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CHARACTERISTIC POLYNOMIAL OF SOME CLUSTER GRAPHS

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ABSTRACT. The characteristic polynomial of a graph G with p vertices is defined as $\phi(G:\lambda) = \det(\lambda I - A(G))$, where A is the adjacency matrix of G and I is the unit matrix. The roots of the characteristic equation $\phi(G:\lambda) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G. The graphs with large number of edges are referred as graph representations of inorganic clusters, called as Cluster graphs. In this paper we obtain the characteristic polynomial of class of cluster graphs.

1. INTRODUCTION

Let G be a simple undirected graph with p vertices and q edges. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. The adjacency matrix of G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The characteristic polynomial of G is $\phi(G:\lambda) = \det(\lambda I - A(G))$, where I is an unit matrix of order p. The roots of an equation $\phi(G:\lambda) = 0$ denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G and their collection is the spectrum of G [1]. Characteristic polynomial of some cluster graphs are obtained in [2–5]. In this paper we obtain the characteristic polynomial of some more cluster graphs.

2. SOME EDGE DELETED CLUSTER GRAPHS

I. Gutman and L. Pavlović [2] introduced four classes of graphs obtained from complete graph by deleting edges. For the sake of continuity we produce these here.

Definition 2.1. [2] Let v be a vertex of the complete graph K_p, $p \geq 3$ and let e_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq p-1$, be its distinct edges, all being incident to v. The graph $Ka_p(k)$ or $Cl(p,k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from K_p. In addition $Ka_p(0) \cong K_p$.

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Definition 2.2. [2] Let f_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor p/2 \rfloor$ be independent edges of the complete graph K_p, $p \geq 3$. The graph Kb_p is obtained by deleting f_i, $i = 1, 2, \ldots, k$ from K_p. The graph $Kb_p(0) \cong K_p$.

Definition 2.3. [2] Let V_k be a k- element subset of the vertex set of the complete graph K_p, $2 \leq k \leq p$, $p \geq 3$. The graph $Kc_p(k)$ is obtained by deleting from K_p all the edges connecting pairs of vertices from V_k. In addition $Kc_p(0) \cong Kc_p(1) \cong K_p$.

Definition 2.4. [2] Let $3 \leq k \leq p$, $p \geq 3$. The graph $Kd_p(k)$ obtained by deleting from K_p, the edges belonging to a k- membered cycle.

Theorem 2.1. [2] *For $p \geq 3$ and $3 \leq k \leq p-1$, it holds that*

$$\phi(Ka_p(k) : \lambda) = (\lambda+1)^{p-3}[\lambda^3 - (p-3)\lambda^2 - (2p-k-3)\lambda + (k-1)(p-1-k)].$$

Theorem 2.2. [2] *For $p \geq 3$ and $0 \leq k \leq \lfloor p/2 \rfloor$ it holds that*

$$\phi(Kb_p(k) : \lambda) = \lambda^k(\lambda+1)^{p-2k-1}(\lambda+2)^{k-1}[\lambda^2 - (p-3)\lambda - 2(p-k-1)].$$

Theorem 2.3. [2] *For $p \geq 3$ and $0 \leq k \leq p$ it holds that*

$$\phi(Kc_p(k) : \lambda) = \lambda^{k-1}(\lambda+1)^{p-k-1}[\lambda^2 - (p-k-1)\lambda - k(p-k)].$$

Theorem 2.4. [2] *For $p \geq 3$ and $3 \leq k \leq p$ it holds that*

$$\phi(Kd_p(k) : \lambda) = (\lambda+1)^{p-k-1}[\lambda^2 - (p-4)\lambda - (3p-2k-3)] \times \prod_{i=1}^{k-1}\left(\lambda + 2\cos\left(\frac{2\pi i}{k}\right) + 1\right).$$

We introduce here another class of graphs obtained from K_p and we denote it by $Ka_p(p, m, k)$. Two subgraphs G_1 and G_2 of graph G are independent subgraphs if $V(G_1) \cap V(G_2)$ is an empty set, where $V(G)$ is the vertex set of the graph G.

Definition 2.5. [4] Let $(K_m)_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$, be independent complete subgraphs with m vertices of the complete graph K_p, $p \geq 3$. The graph $K_{a_p}(m, k)$ obtained from K_p by deleting all the edges of k independent complete subgraphs $(K_m)_i$, $i = 1, 2, \ldots, k$. In addition $K_{a_p}(m, 0) \cong K_{a_p}(0, k) \cong K_{a_p}(0, 0) \cong K_p$.

We note that for $k = 1$, $K_{a_p}(p, m, k) \cong K_{a_p}(p, m)$.

Theorem 2.5. [4] *Let p, m, k be positive integers. Let $mk \leq p$. Then for $p \geq 3$, $0 \leq k \leq \lfloor p/m \rfloor$, $1 \leq m \leq p$, the characteristic polynomial of $K_{a_p}(p, m, k)$ is*

$$\phi(K_{a_p}(p, m, k) : \lambda) = \lambda^{mk-k}(\lambda+1)^{p-mk-1}(\lambda+m)^{k-1}[\lambda^2 - (p-m-1)\lambda - m(p+k-mk-1)].$$

Definition 2.6. [5] Let K_{m_1} and K_{m_2} be independent complete subgraphs of K_p. The graph obtained by deleting all the edges of $k_1 + k_2$ such that $m_1k_1 + m_2k_2 \leq p$ independent complete subgraphs K_{m_1} (k_1 copies) and K_{m_2} (k_2 copies) is denoted by $K_{a_p}(p, (m_1, k_1), (m_2, k_2))$.

Theorem 2.6. [5] *Let p, m_i, k_i, $i = 1, 2$ are positive integers with $\sum_{i=1}^{2} m_ik_i \leq p$. Then for $p \geq 3$, the characteristic polynomial of $K_{a_p}(p, (m_1, k_1), (m_2, k_2))$ is*

$$\begin{aligned}\phi(K_{a_p}(p, (m_1, k_1), (m_2, k_2)) : \lambda) &= \lambda^{\sum_{i=1}^{2} m_ik_i - \sum_{i=1}^{2} k_i}(\lambda + 1)^{p - \sum_{i=1}^{2} m_ik_i - 1} \\ &\times \left\{ \left[(\lambda + m_1 - p) + \frac{k_2(m_2^2 - m_1m_2)}{\lambda + m_2} \right] (\lambda + 1) + (m_1 - 1) \left(\sum_{i=1}^{2} m_ik_i - p \right) \right\} \\ &\times (\lambda + m_1)^{k_1 - 1}(\lambda + m_2)^{k_2}.\end{aligned}$$

Now we proceed to prove the main result.

Definition 2.7. Let K_{m_i}, $i = 1, 2, \ldots, l$ are the independent complete subgraphs of K_p. The graph obtained by deleting all the edges of $\sum_{i=1}^{l} k_i$, such that $\sum_{i=1}^{l} m_ik_i \leq p$ independent complete subgraphs K_{m_1} (k_1 copies), K_{m_2} (k_2 copies), ... , K_{m_l} (k_l copies) is denoted by $K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$.

We note that for $k_1 = k$, $m_1 = m$, $k_i = 0$ for $i = 2, 3, \ldots, l$ it holds that

$$K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l)) \cong K_{a_p}(p, m, k).$$

Theorem 2.7. *Let p, m_i, k_i, $i = 1, 2, \ldots, l$ are positive integers with $\sum_{i=1}^{l} m_ik_i \leq p$. Then for $p \geq 3$, the characteristic polynomial of $K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$ is*

$$\begin{aligned}\phi(K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l)) : \lambda) &= \lambda^{\sum_{i=1}^{l} m_ik_i - \sum_{i=1}^{l} k_i}(\lambda + 1)^{p - \sum_{i=1}^{l} m_ik_i - 1} \\ &\times \left\{ \left[(\lambda + m_1 - p) + \sum_{i=2}^{l} \frac{k_i(m_i^2 - m_im_1)}{\lambda + m_i} \right] (\lambda + 1) + (m_1 - 1) \left(\sum_{i=1}^{l} m_ik_i - p \right) \right\} \\ &\times (\lambda + m_1)^{k_1 - 1} \prod_{i=2}^{l} (\lambda + m_i)^{k_i}.\end{aligned}$$

Proof. Without loss of generality we assume that the vertices of $K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$ are the vertices of K_p and the edges are the edges of K_p that are not there in K_{m_1} (k_1 copies), K_{m_2} (k_2 copies), ... ,K_{m_l} (k_l copies). The characteristic polynomial of $K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$ is

$$
\left|\begin{array}{cccccccccccccccccccccc}
\lambda & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
0 & \lambda & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & 0 & \lambda & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
-1 & -1 & \cdots & -1 & 0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & 0 & \lambda & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & \lambda & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \lambda & \cdots & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & 0 & \cdots & \lambda & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & \lambda & \cdots & 0 & -1 & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & \lambda & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & \lambda & \cdots & -1 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & \lambda
\end{array}\right|
$$

By elementary calculations on the above determinant, we get the result. □

Here we give some examples in Table 1.

$p, (m_1, k_1), (m_2, k_2), \ldots, (m_l k_l)$	The characteristic equations of graphs K_{a_p}
9,(2,1),(3,1),(4,1)	$\lambda^6(\lambda^3 - 26\lambda - 48)$
11,(2,2),(3,1),(4,1)	$\lambda^7(\lambda^4 - 44\lambda^2 - 152\lambda - 144)$
12,(2,1),(3,1),(4,1)	$\lambda^6(\lambda^6 - 56\lambda^4 - 260\lambda^3 - 477\lambda^2 - 392\lambda - 120)$
15,(3,2),(4,1),(5,1)	$\lambda^{11}(\lambda^4 - 83\lambda^2 - 402\lambda - 540)$
14,(2,1),(3,1),(4,1),(5,1)	$\lambda^{10}(\lambda^4 - 71\lambda^2 - 308\lambda - 360)$

Table 1. The characteristic equations of the graphs denoted by $K_{a_p}(p, (m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$

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