Kragujevac Journal of Mathematics
Volume 37(2) (2013), Pages 341–347.

SOLUBLE GROUPS WITH $\mathfrak{F}$-PERMUTABLE SUBGROUPS

CHANGWEN LI[1] AND YAN WANG[2]

Abstract. Let G be a finite group and $\mathfrak{F}$ a class of finite groups. A subgroup H of G is said to be $\mathfrak{F}$-permutable in G if there exists a subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. By using this new concept, we establish some new criteria for a group G to be soluble.

1. Introduction

Throughout this article, all groups considered are finite and G always denotes a group. The terminologies and notations are standard, as in [4] and [9].

Recall that a subgroup H of G is said to be s-permutable, or π-quasinormal [7] in G if H is permutable with every Sylow subgroup P of G (that is, $HP = PH$). A subgroup H of G is said to be c-supplemented [11] in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. By using the s-permutability and c-supplementation of subgroups, people have obtained many interesting results; see, for example, [1, 5, 7, 8, 10–12], etc.

In this article, we give the following more generalized concept.

Definition 1.1. Let H be a subgroup of G and $\mathfrak{F}$ a class of finite groups. We say that H is $\mathfrak{F}$-permutable in G if there exists a subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G.

Recall that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central (see [4]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G

Key words and phrases. s-permutable, $\mathfrak{F}$-permutable, Soluble groups.
2010 *Mathematics Subject Classification.* Primary: 20D10. Secondary: 20D20.
Received: April 8, 2012.
Revised: May 16, 2013.

whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercenter in G if $H \leq Z_{\infty}^{\mathfrak{F}}(G)$. A class $\mathfrak{F}$ of groups is called a formation if it is closed under a homomorphic image and a subdirect product. It is clear that every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual of G and denoted by $G^{\mathfrak{F}}$) with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{S}$ to denote the formation of all soluble groups.

Obviously, all s-permutable subgroups and all c-supplemented subgroups are all $\mathfrak{F}$-permutable subgroups. However, the following examples show that the converse is not true.

Example 1.1. Let $G = C_7 \wr C_3 = [K]C_3$ be a regular wreath product, where K is the base group of $C_7 \wr C_3$ and $|C_i| = i$. Then $K = F(G)$ is the Sylow 7- subgroup of G and the subgroup $H = \{(a_1, a_2, 1) \mid a_1, a_2 \in C_7\}$ is maximal in K. It is clear that H is $\mathfrak{F}$-permutable in G. However, H is not s-permutable in G.

Example 1.2. Let $G = A \rtimes B$, where A is a cyclic group of order 5 and $B = \langle \alpha \rangle$, where $\alpha \in Aut(A)$ with $|\alpha| = 4$. It is easy to see that $\langle \alpha^2 \rangle$ is $\mathfrak{F}$-permutable in G. However, $\langle \alpha^2 \rangle$ is not c-supplemented in G.

2. Preliminaries

Lemma 2.1. *Let A, B and K be subgroups of a group G.*

(1) *If $(|G : A|, |G : B|) = 1$, then $G = AB$ [4, Lemma 3.8.1].*
(2) *If $(|G : A|, |G : B|) = 1$ and K is normal in G, then $K = (K \cap A)(K \cap B)$ [4, Lemma 3.8.2].*
(3) *$K \cap AB = (K \cap A)(K \cap B)$ if and only if $KA \cap KB = K(A \cap B)$ [2, Lemma A.1.2].*

A formation $\mathfrak{F}$ is said to be S-closed (S_n-closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups. The following lemma is well known.

Lemma 2.2. *Let G be a group and $A \leq G$. Let $\mathfrak{F}$ be a non-empty saturated formation. Then*

(1) *If A is normal in G, then $AZ_{\infty}^{\mathfrak{F}}(G)/A \leq Z_{\infty}^{\mathfrak{F}}(G/A)$.*
(2) *If $\mathfrak{F}$ is S-closed, then $Z_{\infty}^{\mathfrak{F}}(G) \cap A \leq Z_{\infty}^{\mathfrak{F}}(A)$.*
(3) *If $\mathfrak{F}$ is S_n-closed and A is normal in G, then $Z_{\infty}^{\mathfrak{F}}(G) \cap A \leq Z_{\infty}^{\mathfrak{F}}(A)$.*
(4) *If $G \in \mathfrak{F}$, then $Z_{\infty}^{\mathfrak{F}}(G) = G$.*

Lemma 2.3. *Suppose that H be a subgroup of G and H is s-permutable in G. Then*

(1) *If $H \leq K \leq G$, then H is s-permutable in K.*
(2) *If N is a normal subgroup of G, then HN is s-permutable in G and HN/N is s-permutable in G/N.*
(3) *H is subnormal in G.*

In view of Lemmas 2.2 and 2.3, we can get the following lemma easily.

Lemma 2.4. *Let G be a group and $H \leq M \leq G$.*

(1) *If H is $\mathfrak{F}$-permutable in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}$-permutable in M.*
(2) *Suppose that $H \trianglelefteq G$. Then M/H is $\mathfrak{F}$-permutable in G/H if and only if M is $\mathfrak{F}$-permutable in G.*
(3) *If $H \trianglelefteq G$, then for every $\mathfrak{F}$-permutable subgroup E of G with $(|H|, |E|) = 1$, HE/H is $\mathfrak{F}$-permutable in G/H.*

Lemma 2.5. [3, Theorem A] *Suppose that G has a Hall π-subgroup, where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.6. [10, Lemma A] *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.7. [8, Lemma 2.4] *Suppose that H is s-permutable in G, and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is s-permutable in G.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing the order of G. If all maximal subgroups of P are $\mathfrak{S}$-permutable in G, then G is soluble.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. Then by the well known Feit-Thompson's theorem, we have that $p = 2$. We now proceed the proof by the following steps.

(1) $O_2(G) = 1$.

Assume that $L = O_2(G) \neq 1$. Obviously, P/L is a Sylow 2-subgroup of G/L. Let M/L be a maximal subgroup of P/L. Then M is a maximal subgroup of P. By the hypothesis and Lemma 2.4(2), M/L is $\mathfrak{S}$-permutable in G/L. The minimal choice of G implies that G/L is soluble. Consequently, G is soluble. This contradiction shows that step (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $E = O_{2'}(G) \neq 1$. Then, obviously, PE/E is a Sylow 2-subgroup of G/E. Suppose that M/E is a maximal subgroup of PE/E. Then there exists a maximal subgroup P_1 of P such that $M = P_1E$. By the hypothesis and Lemma 2.4(3), $M/E = P_1E/E$ is $\mathfrak{S}$-permutable in G/E. The minimal choice of G implies that G/E is soluble. By the well known Feit-Thompson's theorem, we know that E is soluble. It follows that G is soluble, a contradiction.

(3) P is not cyclic.

If P is cyclic, then G is 2-nilpotent by [9, Theorem 10.1.9]. This implies that G is soluble, a contradiction.

(4) If $1 \neq N \trianglelefteq G$, then N is not soluble and $G = PN$.

If N is soluble, then $O_2(N) \neq 1$ or $O_{2'}(N) \neq 1$. Since $O_2(N)$ char $N \trianglelefteq G$, $O_2(N) \leq O_2(G)$. Analogously $O_{2'}(N) \leq O_{2'}(G)$. Hence $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, which contradicts step (1) or step (2). Therefore N is not soluble. Assume that $PN < G$.

By Lemma 2.4(1), every maximal subgroup of P is $\mathfrak{S}$-permutable in PN. Thus PN satisfies the hypothesis. By the minimal choice of G, PN is soluble and so N is. This contradiction shows that $G = PN$.

(5) G has a unique minimal normal subgroup, N say (where N maybe is G).

By step (4), we see that $G = PN$ for every non-identity normal subgroup N of G. It follows that G/N is soluble. Since $\mathfrak{S}$ is closed under subdirect product, G has a unique minimal normal subgroup, N say.

(6) $Z_\infty^{\mathfrak{S}}(G) = 1$.

If $Z_\infty^{\mathfrak{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which contained in $Z_\infty^{\mathfrak{S}}(G)$. Obviously, N is an elementary Abelian r-subgroup for some prime r, which contradicts steps (1) and (2).

(7) Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a subgroup K_1 of G such that P_1K_1 is s-permutable in G and

$$(P_1 \cap K_1)(P_1)_G/(P_1)_G \subseteq Z_\infty^{\mathfrak{S}}(G/(P_1)_G).$$

In view of steps (1) and (6), we get $P_1 \cap K_1 = 1$. This means that $4 \nmid |K_1|$. Hence by [9, Theorem 10.1.9], K_1 has a normal Hall $2'$-subgroup M_1. Evidently, M_1 is also a Hall $2'$-subgroup of P_1K_1. Obviously, there exists a Sylow 2-subgroup $(K_1)_2$ of K_1 such that $P_1(K_1)_2$ is a Sylow 2-subgroup P_1K_1. If $(P_1K_1)_G = 1$, then $P_1(K_1)_2$ is s-permutable in G by Lemma 2.7. Assume that $|(K_1)_2| = 1$. Then P_1 is s-permutable in G. In view of Lemma 2.6, $P_1 \trianglelefteq PO^p(G) = G$, and so $P_1 \leq (P_1K_1)_G = 1$. This shows that P is cyclic, a contradiction. Hence we have $|(K_1)_2| = 2$. Then $P_1(K_1)_2$ is a Sylow 2-subgroup of G. Applying Lemma 2.6 again, $P_1(K_1)_2$ is normal in G, which contradicts $(P_1K_1)_G = 1$. Therefore, $(P_1K_1)_G \neq 1$. By steps (4) and (5), $N \leq P_1K_1$. Since $N \trianglelefteq G$, $N \trianglelefteq P_1K_1$. It is easy to see that $M_1 \cap N$ is also a Hall $2'$-subgroup of N. Since $G = PN$, we have

$$|G : M_1 \cap N| = |PN : M_1 \cap N| = \frac{|P||N|}{|N \cap P||M_1 \cap N|} = |N : M_1 \cap N||P : P \cap N|$$

is a 2-number. This implies that $M_1 \cap N$ is a Hall $2'$-subgroup of G. Thus $M_1 \cap N = M_1$ is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. For any element $x \in G$, both M_1^x and M_1 are Hall $2'$-subgroups of N. Since any two Hall $2'$-subgroups of a group are conjugate by Lemma 2.5, M_1^x and M_1 are conjugate in N. Let $H = N_G(M_1)$. By Frattini argument, $G = NH$. Since $(|N : N \cap P|, |N : M_1|) = 1$, $N = (N \cap P)M_1$ by Lemma 2.1(1). Hence $G = (N \cap P)H$. It follows that

$$P = P \cap (N \cap P)H = (N \cap P)(P \cap H).$$

Since $(|G : P|, |G : M_1|) = 1$, we have $G = PM_1 = PH$ by Lemma 2.1(1). If $P \cap H = P$, then $P \leq H$ and so $G = H$ has a non-identity normal Hall $2'$-subgroup M_1, which contradicts $O_{2'}(G) = 1$. Thus $P \cap H < P$ and so there exists a maximal subgroup P_2 of P such that $P \cap H \leq P_2$. Then $P = (N \cap P)(P \cap H) = (N \cap P)P_2$. By the hypothesis, there exists a subgroup K_2 of G such that P_2K_2 is s-permutable in G and $P_2 \cap K_2 = 1$. Using the same argument as above, we can see that K_2 has a non-identity normal Hall $2'$-subgroup M_2 such that M_2 is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. Moreover, $N \leq P_2K_2$. Hence

$$G = PM_2 = PK_2 = (N \cap P)P_2K_2 = P_2K_2.$$

Since both M_1 and M_2 are Hall $2'$-subgroups of G, by Lemma 2.5 there exists an element $g \in P$ such that $M_2^g = M_1$. Since $(|H : P \cap H|, |H : M_1|) = 1$, $H = (P \cap H)M_1$ by Lemma 2.1(1). Therefore,

$$G = (P_2K_2)^g = P_2N_G(M_2^g) = P_2N_G(M_1) = P_2H = P_2(P \cap H)M_1 = P_2M_1.$$

It follows that $|G| = |P_2||M_1| < |P||M_1| = |G|$. The final contradiction completes the proof. □

Corollary 3.1. *Let M be a maximal subgroup of a group G with $|G : M| = r$, where r is a prime. Let p be the smallest prime dividing $|M|$. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is $\mathfrak{S}$-permutable in G, then G is soluble.*

Proof. If $|G|$ is odd number, then G is soluble by the well known Feit-Thompson's theorem. Now we assume that $2||G|$. If $r = 2$, then M is normal in G. By Lemma 2.4(1), every maximal subgroup of P is $\mathfrak{S}$-permutable in M. Theorem 3.1 implies that M is soluble. It follows that G is soluble. If $r \neq 2$, then $p = 2$ and P is a Sylow 2-subgroup of G. By using our Theorem 3.1, we obtain that G is soluble. □

Theorem 3.2. *A group G is soluble if and only if every Sylow subgroup of G is $\mathfrak{S}$-permutable in G.*

Proof. The necessity is obvious. We need only prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) $P_G = 1$ for any prime p dividing $|G|$ and any Sylow p-subgroup P of G.

If there exists a Sylow p-subgroup P of G such that $P_G \neq 1$, then by Lemma 2.4(1), it is easy to see that G/P_G satisfies the hypothesis of the theorem. Hence the minimal choice of G implies that G/P_G is soluble, and so G is soluble, a contradiction.

(2) $Z_\infty^{\mathfrak{S}}(G) = 1$.

If $Z_\infty^{\mathfrak{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which is contained in $Z_\infty^{\mathfrak{S}}(G)$. Obviously, N is abelian. With the same argument as step (1), we have that G is soluble, a contradiction.

(3) If $1 \neq N \trianglelefteq G$, then G/N is soluble.

Let M/N be a Sylow p-subgroup of G/N, where $p||G/N|$. Then, obviously $M/N = PN/N$, where P is a Sylow p-subgroup of G. By the hypothesis, there exists a subgroup K of G such that PK is s-permutable in G and $P \cap K = 1$. Hence

$$(|PK \cap N : N \cap K|, |PK \cap N : N \cap P|) = 1.$$

By Lemma 2.1(1), $PK \cap N = (P \cap N)(K \cap N)$. In view of Lemma 2.1(3), $PN \cap KN = N(P \cap K) = N$. This implies that $(PN/N) \cap (KN/N) = 1$. By Lemma

2.3(2), $(PN/N)(NK/N) = (PK)N/N$ is s-permutable in G/N. Therefore, $M/N = PN/N$ is $\mathfrak{S}$-permutable in G/N. This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is soluble.

(4) Final contradiction.

Since $\mathfrak{S}$ is closed under subdirect product, by step (3), G has only one minimal normal subgroup, N say. For any prime p dividing the order of N, we claim that every Sylow p-subgroup N_p of N is complemented in N. In fact, let P be a Sylow p-subgroup of G such that $N_p \leq P$. Then, obviously, $N_p = N \cap P$. By the hypothesis, there exists a subgroup K of G such that PK is s-permutable in G and $P \cap K = 1$. Obviously, K is a p-complement of PK. If $(PK)_G = 1$, P is s-permutable in G by Lemma 2.7. It follows that $P \trianglelefteq PO^p(G) = G$ from Lemma 2.6, a contradiction. The unique minimal normality of N implies that $N \leq PK$. Since $(|PK : K|, |PK : P|) = 1$, $N = (N \cap P)(N \cap K) = N_p(N \cap K)$ by Lemma 2.1(2). Then $N_p \cap (N \cap K) = (P \cap N) \cap (N \cap K) = 1$. This shows that every Sylow p-subgroup of N is complemented in N. Hence N is soluble by Hall's theorem [6], which induces that G is soluble. This contradiction completes the proof. □

Corollary 3.2. [11, Theorem 2.4] *A group G is soluble if and only if every Sylow subgroup of G is c-supplemented in G.*

Acknowledgement: The authors would like to thank the referees for their valuable suggestions and useful comments. This research was partly supported by the Natural Science Foundation of China (grant no. 11101369) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] A. Ballester-Bolinches, Y. Wang and X. Guo, *c-supplemented subgroups of finite groups*, Glasgow Math. J. **42** (2000), 383–389.

[2] K. Doerk, T. Hawkes, *Finite Solvable Groups*, Walter de Gruyter, New York, 1992.

[3] F. Gross, *Conjugacy of odd order Hall subgroups*, Bull. London Math. Soc. **19** (1987), 311-319.

[4] W. Guo, *The Theory of Classes of Groups, Science Press-Kluwer Academic Publishers*, Beijing-New York-Dordrecht-Boston-London, 2000.

[5] X. Guo and K. P. Shum, *Finite p-nilpotent groups with some subgroups c-supplemented*, J. Aust. Math. Soc. **78** (2005), 429–439.

[6] P. Hall, *A characteristic property of soluble groups*, J. London Math. Soc. **12** (1937), 198–200.

[7] O. H. Kegel, *Sylow Gruppen und subnormalteiler endlicher Gruppen*, Math. Z. **78** (1962), 205–221.

[8] Y. Li, Y. Wang and H. Wei, *On p-nilpotency of finite groups with some subgroups π-quasinormally embedded*, Acta Math. Hungar. **108** (2005), 283–298.

[9] D. J. S. Robinson, *A Course in Theory of Group*, Spinger-Verlag, New York, 1982.
[10] P. Schmidt, *Subgroups permutable with all Sylow subgroups*, J. Algebra **207** (1998), 285–293.
[11] Y. Wang, *Finite groups with some subgroups of Sylow subgroups c-supplemented*, J. Algebra **224** (2000), 467–478.
[12] H. Wei, Y. Wang and Y. Li, *c-supplemented maximal and minimal subgroups of Sylow subgroups of finite groups*, Proc. Amer. Math. Soc. **132** (2004), 2197–2204.

[1]School of Mathematics and Statistics,
Jiangsu Normal University,
Xuzhou, 221116, China
E-mail address: lcw2000@126.com

[2]Kewen Institute,
Jiangsu Normal University,
Xuzhou, 221116, China
E-mail address: 439785527@qq.com