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ON (p, q)-TH ORDER OF A FUNCTION OF SEVERAL COMPLEX VARIABLES ANALYTIC IN THE UNIT POLYDISC

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Abstract. In this paper, we study the maximum modulus and the coefficients of the power series expansion of a function of several complex variables analytic in the unit polydisc.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M(r) = M(r, f)$ be the maximum of $|f(z)|$ on $|z| = r$. In [12], Sons was define the order ρ and the lower order λ as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\ \log M(r, f)}{-\log(1-r)}.$$

Maclane [10] and Kapoor [9],proved the following results which characterized the order and lower order of a function f analytic in U, in terms of the coefficients c_n.

Theorem 1.1. [10] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having order ρ $(0 \le \rho \le \infty)$. Then*

$$\frac{\rho}{1+\rho} = \lim\sup_{n\to\infty} \frac{\log^+\ \log^+ |c_n|}{\log\ n}.$$

Theorem 1.2. [9] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having lower order λ $(0 \le \lambda \le \infty)$. Then*

$$\frac{\lambda}{1+\lambda} \ge \lim\inf_{n\to\infty} \frac{\log^+\ \log^+ |c_n|}{\log\ n}.$$

In the paper we use the following definitions and notations.

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Notation 1.1. [11] $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer $m, \log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

Notation 1.2. [1] For $0 < x < \infty$ we write $\log^{*(0)} x = x$, $\log^{*(1)} x = \log(1 + x)$, $\log^{*(2)} x = \log(1 + \log(1 + x))$, $\log^{*(3)} x = \log(1 + \log(1 + \log(1 + x)))$ etc.

Definition 1.1. [8] If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, its p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix} \rho_p \\ \lambda_p \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \quad p \geq 2.$$

Using the definitions of p-th order and lower p-th order Banerjee [1] generalized the Theorem 1.1 and the Theorem 1.2 in the following manner.

Theorem 1.3. [1] *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be analytic in* U *and having* p*-th order* ρ_p $(0 \leq \rho_p \leq \infty)$. *Then*

$$\frac{\rho_p}{1+\rho_p} = \limsup_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log\ n}.$$

Theorem 1.4. [1] *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be analytic in* U *and having lower* p*-th order* λ_p $(0 \leq \lambda_p \leq \infty)$. *Then*

$$\frac{\lambda_p}{1+\lambda_p} \geq \liminf_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log\ n}.$$

Definition 1.2. [2] Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc

$$P : \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

then order of f is denoted by ρ and is defined by

$$\rho = \inf\left\{\mu > 0 : F(r_1, r_2) < \exp\left(\frac{1}{1-r_1}.\frac{1}{1-r_2}\right)^{\mu}; \text{for all } 0 < r_0(\mu) < r_1, r_2 < 1\right\}.$$

Equivalent formula for ρ is

$$\rho = \limsup_{r_1, r_2\to 1} \frac{\log\log F(r_1, r_2)}{-\log(1-r_1)(1-r_2)}.$$

Recently Banerjee and Dutta [3] introduced the definition of p-th order and lower p-th order of functions of two complex variables analytic in the unit polydisc and generalized the above results.

Definition 1.3. [3] Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U = \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

and

$$F(r_1,r_2)=\max\{|f(z_1,z_2)|:|z_j|\le r_j; j=1,2\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p\\ \lambda_p\end{matrix}=\lim_{r_1,r_2\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}F(r_1,r_2)}{-\log(1-r_1)(1-r_2)}, p\ge 2.$$

Note 1.1. When $p=2$, Definition 1.3 coincides with Definition 1.2.

Theorem 1.5. [3] *Let $f(z_1,z_2)$ be analytic in U and having p-th order ρ_p $(0\le\rho_p\le\infty)$. Then*

$$\frac{\rho_p}{1+\rho_p}=\lim\sup_{m,n\to\infty}\frac{\log^{+[p]}|c_{mn}|}{\log\ mn}.$$

Theorem 1.6. [3] *Let $f(z_1,z_2)$ be analytic in U and having lower p-th order λ_p $(0\le\lambda_p\le\infty)$. Then*

$$\frac{\lambda_p}{1+\lambda_p}\ge\lim\inf_{m,n\to\infty}\frac{\log^{+[p]}|c_{mn}|}{\log\ mn}.$$

In a resent paper Dutta [6] introduced the following definitions of (p,q)-th order and lower (p,q)-th order of functions of two complex variables analytic in the unit polydisc and proved a similar analytic expression.

Definition 1.4. Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty}c_{mn}z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U=\{(z_1,z_2):|z_j|\le 1; j=1,2\}$$

and

$$F(r_1,r_2)=\max\{|f(z_1,z_2)|:|z_j|\le r_j; j=1,2\},$$

be its maximum modulus. Then the (p,q)-th order ρ_q^p and the lower (p,q)-th order λ_q^p are define as

$$\begin{matrix}\rho_q^p\\ \lambda_q^p\end{matrix}=\lim_{r_1,r_2\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}F(r_1,r_2)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)},\quad p\ge q+1\ge 2.$$

Note 1.2. When $q=1$, Definition 1.4 accords with the Definition 1.3.

Theorem 1.7. [6] *Let $f(z_1,z_2)$ be analytic in U and having the (p,q)-th order ρ_q^p $(0\le\rho_q^p\le\infty)$. Then*

$$\frac{\rho_q^p}{1+\rho_q^p}=\lim\sup_{m,n\to\infty}\frac{\log^{+[p]}|c_{mn}|}{\log^{[q]}mn}.$$

In papers [4] and [5] Dutta introduced the definition of order and lower order of functions of several complex variables analytic in the unit polydisc and generalized the above results for functions of several complex variables analytic in the unit polydisc.

Definition 1.5. [4] Let $f(z_1, z_{2,} \dots z_n) = \sum_{m_1,m_2,\dots m_n=0}^{\infty} c_{m_1 m_2 \dots\ m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$ be a function of n complex variables $z_1,\ z_2, \dots z_n$ holomorphic in the unit polydisc

$$U = \{(z_1, z_2 \dots z_n) : |z_j| \leq 1; j = 1, 2, \dots n\}$$

and

$$F(r_1, r_2, \dots r_n) = \max\{|f(z_1, z_2, \dots z_n)| : |z_j| \leq r_j; j = 1, 2, \dots n\},$$

be its maximum modulus. Then the order ρ and lower order λ are defined as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r_1,r_2,\dots r_n \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log F(r_1, r_2, \dots r_n)}{-\log(1-r_1)(1-r_2)\dots\ (1-r_n)}.$$

When $n = 2$, Definition 1.5 coincides with Definition 1.2.

Theorem 1.8. [4] *Let $f(z_1, z_2 \dots z_n)$ be analytic in U and having order ρ $(0 \leq \rho \leq \infty)$. Then*

$$\frac{\rho}{1+\rho} = \lim \sup_{m_1,m_2,\dots,m_n \to \infty} \frac{\log^+ \log^+ |c_{m_1 m_2 \dots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

Theorem 1.9. [4] *Let $f(z_1, z_2, \dots z_n)$ be analytic in U and having lower order λ $(0 \leq \lambda \leq \infty)$. Then*

$$\frac{\lambda}{1+\lambda} \geq \lim \inf_{m_1,m_2,\dots\ m_n \to \infty} \frac{\log^+ log^+ |c_{m_1 m_2 \dots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

Definition 1.6. [5] Let $f(z_1, z_2 \dots\ z_n) = \sum_{m_1,m_2,\dots m_n=0}^{\infty} c_{m_1 m_2 \dots\ m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$ be a function of n complex variables $z_1,\ z_2, \dots z_n$ holomorphic in the unit polydisc

$$U = \{(z_1, z_2, \dots z_n) : |z_j| \leq 1; j = 1, 2, \dots n\}$$

and

$$F(r_1, r_2, \dots r_n) = \max\{|f(z_1, z_2, \dots z_n)| : |z_j| \leq r_j; j = 1, 2, \dots n\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r_1,r_2,\dots r_n \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} F(r_1, r_2, \dots r_n)}{-\log(1-r_1)(1-r_2)\dots\ (1-r_n)}, p \geq 2.$$

When $n = 2$, Definition 1.6 coincides with Definition 1.3 also if $p = 2$, Definition 1.6 coincides with Definition 1.5.

Theorem 1.10. [5] *Let $f(z_1, z_2, \dots z_n)$ be analytic in U and having the p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then*

$$\frac{\rho_p}{1+\rho_p} = \lim \sup_{m_1,m_2,\dots\ m_n \to \infty} \frac{\log^{+[p]} |c_{m_1 m_2 \dots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

Theorem 1.11. [5] *Let $f(z_1, z_2, \dots z_n)$ be analytic in U and having the lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then*

$$\frac{\lambda_p}{1+\lambda_p} = \lim_{m_1,m_2,\dots\, m_n \to \infty} \inf \frac{\log^{+[p]} |c_{m_1 m_2 \dots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

In this paper we introduce the following definitions of (p,q)-th order and lower (p,q)-th order of functions of several complex variables analytic in the unit polydisc and prove a similar analytic expression.

Definition 1.7. Let $f(z_1, z_2, \dots z_n) = \sum_{m_1,m_2,\dots m_n=0}^{\infty} c_{m_1 m_2 \dots\, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$ be a function of n complex variables $z_1,\ z_2, \dots z_n$ holomorphic in the unit polydisc

$$U = \{(z_1, z_2, \dots z_n) : |z_j| \leq 1; j = 1, 2, \dots n\}$$

and

$$F(r_1, r_2, \dots r_n) = \max\{|f(z_1, z_2, \dots z_n)| : |z_j| \leq r_j; j = 1, 2, \dots n\},$$

be its maximum modulus. Then the (p,q)-th order ρ_q^p and the lower (p,q)-th order λ_q^p are define as

$$\begin{matrix}\rho_q^p \\ \lambda_q^p\end{matrix} = \lim_{r_1,r_2,\dots r_n \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} F(r_1, r_2, \dots r_n)}{\log^{[q]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)},\ p \geq q+1 \geq 2.$$

Note 1.3. When $q = 1$, Definition 1.7 accords with the Definition 1.6 and if $n = 2$, Definition 1.7 coincides with Definition 1.4.

Here $f(z_1, z_2, \dots z_n) = \sum_{m_1,m_2,\dots m_n=0}^{\infty} c_{m_1 m_2 \dots m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$ denotes a function of several complex variables analytic in the unit polydisc U. We do not explain the standard notations and definitions of the theory of analytic functions as available in [7, 13] and [14].

2. Lemmas

The following lemmas will be needed in the rest of the paper.

Lemma 2.1. *Let the maximum modulus $F(r_1, r_2, \dots r_n)$ of a function $f(z_1, z_2, \dots z_n)$ analytic in U, satisfy*

$$\log^{[p-1]} F(r_1, r_2, \dots r_n) < \left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^A \tag{2.1}$$

$0 < A < \infty$ *for all* r_j *such that* $r_0(A) < r_j < 1; j = 1, 2, \ldots n$. *Then for all* $m_j > m_{j_0}(A) > 1; j = 1, 2, \ldots n$,

$$\log^{[p-1]} |c_{m_1 m_2 \ldots m_n}| \leq [n + 1 + O(1)] \left(\log^{[q-1]} \left(\prod_{j=1}^{n} m_j \right) \right)^{\frac{A}{A+1}}.$$

Proof. Define n sequences $\{r_{jm_j}\}$ by

$$(1 - r_{jm_j})^{-1} = \exp^{[q-1]} \left\{ \left(\log^{[q-1]} m_j \right)^{\frac{1}{n(A+1)}} \right\}; j = 1, 2, \ldots n.$$

Then $r_{jm_j} \to 1$ as $m_j \to \infty$; for all $j = 1, 2, \ldots n$. From Cauchy's inequality,

$$\begin{aligned}
|c_{m_1 m_2 \ldots m_n}| &= \frac{1}{\prod_{j=1}^{n} (m_j!)} \left| \frac{\partial^{m_1 + m_2 + \ldots + m_n} f(0, 0, \ldots 0)}{\partial z_1^{m_1} \partial z_2^{m_2} \ldots \partial z_n^{m_n}} \right| \\
&= \left| \frac{1}{(2\pi i)^n} \int_{|z_1| = r_1} \int_{|z_2| = r_2} \cdots \int_{|z_n| = r_n} \frac{f(z_1, z_2, \ldots z_n) dz_1 dz_2 \ldots dz_n}{z_1^{m_1+1} z_2^{m_2+1} \ldots z_n^{m_n+1}} \right| \\
&\leq \frac{F(r_1, r_2, \ldots r_n)}{r_1^{m_1} r_2^{m_2} \ldots r_n^{m_n}} \\
&= \frac{F(r_1, r_2, \ldots r_n)}{\prod_{j=1}^{n} r_j^{m_j}}. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2) we have for all $m_j > m_{j_0}(A) > 1$; $j = 1, 2, \ldots n$,

$$\begin{aligned}
\log |c_{m_1 m_2 \ldots m_n}| &\leq \log F(r_{1m_1}, r_{2m_2}, \ldots r_{nm_n}) - \sum_{j=1}^{n} m_j \log r_{jm_j} \\
&< \exp^{[p-2]} \left\{ \log^{[q-1]} \left(\frac{1}{\prod_{j=1}^{n} (1 - r_{jm_j})} \right) \right\}^A + \left[\sum_{j=1}^{n} m_j (1 - r_{jm_j}) \right] [1 + O(1)] \\
&= \exp^{[p-2]} \left[\log^{[q-1]} \left\{ \prod_{j=1}^{n} \left(\exp^{[q-1]} \left(\log^{[q-1]} m_j \right)^{\frac{1}{n(A+1)}} \right) \right\} \right]^A \\
&\quad + \left[\sum_{j=1}^{n} \frac{m_j}{\exp^{[q-1]} \left\{ \left(\log^{[q-1]} m_j \right)^{\frac{1}{n(A+1)}} \right\}} \right] [1 + O(1)].
\end{aligned}$$

$$\therefore \log|c_{m_1m_2\ldots m_n}| \le \exp^{[p-2]}\left[\log^{[q-1]}\left\{\exp^{[q-1]}\left(\prod_{j=1}^{n}\left(\log^{[q-1]} m_j\right)\right)^{\frac{1}{n(A+1)}}\right\}\right]^{A}$$
$$+\left[\sum_{j=1}^{n}\frac{m_j}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} m_j\right)^{\frac{1}{n(A+1)}}\right\}}\right][1+O(1)]$$
$$\le \exp^{[p-2]}\left(\log^{[q-1]}\left(\prod_{j=1}^{n} m_j\right)\right)^{\frac{A}{A+1}}$$
$$+\left[\sum_{j=1}^{n}\frac{m_j}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} m_j\right)^{\frac{1}{n(A+1)}}\right\}}\right][1+O(1)]$$
$$\le \left[\exp^{[p-2]}\left(\log^{[q-1]}\left(\prod_{j=1}^{n} m_j\right)\right)^{\frac{A}{A+1}}\right][n+1+O(1)]$$
$$\le \exp^{[p-2]}\left\{[n+1+O(1)]\left(\log^{[q-1]}\left(\prod_{j=1}^{n} m_j\right)\right)^{\frac{A}{A+1}}\right\}.$$

Therefore

$$\log^{[p-1]}|c_{m_1m_2\ldots m_n}| \le [n+1+O(1)]\left(\log^{[q-1]}\left(\prod_{j=1}^{n} m_j\right)\right)^{\frac{A}{A+1}}.$$

This proves the lemma. □

Lemma 2.2. *Let $f(z_1, z_2, \ldots z_n)$ be analytic in U and satisfy*

$$\log^{[p-1]}|c_{m_1m_2\ldots m_n}| < \prod_{j=1}^{n}\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m_j\right)^{D}\right\}\right],$$

$0 < C < \infty,\ 0 < D < 1$, *for all $m_j > m_{j_0}(C, D); j = 1, 2, \ldots n$. Then for all r_j such that $r_{j0}(C, D) < r_j < 1; j = 1, 2, ...n$,*

$$\log^{[p-1]} F(r_1, r_2, \ldots r_n) < T(C, D)\left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^{\frac{D}{1-D}},$$

where

$$T(C, D) = C^{\frac{n}{1-D}} D^{\frac{nD}{1-D}}[2 + o(1)].$$

Proof. For all $m_j > m_{j_0}(C, D); j = 1, 2, \dots n$,

$$\log^{[p-1]} |c_{m_1 m_2 \dots m_n}| < \prod_{j=1}^{n} \left[\exp^{[p-1]} \left\{ C \left(\log^{[q-1]} m_j \right)^D \right\} \right].$$

Now for $|z_j| = r_j < 1; j = 1, 2, \dots n$, we have

$$\begin{aligned} F(r_1, r_2, \dots r_n) &< \sum_{m_1, m_2, \dots m_n = 0}^{\infty} \left| c_{m_1 m_2 \dots m_n} \right| r_1^{m_1} r_2^{m_2} \dots r_n^{m_n} \\ &< K(m_{1_0}, m_{2_0}, \dots m_{n_0}) + \sum_{\substack{m_1 = m_{1_0} + 1 \\ m_2 = m_{2_0} + 1 \\ \vdots \\ m_n = m_{n_0} + 1}}^{\infty} \left\{ \prod_{j=1}^{n} \exp^{[p-1]} \left(C m_j^D \right) r_j^{m_j} \right\} \\ &\leq K(m_{1_0}, m_{2_0}, \dots m_{n_0}) + \prod_{j=1}^{n} \left[\sum_{m_j = m_{j_0} + 1}^{\infty} \exp^{[p-1]} \left(C m_j^{\frac{B}{B+1}} \right) r_j^{m_j} \right], \end{aligned}$$

where $B = \frac{D}{1-D}$. Choose

$$M_j = M(r_j) = \left[\exp^{[q-1]} \left(\frac{2^{2p-3} C}{\log^{*(p-2)} \left(\log \frac{1}{r_j} \right)} \right)^{B+1} \right]; j = 1, 2, \dots n$$

where $[x]$ denotes the greatest integer not greater then x. Clearly $M(r_j) \to \infty$ as $r_j \to 1$for $j = 1, 2, \dots n$. The above estimate of $F(r_1, r_2, \dots r_n)$ for all r_j; $j = 1, 2, \dots n$ sufficiently close to 1 gives,

$$F(r_1, r_2, \dots r_n) < K(m_{1_0}, m_{2_0}, \dots\ m_{n_0}) + \prod_{j=1}^{n} \left[M(r_j) H(r_j) + \sum_{m_j = M_j + 1}^{\infty} r_j^{m_j/2} \right] \tag{2.3}$$

where

$$H(r_j) = \max_{m_j} \left\{ \exp^{[p-1]} \left(C \left(\log^{[q-1]} m_j \right)^{\frac{B}{B+1}} \right) r_j^{m_j} \right\}; j = 1, 2, \dots n$$

for if $m_j \geq M_j + 1$, then

$$m_j > \ \exp^{[q-1]} \left(\frac{2^{2p-3} C}{\log^{*(p-2)} \left(\log \frac{1}{r_j} \right)} \right)^{B+1}.$$

So

$$C\left(\log^{[q-1]} m_j\right)^{\frac{B}{B+1}} < \frac{\log^{[q-1]} m_j}{2^{2p-3}} \ \log^{*(p-2)}\left(\log \frac{1}{r_j}\right)$$
$$\leq \frac{m_j}{2^{2p-3}} \ \log^{*(p-2)}\left(\log \frac{1}{r_j}\right)$$
$$= \log\left[1 + \log^{*(p-3)}\left(\log \frac{1}{r_j}\right)\right]^{\frac{m_j}{2^{2p-3}}}$$
$$\leq \log\left[1 + \frac{m_j}{2^{2p-4}} \ \log^{*(p-3)}\left(\log \frac{1}{r_j}\right)\right].$$

Hence

$$\exp\left\{C\left(\log^{[q-1]} m_j\right)^{\frac{B}{B+1}}\right\} \leq 1 + \frac{m_j}{2^{2p-4}} \ \log^{*(p-3)}\left(\log \frac{1}{r_j}\right)$$
$$\leq \frac{m_j}{2^{2p-5}} \ \log^{*(p-3)}\left(\log \frac{1}{r_j}\right)$$
$$\leq \log\left[1 + \frac{m_j}{2^{2p-6}} \ \log^{*(p-4)}\left(\log \frac{1}{r_j}\right)\right].$$

Therefore

$$\exp^{[2]}\left\{C\left(\log^{[q-1]} m_j\right)^{\frac{B}{B+1}}\right\} \leq 1 + \frac{m_j}{2^{2p-6}} \ \log^{*(p-4)}\left(\log \frac{1}{r_j}\right)$$
$$\leq \frac{m_j}{2^{2p-7}} \ \log^{*(p-4)}\left(\log \frac{1}{r_j}\right).$$

Taking repeated exponential, we obtain

$$\exp^{[p-2]}\left\{C \ \left(\log^{[q-1]} m_j\right)^{\frac{B}{B+1}}\right\} < \frac{m_j}{2} \ \log \frac{1}{r_j}$$

i.e.

$$\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m_j\right)^{\frac{B}{B+1}}\right\} r_j^{m_j} < r_j^{\frac{m_j}{2}}$$

for all $j = 1, 2, \dots n$. Therefore the infinite series $\sum_{m_j=M_j+1}^{\infty} r_j^{\frac{m_j}{2}}$ in (2.3) is bounded by $r_j^{\frac{M_j+1}{2}} \left(\frac{1}{1-r_j^{\frac{1}{2}}} \right)$ for all $j = 1, 2, \dots n$. Since $B > 0$, we have

$$
\begin{aligned}
-\frac{M_j+1}{2} \log \frac{1}{r_j} - \log\left(1 - r_j^{\frac{1}{2}}\right) &< -\frac{1}{2}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}\left(\log \frac{1}{r_j}\right)}\right)^{B+1} \log \frac{1}{r_j} \\
&\quad - \log(1-r_j) + \log\left(1 + r_j^{\frac{1}{2}}\right) \\
&< -\frac{1}{2}\left(\frac{2^{2p-3}C}{\log \frac{1}{r_j}}\right)^{B+1} \log \frac{1}{r_j} - \log(1-r_j) + \log\left(1 + r_j^{\frac{1}{2}}\right) \\
&\to -\infty \text{ as } r_j \to 1.
\end{aligned}
$$

Thus for r_j sufficiently close to 1, $\sum_{m_j=M_j+1}^{\infty} r_j^{m_j/2} = o(1)$ for all $j = 1, 2, \dots .n$. The maximum of $\exp^{[p-1]}\left\{ C \left(\log^{[q-1]} m_j \right)^{\frac{B}{B+1}} \right\} r_j^{m_j}$ assume at the point

$$
m_j = \exp^{[q-1]}\left\{ \frac{BC}{B+1} \log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B+1}{n}}
$$

and $H(r_j)$ is given by

$$
\begin{aligned}
\log H(r_j) &= \exp^{[p-2]}\left\{ C \left(\log^{[q-1]} m_j \right)^{\frac{B}{B+1}} \right\} + m_j \log r_j \\
&= \exp^{[p-2]}\left[\frac{C.B^B.C^B}{(B+1)^B} \left\{ \log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B}{n}} \right] \\
&\quad - \exp^{[q-1]}\left\{ \frac{BC}{B+1} \log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B+1}{n}} \log \frac{1}{r_j}
\end{aligned}
$$

$$
\text{(2.4)} \qquad \le \exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B} \left\{ \log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B}{n}} \right].
$$

Thus for $r_j; j = 1, 2, \ldots .n$ sufficiently close to 1, from (2.3)

$$F(r_1, r_2, \ldots r_n) < \prod_{j=1}^{n}\left[M(r_j)H(r_j) + o(1)\right]\left[1 + \frac{K(m_{1_0}, m_{2_0}, \ldots m_{n_0})}{\prod_{j=1}^{n}\left[M(r_j)H(r_j) + o(1)\right]}\right]$$

$$= \prod_{j=1}^{n}\left[M(r_j)H(r_j) + o(1)\right][1 + O(1)].$$

Therefore

$$\log F(r_1, r_2, \ldots r_n) < \sum_{j=1}^{n}\{\log M(r_j) + logH(r_j)\} + O(1)$$

$$\leq \sum_{j=1}^{n}\exp^{[q-2]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_j})}\right)^{B+1}$$

$$+ \sum_{j=1}^{n}\exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B}{n}}\right] + O(1)$$

[from (2.4)]

$$\leq 2\sum_{j=1}^{n}\exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B}{n}}\right] + O(1)$$

$$\leq \exp^{[p-2]}\left[\frac{C^{n(B+1)}.B^{nB}}{(B+1)^{nB}}\prod_{j=1}^{n}\left\{\log^{[q-1]}\left(\frac{1}{1-r_j}\right)\right\}^{\frac{B}{n}}\right][2 + o(1)]$$

$$\leq \exp^{[p-2]}\left[\frac{C^{n(B+1)}.B^{nB}}{(B+1)^{nB}}\left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^{B}\right][2 + o(1)].$$

i.e.

$$\log^{[p-1]} F(r_1, r_2, \ldots\ r_n) \leq \frac{C^{n(B+1)}.B^{nB}}{(B+1)^{nB}}[2 + o(1)]\left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^{B}$$

$$= C^{\frac{n}{1-D}}D^{\frac{nD}{1-D}}[2 + o(1)]\left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^{\frac{D}{1-D}}$$

$$= T(C,D)\left\{\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)\right\}^{\frac{D}{1-D}}$$

where

$$T(C,D) = C^{\frac{n}{1-D}} D^{\frac{nD}{1-D}}[2+o(1)].$$

This proves the lemma. □

3. New result

In this section, we prove the following theorem.

Theorem 3.1. *Let $f(z_1, z_2 \dots z_n)$ be analytic in U and having the (p,q)-th order ρ_q^p $(0 \le \rho_q^p \le \infty)$. Then*

$$\frac{\rho_q^p}{1+\rho_q^p} = \lim \sup_{m_1,m_2,\dots\, m_n\to\infty} \frac{\log^{+[p]}|c_{m_1m_2\dots m_n}|}{\log^{[q]}\left(\prod_{j=1}^{n} m_j\right)}. \tag{3.1}$$

Proof. If $|c_{m_1m_2\dots m_n}|$ is bounded by K for all $m_j; j = 1,2,\dots n$ then the sum $\sum_{m_1,m_2,\dots m_n=0}^{\infty} c_{m_1m_2\dots\, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$ is bounded by $\frac{K}{\prod_{j=1}^{n}(1-r_j)}$. Therefore

$$\begin{aligned}
F(r_1, r_2, \dots r_n) &\le \sum_{m_1,m_2,\dots m_n=0}^{\infty} \left|c_{m_1m_2\dots m_n}\right| r_1^{m_1} r_2^{m_2} \dots r_n^{m_n} \\
&\le \frac{K}{\prod_{j=1}^{n}(1-r_j)} \\
&< \exp^{[p-1]}\left[\log^{[q-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)^{\epsilon}\right] \text{for } p \ge q+1
\end{aligned}$$

for any $0 < \epsilon < 1$ and $r_j; j = 1,2,\dots n$ sufficiently close to 1.

Therefore

$$\rho_q^p = \limsup_{r_1,r_2\dots r_n\to 1} \frac{\log^{[p]} F(r_1, r_2, \dots r_n)}{\log^{[q]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)} \le \epsilon$$

since $0 < \epsilon < 1$ arbitrary, $\rho_q^p = 0$ and so (3.1) is satisfied. Thus we need to consider only the case

$$\limsup_{m_1,m_2,\dots m_n\to\infty} |c_{m_1m_2\dots\, m_n}| = \infty.$$

In this regard, all the $\log^+$ in (3.1) may be replaced by log. First let $0 < \rho_q^p < \infty$. Then for all $r_j; j = 1, 2, \ldots n$ sufficiently close to 1 and for arbitrary $\varepsilon > 0$, we get from the definition of (p,q)-th order,

$$\log^{[p-1]} F(r_1, r_2, \ldots r_n) \leq \left\{ \log^{[q-1]} \left(\frac{1}{\prod_{j=1}^{n} (1 - r_j)} \right) \right\}^{\rho_q^p + \varepsilon}$$
$$= \left\{ \log^{[q-1]} \left(\frac{1}{\prod_{j=1}^{n} (1 - r_j)} \right) \right\}^{\mu},$$

where $\mu = \rho_q^p + \varepsilon$.

Using Lemma 2.1 with $A = \mu$ it follows the above inequality that for $m_j > m_{j_0}(\mu)$; $j = 1, 2, \ldots n$,

$$\log^{[p-1]} |c_{m_1 m_2 \ldots m_n}| \leq [n + 1 + O(1)] \left(\log^{[q-1]} \left(\prod_{j=1}^{n} m_j \right) \right)^{\frac{\mu}{\mu+1}}$$
$$\log^{[p]} |c_{m_1 m_2 \ldots m_n}| \leq \log[n + 1 + O(1)] + \frac{\mu}{\mu + 1} \log^{[q]} \left(\prod_{j=1}^{n} m_j \right).$$

Therefore,

$$\limsup_{m_1, m_2, \ldots m_n \to \infty} \frac{\log^{[p]} |c_{m_1 m_2 \ldots m_n}|}{\log^{[q]} \left(\prod_{j=1}^{n} m_j \right)} \leq \frac{\mu}{1 + \mu}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim \sup_{m_1, m_2, \ldots m_n \to \infty} \frac{\log^{[p]} |c_{m_1 m_2 \ldots m_n}|}{\log^{[q]} \left(\prod_{j=1}^{n} m_j \right)} \leq \frac{\rho_q^p}{1 + \rho_q^p}. \tag{3.2}$$

Since f is analytic in U, the above inequality is trivially true if $\rho_q^p = \infty$ and the right hand side is interpreted as 1 in this case. Conversely, if

$$\theta = \limsup_{m_1, m_2, \ldots m_n \to \infty} \frac{\log^{[p]} |c_{m_1 m_2 \ldots m_n}|}{\log^{[q]} \left(\prod_{j=1}^{n} m_j \right)}$$

then $0 \leq \theta \leq 1$. First let $\theta < 1$ and choose $\theta < \theta' < 1$. Then for all sufficiently large $m_j; j = 1, 2, \ldots n$

$$\log^{[p-1]} |c_{m_1 m_2 \ldots m_n}| \leq \left(\log^{[q-1]} \left(\prod_{j=1}^{n} m_j \right) \right)^{\theta'}.$$

Using Lemma 2.2 with $C = 1,\ D = \theta^{'}$, it follows from the above inequality that for all r_j such that $r_0(\theta^{'}) < r_j < 1; j = 1, 2, \ldots n$,

$$\log^{[p-1]} F(r_1, r_2, \ldots r_n) \leq \theta^{'\frac{n\theta^{'}}{1-\theta^{'}}} \left\{ \log^{[q-1]} \left(\frac{1}{\prod_{j=1}^{n}(1-r_j)} \right) \right\}^{\frac{\theta^{'}}{1-\theta^{'}}} [2 + o(1)].$$

Therefore,

$$\log^{[p]} F(r_1, r_2, \ldots r_n) \leq \frac{n\theta^{'}}{1-\theta^{'}}\ \log(\theta^{'}) + \frac{\theta^{'}}{1-\theta^{'}} \log\left\{ \log^{[q-1]} \left(\frac{1}{\prod_{j=1}^{n}(1-r_j)} \right) \right\} + \log[2+o(1)]$$

i.e.

$$\lim \sup_{r_1, r_2 \ldots r_n \to 1} \frac{\log^{[p]} F(r_1, r_2, \ldots r_n)}{\log^{[q]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)} \right)} \leq \frac{\theta^{'}}{1-\theta^{'}} \lim \sup_{r_1, r_2 \ldots r_n \to 1} \frac{\log^{[q]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)} \right)}{\log^{[q]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)} \right)}.$$

Therefore,

$$\rho_q^p \leq \frac{\theta^{'}}{1-\theta^{'}}.$$

Since $\theta^{'} > \theta$ is arbitrary, it follows that

$$\frac{\rho_q^p}{1+\rho_q^p} \leq \theta = \lim \sup_{m_1, m_2, \ldots m_n \to \infty} \frac{\log^{[p]} |c_{m_1 m_2 \ldots m_n}|}{\log^{[q]}\left(\prod_{j=1}^{n} m_j \right)}. \tag{3.3}$$

If $\theta = 1$, the above inequality is obviously true. Inequality (3.2) and (3.3) together give (3.1) when

$$\lim \sup_{m_1, m_2, \ldots m_n \to \infty} |c_{m_1 m_2 \ \ldots m_n}| = \infty.$$

This proves the theorem. □

Conjecture 3.2. Is it possible to prove similar result for lower (p, q)-th order of a function analytic in a unit polydisc?

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