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A NOTE ON PRIME QUASI-IDEALS IN TERNARY SEMIRINGS

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Abstract. In this paper we introduce the concepts of prime and semiprime quasi-ideals in ternary semirings and give its characterization. Furthermore, we show that a quasi-ideal Q of a ternary semiring S is S-prime if and only if $RML \subseteq Q$ implies $R \subseteq Q$ or $M \subseteq Q$ or $L \subseteq Q$ for any right ideal R, lateral ideal M and left ideal L of S.

1. Introduction and preliminaries

Lehmer [2] initiated the concept of ternary algebraic systems called triplexes in 1932. The notion of semiring was introduced by Vandiver [3] in 1934. In fact semiring is a generalization of ring. In 1971 Lister [9] characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. In 2003 Dutta and Kar [6] introduced the notion of ternary semiring, regular ternary semiring and k-regular ternary semiring and studied some of their properties. Steinfeld [4] surveyed widely the generalization for the notion of quasi-ideal in rings and semigroups in 1978. Dixit and Dewan [8] studied about the quasi-ideals and bi-ideals of ternary semigroups. S. Kar [5] introduced the notions of quasi-ideal and bi-ideal in ternary semiring and obtained their properties. In this paper we introduce the notions of prime and semiprime quasi-ideals in ternary semirings. Recall the following [5–7].

A non-empty set S together with a binary operation called addition and ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$,

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(iv) $ab(c+d) = abc + abd$, for all $a, b, c, d, e \in S$.

A ternary semiring S is said to be commutative if $abc = bac = bca$ for all $a, b, c \in S$.

Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$ then '0' is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Throughout this paper, S will always denote a ternary semiring with zero and unless, otherwise stated a ternary semiring means a ternary semiring with zero.

An additive subsemigroup T of S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

An additive subsemigroup I of S is called a left (right,lateral) ideal of S if s_1s_2i (respectively $is_1s_2, s_1is_2) \in I$ for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, a right, a lateral ideal of S then I is called an ideal of S. Let S be a ternary semiring and $a \in S$. Then the principal left ideal generated by a is given by $< a >_l = \{\sum r_is_ia + na : r_i, s_i \in S : n \in Z_0^+\}$, right ideal generated by a is given by $< a >_r = \{\sum ar_is_i + na : r_i, s_i \in S : n \in Z_0^+\}$, lateral ideal generated by a is given by $< a >_m = \{\sum r_ias_i + \sum p_jq_jar_js_j + na : r_i, s_i, p_j, q_j, r_j, s_j \in S : n \in Z_0^+\}$ and ideal generated by a is given by $< a >_i = \{\sum r_is_ia + \sum ap_iq_i + \sum u_kav_k + \sum {p_k}'{q_k}'a{r_k}'{s_k}' + na : r_i, s_i, p_i, q_i, u_k, v_k, {p_k}', {q_k}', {r_k}', {s_k}' \in S : n \in Z_0^+\}$ where $\sum$ denote the finite sum and $Z_0^+$is the set of all positive integer with zero. An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$. A proper ideal P of a ternary semiring S is called a prime ideal of S if $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three ideals A, B, C of S.

2. Prime quasi-ideals in ternary semirings

In this section, we introduce the notions of prime and semiprime quasi-ideals in ternary semirings and some relevant counter examples are also indicated.

Definition 2.1. A proper quasi-ideal Q of a ternary semiring S is prime if $ABC \subseteq Q$ implies $A \subseteq Q$ or $B \subseteq Q$ or $C \subseteq Q$ for any three quasi-ideals A, B and C of S. A proper quasi-ideal Q of S is semiprime if $A^3 \subseteq Q$ implies $A \subseteq Q$ for any quasi-ideal A of S.

Remark 2.1. Every prime quasi-ideal of a ternary semiring S is a semiprime quasi-ideal of S. But converse need not be true.

Example 2.1. Let $S = M_2(Z_0^-)$ be a ternary semiring of 2×2 square matrices over Z_0^-. Let $Q = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^-\right\}$. Then Q is a semiprime quasi-ideal of S. But Q is not prime quasi-ideal of S since $A = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in Z_0^-\right\}$, $B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in Z_0^-\right\}$

and $C = \left\{ \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} : d \in Z_0^- \right\}$ are quasi-ideals of S such that $ABC \subseteq Q$ but $A \nsubseteq Q, B \nsubseteq Q$ and $C \nsubseteq Q$.

Definition 2.2. A proper quasi-ideal Q of a ternary semiring S is called weakly prime if $Q \subseteq A, Q \subseteq B, Q \subseteq C$ and $ABC \subseteq Q$ implies $A = Q$ or $B = Q$ or $C = Q$ for any quasi-ideals A, B and C of S.

Remark 2.2. Every prime quasi-ideal of a ternary semiring S is a weakly prime quasi-ideal of S. But converse need not be true which can be illustrated as follows:

Example 2.2. Let $S = M_2(Z_0^-)$ be a ternary semiring of 2×2 square matrices over Z_0^-. Let $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 30Z_0^- \right\}$. Then Q is weakly prime quasi-ideal of S. But Q is not prime quasi-ideal of S since $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 2Z_0^- \right\}, B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 3Z_0^- \right\}$ and $C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 5Z_0^- \right\}$ are quasi-ideals of S such that $ABC \subseteq Q$. But $A \nsubseteq Q, B \nsubseteq Q$ and $C \nsubseteq Q$.

Proposition 2.1. *Let S be a ternary semiring and $a \in S$. Then the principal quasi-ideal generated by a is given by* $\langle a \rangle_q = \{[aSS \cap (SaS + SSaSS) \cap SSa] + na : n \in Z_0^+\}$.

Proposition 2.2. *Let S be a ternary semiring and Q be a quasi-ideal of S. If Q is prime, then Q is a right or lateral or left ideal of S.*

Proof. Let Q be a prime quasi-ideal of S. Then $(QSS)(SQS + SSQSS)(SSQ) \subseteq QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$. Since Q is prime, we have $QSS \subseteq Q$ or $SQS + SSQSS \subseteq Q$ or $SSQ \subseteq Q$. Hence Q is a right or lateral or left ideal of S. □

Proposition 2.3. *Let S be a commutative ternary semiring and Q be a quasi-ideal of S. Then Q is prime if and only if $xyz \in Q$ implies $x \in Q$ or $y \in Q$ or $z \in Q$.*

Proof. Suppose Q is a prime quasi-ideal of S. Let $xyz \in Q$. Then Q is an ideal of S (by Proposition 2.2). Let $a \in \langle x \rangle_q \langle y \rangle_q \langle z \rangle_q$. Then

$$a = [(xSS \cap (SxS + SSxSS) \cap SSx) + nx].[(ySS \cap (SyS + SSySS) \cap SSy) + ny].$$
$$[(zSS \cap (SzS + SSzSS) \cap SSz) + nz].$$

Since $xyz \in Q$ and Q is an ideal of S, therefore we have $a \in Q$. Thus $\langle x \rangle_q \langle y \rangle_q \langle z \rangle_q \subseteq Q$. Since Q is a prime quasi-ideal of S, therefore $x \in Q$ or $y \in Q$ or $z \in Q$.

Converse is obvious. □

Proposition 2.4. *Let S be a ternary semiring and Q be a quasi-ideal of S. Then Q is prime if and only if* $[(xSS) \cap (SxS + SSxSS) \cap (SSx)][ySS \cap (SyS + SSySS) \cap SSy][zSS \cap (SzS + SSzSS) \cap SSz] \subseteq Q$ *implies $x \in Q$ or $y \in Q$ or $z \in Q$.*

Proof. Suppose Q is a prime quasi-ideal of S and let $[(xSS) \cap (SxS + SSxSS) \cap SSx][ySS \cap (SyS + SSySS) \cap SSy][zSS \cap (SzS + SSzSS) \cap SSz] \subseteq Q$ for some $x, y, z \in S$. Clearly, $[xSS \cap (SxS + SSxSS) \cap SSx]$, $[ySS \cap (SyS + SSySS) \cap SSy]$ and $[zSS \cap (SzS + SSzSS) \cap SSz]$ are quasi-ideals of S. Since Q is prime, therefore we have $xSS \cap (SxS + SSxSS) \cap SSx \subseteq Q$ or $ySS \cap (SyS + SSySS) \cap SSy \subseteq Q$ or $zSS \cap (SzS + SSzSS) \cap SSz \subseteq Q$. If $xSS \cap (SxS + SSxSS) \cap SSx \subseteq Q$, then we have $\langle x \rangle_q \subseteq Q$. This implies $x \in Q$. Similarly we have $y \in Q$ or $z \in Q$.

Converse is obvious. □

Theorem 2.1. *Let S be a ternary semiring. If the quasi-ideals of S with respect to the inclusion relation forms a chain, then every weakly prime quasi-ideal is a prime quasi-ideal.*

Proof. Let Q be a weakly prime quasi-ideal of S. Let A, B and C are quasi-ideals of S such that $ABC \subseteq Q$. Suppose $A \nsubseteq Q, B \nsubseteq Q$ and $C \nsubseteq Q$. By hypothesis $Q \subseteq A, Q \subseteq B$ and $Q \subseteq C$. Since Q is weakly prime, therefore $A = Q$ or $B = Q$ or $C = Q$, a contradiction. Therefore $A \subseteq Q$ or $B \subseteq Q$ or $C \subseteq Q$. Hence Q is a prime quasi-ideal of S. □

Theorem 2.2. *Let S be a ternary semiring. Then the following are equivalent:*

(i) *The quasi-ideals of S are idempotent.*
(ii) *If A, B, C are quasi-ideals of S such that $A \cap B \cap C \neq \emptyset$, then $A \cap B \cap C \subseteq ABC$.*
(iii) *$\langle a \rangle_q = [\langle a \rangle_q]^3$ for all $a \in S$.*

Proof. (i)⇒(ii) Let A, B and C are quasi-ideals of S such that $A \cap B \cap C \neq \emptyset$. Then it is easy to show that $A \cap B \cap C$ is a quasi-ideal of S. Since every quasi-ideal of S is an idempotent, therefore

$$\begin{aligned} A \cap B \cap C &= (A \cap B \cap C)^3 \\ &= (A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C) \subseteq ABC. \end{aligned}$$

(ii)⇒(iii) It is straight forward.
(iii)⇒(i) It is obvious. □

Definition 2.3. A non-empty subset A of a ternary semiring S is called an m_q-system if for any $a, b, c \in A$ there exist $a_1 \in \langle a \rangle_q, b_1 \in \langle b \rangle_q$ and $c_1 \in \langle c \rangle_q$ such that $a_1 b_1 c_1 \in A$.

Definition 2.4. A non-empty subset B of a ternary semiring S is called an n_q-system if for any $b \in B$, there exist $b_1, b_2, b_3 \in \langle b \rangle_q$ such that $b_1 b_2 b_3 \in B$.

Remark 2.3. Every m_q-system is an n_q-system. But converse need not be true.

Example 2.3. Let $S = Z_6^-$ be the ternary semiring under addition and multiplication modulo 6. Let $A = \{-2, -3\}$. Then A is an n_q-system but not an m_q-system.

Theorem 2.3. *Let S be a ternary semiring and Q be a quasi-ideal of S. Then*

(i) *Q is a prime quasi-ideal if and only if its complement is an m_q-system.*
(ii) *Q is a semiprime quasi-ideal if and only if its complement is an n_q-system.*

Proof. (i) Assume that Q is a prime quasi-ideal of S. Let $a, b, c \in S \setminus Q$. Suppose $a_1b_1c_1 \notin S \setminus Q$ for all $a_1 \in \langle a\rangle_q, b_1 \in \langle b\rangle_q$ and $c_1 \in \langle c\rangle_q$. Then $\langle a\rangle_q\langle b\rangle_q\langle c\rangle_q \subseteq Q$. Since Q is a prime quasi-ideal of S, therefore $a \in Q$ or $b \in Q$ or $c \in Q$. This is a contradiction. Hence $a_1b_1c_1 \in S \setminus Q$ for some $a_1 \in \langle a\rangle_q, b_1 \in \langle b\rangle_q$ and $c_1 \in \langle c\rangle_q$.

Conversely, let A, B and C be quasi-ideals of S such that $ABC \subseteq Q$. Assume that $A \nsubseteq Q, B \nsubseteq Q$ and $C \nsubseteq Q$. Let $a \in A \setminus Q, b \in B \setminus Q$ and $c \in C \setminus Q$. Then $a, b, c \in S \setminus Q$. Since $S \setminus Q$ is an m_q-system, therefore $a_1b_1c_1 \in S \setminus Q$ for some $a_1 \in \langle a\rangle_q, b_1 \in \langle b\rangle_q$ and $c_1 \in \langle c\rangle_q$. But $a_1b_1c_1 \in \langle a\rangle_q\langle b\rangle_q\langle c\rangle_q \subseteq ABC \subseteq Q$. This is a contradiction. Hence $A \subseteq Q$ or $B \subseteq Q$ or $C \subseteq Q$.

(ii) Similar to (i) □

Definition 2.5. A quasi-ideal Q of a ternary semiring S is S-prime if $xSySz \subseteq Q$ implies $x \in Q$ or $y \in Q$ or $z \in Q$. A quasi-ideal Q of a ternary semiring S is called S-semiprime if $xSxSx \subseteq Q$ implies $x \in Q$.

Theorem 2.4. *A quasi-ideal Q of a ternary semiring S is S-prime if and only if $RML \subseteq Q$ implies $R \subseteq Q$ or $M \subseteq Q$ or $L \subseteq Q$ for any right ideal R, lateral ideal M and left ideal L of S.*

Proof. Let Q be a S-prime quasi-ideal of S and $RML \subseteq Q$. Suppose $R \nsubseteq Q$ and $M \nsubseteq Q$. Then there exists $x \in R\backslash Q$ and $y \in M\backslash Q$. Let $z \in L$. Then $xSySz \subseteq RSMSL \subseteq RML \subseteq Q$. Since Q is S-prime, therefore $x \in Q$ or $y \in Q$ or $z \in Q$. Now $x \notin Q$ and $y \notin Q$. Thus $z \in Q$. Consequently $L \subseteq Q$.

Conversely, suppose $xSySz \subseteq Q$. Then $(xSS)(SyS)(SSy) \subseteq xSySz \subseteq Q$. Since xSS is a right ideal, SyS is a lateral ideal and SSz is a left ideal, therefore by hypothesis, $xSS \subseteq Q$ or $SyS \subseteq Q$ or $SSz \subseteq Q$. If $xSS \subseteq Q$, then $x^3 \in xSS \subseteq Q$.

Now

$$\begin{aligned}\langle x\rangle_r\langle x\rangle_m\langle x\rangle_l &= (Z_0x + xSS)(Z_0x + SxS + SSxSS)(Z_0x + SSx)\\ &\subseteq x^3 + xSS \subseteq Q.\end{aligned}$$

By hypothesis, $\langle x\rangle_r \subseteq Q$ or $\langle x\rangle_m \subseteq Q$ or $\langle x\rangle_l \subseteq Q$. Therefore $x \in Q$. Similarly if $SyS \subseteq Q$ then $y \in Q$ and if $SSz \subseteq Q$ then $z \in Q$. Hence Q is S-prime. □

Let Q be a quasi-ideal of a ternary semiring S. Then define

$$\begin{aligned} L(Q) &= \{x \in Q : SSx \subseteq Q\}\\ M(Q) &= \{x \in Q : SxS + SSxSS \subseteq Q\}\\ R(Q) &= \{x \in Q : xSS \subseteq Q\}\\ I_L &= \{y \in L(Q) : ySS \subseteq L(Q)\}\\ {}_MI_M &= \{y \in M(Q) : SyS + SSySS \subseteq M(Q)\}\\ I_R &= \{y \in R(Q) : SSy \subseteq R(Q)\}\end{aligned}$$

Proposition 2.5. *Let Q be a quasi-ideal of a ternary semiring S. Then $L(Q)$ (resp.$M(Q), R(Q)$) is a left (resp. lateral, right) ideal of S contained in Q if $L(Q)$ (resp. $M(Q), R(Q)$) is nonempty.*

Proof. Let $x \in L(Q)$ and $s_1, s_2 \in S$. Then $s_1s_2x \in SSx \subseteq Q$. Now $SSs_1s_2x \subseteq SSx \subseteq Q$. Thus we have $s_1s_2x \in L(Q)$. Consequently $SSL(Q) \subseteq L(Q)$. So $L(Q)$ is a left ideal of S. Similarly we can prove that $M(Q)$ is a lateral ideal and $R(Q)$ is a right ideal of S. □

Proposition 2.6. *Let Q be a quasi-ideal of a ternary semiring S. If I_L (resp. ${}_RI,{}_MI_M$) is non-empty then I_L (resp ${}_RI,{}_MI_M$) is the largest ideal of S contained in Q. Moreover $I_L ={}_R I ={}_M I_M$.*

Proof. Let $x \in I_L$. Then $I_L \subseteq L(Q) \subseteq Q$ implies $x \in L(Q)$ and $x \in Q$. That is $SSx \subseteq Q$. Then $SSs_1s_2x \subseteq SSx \subseteq Q$ for some $s_1, s_2 \in S$. This implies $s_1s_2x \in L(Q)$. Since $L(Q)$ is a left ideal of S (by Proposition 2.5) and $xSS \subseteq L(Q)$ therefore $s_1s_2xSS \subseteq SSL(Q) \subseteq L(Q)$. Thus $s_1s_2x \in I_L$. That is $SSI_L \subseteq I_L$. Hence I_L is a left ideal of S. Similarly, we can show that I_L is a right ideal and a lateral ideal of S. Hence I_L is an ideal of S contained in Q.

Let I be any ideal of S contained in Q. Then $SSI \subseteq I \subseteq Q$. This implies $I \subseteq L(Q)$. Now $ISS \subseteq I \subseteq L(Q)$. This implies $I \subseteq I_L$. Hence I_L is the largest ideal of S contained in Q. Similarly we can prove that ${}_RI$ and ${}_MI_M$ are the largest ideals of S contained in Q. Since $I_L,{}_R I$ and ${}_MI_M$ are the largest ideals of S contained in Q, therefore $I_L ={}_R I ={}_M I_M$.

Notation. we denote $I_Q = I_L ={}_R I ={}_M I_M$. □

Proposition 2.7. *Let Q be a S-prime quasi-ideal of a ternary semiring S. Then I_Q is a prime ideal of S.*

Proof. Let Q be a S-prime quasi-ideal of S. Suppose $RML \subseteq I_Q$ for any ideals R, M, L of S. Now $I_Q \subseteq L(Q) \subseteq Q$ implies $RML \subseteq Q$. Since Q is S-prime, therefore $R \subseteq Q$ or $M \subseteq Q$ or $L \subseteq Q$ (by Theorem 2.4). Also I_Q is the largest ideal contained in Q, therefore $R \subseteq I_Q$ or $M \subseteq I_Q$ or $L \subseteq I_Q$. Hence I_Q is a prime ideal of S. □

Corollary 2.1. *Let Q be a semiprime quasi-ideal of a ternary semiring S. Then I_Q is a semiprime ideal of S.*

Proposition 2.8. *Let B be a bi-ideal of a ternary semiring S. If B is S-semiprime then B is a quasi-ideal of S.*

Proof. Let $x \in (BSS) \cap (SBS+SSBSS) \cap SSB$. Then $xSxSx \in (BSS)S(SSBSS)S(SSB) \subseteq BSBSB \subseteq B$. Since B is S-semiprime, therefore $x \in B$. Consequently, $(BSS) \cap (SBS + SSBSS) \cap SSB \subseteq B$. Hence B is a quasi-ideal of S. □

Proposition 2.9. *If a ternary semiring S is regular, then every quasi-ideal of S is S-semiprime.*

Proof. Suppose S is regular and Q be a quasi-ideal of S. Let $aSaSa \subseteq Q$ for $a \in S$. Since S is regular, therefore for $a \in S$ there exists $x \in S$ such that $a = axa$. Thus $a = axaxa \in aSaSa \subseteq Q$. Therefore $a \in Q$. Hence Q is S-semiprime. □

Remark 2.4. Since every quasi-ideal of a ternary semiring is a bi-ideal, therefore these results are also valid for bi-ideals in ternary semirings.

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