \Lambda_F^2$, and hence uniformly continuous, where $\Lambda^{2I}_{\chi_F^2}$ analytic I double gai sequence space.*

Proof. Let $x, y \in \Lambda^{2I}_{\chi_F^2}, x \neq y$. Then the sets

$$A_x = \left\{m, n \in \mathbb{N} : \left((m+n)!\left|x_{mn} - h(x)\right|\right)^{1/m+n} \geq d(x,y)\right\} \in I,$$
$$A_y = \left\{m, n \in \mathbb{N} : \left((m+n)!\left|y_{mn} - h(x)\right|\right)^{1/m+n} \geq d(x,y)\right\} \in I.$$

Thus the sets

$$B_x = \left\{m, n \in \mathbb{N} : \left((m+n)!\left|x_{mn} - h(x)\right|\right)^{1/m+n} < d(x,y)\right\} \in \Lambda^{2I}_{\chi_F^2},$$
$$B_y = \left\{m, n \in \mathbb{N} : \left((m+n)!\left|y_{mn} - h(x)\right|\right)^{1/m+n} < d(x,y)\right\} \in \Lambda^{2I}_{\chi_F^2}.$$

Here also $B = B_x \bigcap B_y \in \Lambda^{2I}_{\chi_F^2}$, so that $B \neq \phi$. Now taking $m, n \in B$,

$$\begin{aligned}\left((m+n)!\left|h(x) - h(y)\right|\right)^{1/m+n} &\leq \left((m+n)!\left|h(x) - x_{mn}\right|\right)^{1/m+n} \\ &\quad + \left((m+n)!\left|x_{mn} - y_{mn}\right|\right)^{1/m+n} \\ &\quad + \left((m+n)!\left|y_{mn} - h(y)\right|\right)^{1/m+n} \\ &\leq 3d(x,y).\end{aligned}$$

Thus h is a Lipschitz function. □

Proposition 3.5. *If $x, y \in \Lambda^{2I}_{\chi_F^2}$. then $(x \cdot y) \in \Lambda^{2I}_{\chi_F^2}$ and $h(xy) = h(x)h(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \left\{ m,n \in \mathbb{N} : \left((m+n)! \left|x_{mn} - h(x)\right|\right)^{1/m+n} < \epsilon \right\} \in \Lambda^{2I}_{\chi^2_F},$$

$$B_y = \left\{ m,n \in \mathbb{N} : \left((m+n)! \left|y_{mn} - h(y)\right|\right)^{1/m+n} < \epsilon \right\} \in \Lambda^{2I}_{\chi^2_F}.$$

Now,

$$\begin{aligned} &\left((m+n)! \left|x_{mn}y_{mn} - h(x)\,h(y)\right|\right)^{1/m+n} = \\ &\quad \left((m+n)! \left|x_{mn}y_{mn} - x_{mn}h(y) + x_{mn}h(y) - h(x)\,h(y)\right|\right)^{1/m+n} \le \\ &\quad (m+n)! \left|x_{mn}\right|^{1/m+n} \left|y_{mn} - h(y)\right|^{1/m+n} + (m+n)! \left|h(y)\right|^{1/m+n} \left|x_{mn} - h(x)\right|^{1/m+n} \end{aligned}$$

As $\Lambda^{2I}_{\chi^2_F} \subseteq \Lambda^2_F$, there exists an $M \in \mathbb{R}$ such that $|x_{mn}|^{1/m+n} < M$ and $|h(y)|^{1/m+n} < M$.

Therefore using above equation we get

$$\left((m+n)! \left|x_{mn}y_{mn} - h(x)\,h(y)\right|\right)^{1/m+n} \le M\epsilon + M\epsilon = 2M\epsilon$$

for all $m,n \in B_x \bigcap B_y \in \Lambda^{2I}_{\chi^2_F}$. Hence $(x \cdot y) \in \Lambda^{2I}_{\chi^2_F}$ and $h(xy) = h(x)\,h(y)$. □

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