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DERIVED GRAPHS OF SUBDIVISION GRAPHS

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Abstract. The derived graph $[G]^\dagger$ of a graph G is the graph having the same vertex set as G, two vertices of $[G]^\dagger$ being adjacent if and only if their distance in G is two. In this paper the derived graphs of the subdivision graphs, their spectra and energies are determined.

1. Introduction

Let G be an undirected, simple graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, \dots, v_n$. The *distance* $d_G(v_i, v_j)$ between the vertices v_i and v_j is the length of a shortest path between them. If there is no path between v_i and v_j then we formally assume that $d_G(v_i, v_j) = \infty$.

The *adjacency matrix* $A(G) = [a_{ij}]$ of the graph G is the square matrix of order n in which $a_{ij} = 1$ if v_i is adjacent to v_j , and $a_{ij} = 0$ otherwise. The characteristic polynomial $\phi(G, \lambda) = \det(\lambda I - A(G))$ is the *characteristic polynomial* of G. The eigenvalues of $A(G)$, denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$, are said to be the *eigenvalues* of the graph G and to form its *spectrum* [3].

The energy $E(G)$ of a graph G is defined as [4]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

Details on graphs energy are found in the recent monograph [6].

If G_1 and G_2 are two graphs with disjoint vertex sets, then their union, denoted by $G_1 \cup G_2$, is the graph whose vertex set is the union of the vertex sets of G_1 and G_2 ,

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and whose edge set is the union of the edge sets of G_1 and G_2. Evidently,

$$\phi(G_1 \cup G_2, \lambda) = \phi(G_1, \lambda)\phi(G_2, \lambda) \tag{1.1}$$

and therefore

$$E(G_1 \cup G_2) = E(G_1) + E(G_2) \ . \tag{1.2}$$

If $G = H_1 \cup H_2 \cup \cdots \cup H_p$ and $H_1 \cong H_2 \cong \cdots \cong H_p \cong H$, then we write $G = p\,H$.

Definition 1.1. Let G be a simple graph. Its derived graph $[G]^\dagger$ is the graph whose vertices are same as the vertices of G and two vertices in $[G]^\dagger$ are adjacent if and only if the distance between them in G is two.

Directly from this definition follows that $[G_1 \cup G_2]^\dagger = [G_1]^\dagger \cup [G_2]^\dagger$.

Spectra and energy of derived graphs of some graphs were earlier established in [1, 2, 5]. We now continue these studies by obtaining expressions for the derived graphs of subdivision graphs, their spectra, and energies.

2. Derived graphs of subdivision graphs

The ordinary *subdivision graph* $S(G)$ of the graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G. For $k \geq 1$, the k-th subdivision graph $S_k(G)$ is obtained from G by inserting k new vertices of degree 2 on each edge of G. Thus $S_0(G) \cong G$ and $S_1(G) \cong S(G)$.

For $k \geq 1$, $S_k(G_1 \cup G_2) = S_k(G_1) \cup S_k(G_2)$.

The *line graph* $L(G)$ of G is the graph whose vertices are in one-to-one correspondence with the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G share a common vertex. For $k \geq 1$, the k-th iterated line graph of G is $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$.

For $k \geq 1$, $L^k(G_1 \cup G_2) = L^k(G_1) \cup L^k(G_2)$.

Theorem 2.1. *Let G be any simple graph. Then $[S(G)]^\dagger \cong G \cup L(G)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G, and let $u_1, u_2, \ldots, u_m$ be the vertices of $S(G)$ inserted on the edges of G.

Two vertices v_i and v_j of $S(G)$ are at distance two if and only if v_i and v_j are adjacent in G. Therefore, the vertices $v_1, v_2, \ldots, v_n$ induce a subgraph of $[S(G)]^\dagger$ isomorphic to G.

Two vertices u_i and u_j of $S(G)$ are at distance two if and only if they are inserted on incident edges of G. Therefore, the vertices $u_1, u_2, \ldots, u_m$ induce a subgraph of $[S(G)]^\dagger$ isomorphic to $L(G)$.

Theorem 2.1 follows now from the fact that no two vertices v_i and u_j of $S(G)$ are at distance two. □

Theorem 2.2. *Let G be any simple graph. Let H_0 be its ordinary subdivision graph, and let $H_{k+1} = S([H_k]^\dagger)$ for $k = 0, 1, 2, \ldots$. Then for $k = 1, 2, \ldots,$*

$$[H_{k-1}]^\dagger = G \cup \left\{ \bigcup_{i=1}^{k} \binom{k}{i} L^i(G) \right\}.$$

Proof. We prove Theorem 2.2 by induction on k. For $k = 1$, from Theorem 2.1,

$$[H_0]^\dagger = [S(G)]^\dagger = G \cup L(G) \ .$$

Assume that for $k \geq 2$,

$$[H_{k-2}]^\dagger = G \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k-1}{i} L^i(G) \right\}.$$

Then,

$$\begin{aligned}
[H_{k-1}]^\dagger &= [S([H_{k-2}]^\dagger)]^\dagger = [H_{k-2}]^\dagger \cup L([H_{k-2}]^\dagger) \\
&= \left\{ G \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k-1}{i} L^i(G) \right\} \right\} \cup \left\{ L \left\{ G \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k-1}{i} L^i(G) \right\} \right\} \right\} \\
&= \left\{ G \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k-1}{i} L^i(G) \right\} \right\} \cup \left\{ L(G) \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k-1}{i} L^{i+1}(G) \right\} \right\} \\
&= G \cup \left\{ \bigcup_{i=1}^{k-1} \left(\binom{k-1}{i} + \binom{k-1}{i-1} \right) L^i(G) \right\} \cup L^k(G) \\
&= G \cup \left\{ \bigcup_{i=1}^{k-1} \binom{k}{i} L^i(G) \right\} \cup L^k(G) = G \cup \left\{ \bigcup_{i=1}^{k} \binom{k}{i} L^i(G) \right\}.
\end{aligned}$$

□

Theorem 2.3. *Let G be any simple graph. Then $[S_{2k+1}(G)]^\dagger \cong S_k(G) \cup L(S_k(G))$ holds for all $k \geq 0$.*

Proof. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$. For each edge $e_i = uv$ of G, there are $2k+1$ newly added vertices in $S_{2k+1}(G)$. For the edge $e_i = uv$ of G, let $u_i^1, u_i^2, \ldots, u_i^{2k+1}$ be the subdivision vertices on this edge in $S_{2k+1}(G)$, where u is adjacent to u_i^1, u_i^ℓ is adjacent to $u_i^{\ell+1}$, $\ell = 1, 2, \ldots, 2k$, and u_i^{2k+1} is adjacent to v.

Therefore $d_{S_{2k+1}(G)}(u, u_i^2) = 2$, $d_{S_{2k+1}(G)}(u_i^\ell, u_i^{\ell+2}) = 2$, $d_{S_{2k+1}(G)}(u_i^{2k}, v) = 2$, $i = 1, 2, \ldots, m$ and $\ell = 1, 2, \ldots, 2k$. Thus the pair of vertices which are at distance two in $S_{2k+1}(G)$ form $S_k(G)$ in $[S_{2k+1}(G)]^\dagger$.

Consider any two edges $e_i = uv$ and $e_j = xy$ of G. Let the vertices on e_i in $S_{2k+1}(G)$ be $u_i^1, u_i^2, \ldots, u_i^{2k+1}$ in that order, and let the vertices on e_j in $S_{2k+1}(G)$ be $u_j^1, u_j^2, \ldots, u_j^{2k+1}$. If the edges e_i and e_j are adjacent in G, say $u = x$, then $d_{S_{2k+1}(G)}(u_i^1, u_j^1) = 2$, and $d_{S_{2k+1}(G)}(u_i^\ell, u_i^{\ell+2}) = 2$, $\ell = 1, 2, \ldots, 2k-1$, $i, j = 1, 2, \ldots, m$. Hence these pairs of vertices form $L(S_k(G))$ in $[S_{2k+1}(G)]^\dagger$. No vertex of $S_k(G)$ is adjacent to the vertex of $L(S_k(G))$ in $[S_{2k+1}(G)]^\dagger$. Therefore $[S_{2k+1}(G)]^\dagger \cong S_k(G) \cup L(S_k(G))$. □

The *degree* $\deg_G(v)$ of a vertex v of the graph G is the number of edges incident to it.

Theorem 2.4. *Let G be a graph of order n, with m edges, and with degree sequence $d_1, d_2, \ldots, d_n$. Then for $k \geq 1$, the degree sequence of $[S_{2k}(G)]^\dagger$ is d_i ($d_i + 1$ times), $i = 1, 2, \ldots, n$ and 2 ($2m(k-1)$ times).*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and let $\deg_G(v_i) = d_i$, $i = 1, 2, \ldots, n$. For each v_i there are d_i edges incident to it. Let $e_1, e_2, \ldots, e_{d_i}$ be the edges incident to a vertex v_i in G. For the edge $e_i = uv$, let $u_i^1, u_i^2, \ldots, u_i^{2k}$ be the subdivision vertices on this edge in $S_{2k}(G)$, where u_i^1 is adjacent to u, u_i^ℓ is adjacent to $u_i^{\ell+1}$, $\ell = 1, 2, \ldots, 2k-1$ and u_i^{2k} is adjacent to v.

In $[S_{2k}(G)]^\dagger$, the vertices u_i^1 and u_j^1 are adjacent to each other, $i, j = 1, 2, \ldots, d_i$ and u_i^1 is adjacent to u_i^3 , $i = 1, 2, \ldots, d_i$.

Hence the degree of u_i^1 in $[S_{2k}(G)]^\dagger$ is d_i . Also, v_i is adjacent to u_i^2, $i = 1, 2, \ldots, d_i$. Hence the degree of v_i in $[S_{2k}(G)]^\dagger$ is d_i .

Therefore, in $[S_{2k}(G)]^\dagger$ there are $d_i + 1$ vertices of degree d_i , $i = 1, 2, \ldots, n$.

The number of vertices of $[S_{2k}(G)]^\dagger$ is $n + 2mk$. The remaining

$$n + 2mk - \sum_{i=1}^{n} (d_i + 1) = n + 2mk - (2m + n) = 2m(k-1)$$

vertices are of degree 2. □

3. Energy of derived graphs

Results stated in this section are straightforward consequences of the Eqs. (1.1) and (1.2) and the Theorems 2.1-2.3.

Theorem 3.1. *Using the same notation as in Theorem 2.1,*

$$\phi([S(G)]^\dagger, \lambda) = \phi(G, \lambda)\, \phi(L(G), \lambda)$$

$$E([S(G)]^\dagger) = E(G) + E(L(G)).$$

Theorem 3.2. *Using the same notation as in Theorem 2.2, for $k \geq 1$,*

$$\phi([H_{k-1}]^\dagger, \lambda) = \phi(G, \lambda) \prod_{i=1}^{k} \left(\phi(L^i(G), \lambda)\right)^{\binom{k}{i}}$$

$$E([H_{k-1}]^\dagger) = E(G) + \sum_{i=1}^{k} \binom{k}{i} E(L^i(G)).$$

Theorem 3.3. *Using the same notation as in Theorem 2.3,*

$$\phi([S_{2k+1}(G)]^\dagger, \lambda) = \phi(S_k(G), \lambda)\, \phi(L(S_k(G)), \lambda)$$

$$E([S_{2k+1}(G)]^\dagger) = E(S_k(G)) + E(L(S_k(G))).$$

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