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A NEW OSTROWSKI-GRÜSS TYPE INEQUALITY

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Abstract. In the present note, we establish a similar inequality of Ostrowski-Grüss type for functions whose first derivative is absolutely continuous and second derivative is bounded both above and below almost everywhere.

1. Introduction

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss' inequality (see [15, p. 296]).

Theorem 1.1. *Let $f, g : [a, b] \to R$ be two integrable functions such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, ϕ, Φ, γ and Γ are constants. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi-\phi),$$

where the constant $\frac{1}{4}$ is sharp.

Another celebrated integral inequality which provides an approximation of the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ in terms of the values of f at a certain point $x \in [a, b]$, is Ostrowski's inequality [16, p. 468].

Theorem 1.2. *Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b), whose derivative $f' : [a, b] \to R$ is bounded on (a, b), i.e.*

$$\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty.$$

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Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f^{'}\|_\infty$$

for all $x \in [a, b]$*. The constant* $\frac{1}{4}$ *is sharp.*

In the years thereafter, numerous generalizations, extensions and variants of Grüss inequality have appeared in the literature (see [4–12]). Dragomir and Wang [8] proved the following Ostrowski type inequality in terms of the lower and upper bounds of the first derivative.

Theorem 1.3. *Let* $f : [a, b] \to R$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*, whose derivative satisfies the condition:* $\gamma \leq f^{'}(x) \leq \Gamma$ *for all* $x \in (a, b)$*. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma)$$

for all $x \in [a, b]$*.*

In Theorem 1.3, if we take $x = a$, then it follows that

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) + f(a)}{2} \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma). \tag{1.1}$$

Cerone et al. [2] obtained the following result for twice differentiable mappings in terms of the upper and lower bounds of the second derivative.

Theorem 1.4. *Let* $f : [a, b] \to R$ *be continuous on* $[a, b]$ *and twice differentiable on* (a, b)*, and assume that the second derivative* $f^{''} : (a, b) \to R$ *satisfies the condition* $\phi \leq f^{''}(x) \leq \Phi$ *for all* $x \in (a, b)$*. If we denote* $t = \frac{a+b}{2}$ *then we have the inequality*

$$\begin{aligned} &\left| f(x) - (x-t) f^{'}(x) - \frac{1}{b-a} \int_a^b f(t)dt + \left[\frac{(b-a)^2}{24} + \frac{1}{2}(x-t)^2 \right] \frac{f^{'}(b) - f^{'}(a)}{b-a} \right| \\ &\leq \frac{1}{8}(\Phi - \phi) \left[\frac{1}{2}(b-a) + |x-t| \right]^2 \end{aligned}$$

for all $x \in [a, b]$*.*

Motivated by the works of Cerone et al. [2], the purpose of the present note is to establish a similar inequality of Ostrowski-Grüss type to the inequality in Theorem 1.4 for functions whose first derivative is absolutely continuous and second derivative is bounded both above and below almost everywhere. From our inequality, we can obtain an analogue (see Remark 2.1) to (1.1).

2. Main results

Theorem 2.1. *Let $f : [a,b] \to (-\infty,\infty)$ be a function such that the derivative f' is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in (-\infty,\infty)$ such that $\gamma \le f''(x) \le \Gamma$ a.e. on $[a,b]$. Then we have*

$$\Big|[f'(b)+f'(a)]\Big[\frac{1}{6}(b+a)+\frac{1}{4}\Big]+\frac{bf'(b)+af'(a)}{6}+\frac{f(a)(1+2a)-f(b)(1+2b)}{2(b-a)}$$

$$+\frac{1}{b-a}\int_a^b f(x)dx\Big| \le \begin{cases} \frac{\Gamma-\gamma}{2(b-a)}\left(\frac{1}{6}+\frac{2}{3}C\right)\sqrt{1+4C}, & -\frac{3}{2}-2b \le a \le -\frac{3}{4}-\frac{b}{2}; \\ -\frac{\Gamma-\gamma}{2(b-a)}\int_a^{x_2}(x^2+x-C)dx, & -\frac{3}{4}-\frac{b}{2} \le a \le b; \\ -\frac{\Gamma-\gamma}{2(b-a)}\int_{x_1}^{b}(x^2+x-C)dx, & a \le -\frac{3}{2}-2b, \end{cases}$$

where

$$C := \frac{1}{b-a}\int_a^b (x+x^2)dx = \frac{1}{2}(b+a)+\frac{1}{3}(b^2+ab+a^2),$$

and

$$x_1 = \frac{-1-\sqrt{1+4C}}{2}, \quad x_2 = \frac{-1+\sqrt{1+4C}}{2}.$$

Proof. It is not difficult to check that

$$\begin{aligned} &[f'(b)+f'(a)]\Big[\frac{1}{6}(b+a)+\frac{1}{4}\Big]+\frac{bf'(b)+af'(a)}{6}+\frac{1}{b-a}\int_a^b f(x)dx \\ &\quad+\frac{f(a)(1+2a)-f(b)(1+2b)}{2(b-a)} \\ &\quad=\frac{1}{2}\left\{\frac{1}{b-a}\int_a^b (x+x^2)f''(x)dx-\frac{1}{(b-a)^2}\int_a^b (x+x^2)dx\int_a^b f''(x)dx\right\} \\ &\quad=\frac{1}{2(b-a)}\int_a^b\left\{(x+x^2)-\frac{1}{b-a}\int_a^b (x+x^2)dx\right\}f''(x)dx. \end{aligned} \tag{2.1}$$

Let

$$A = \left\{x \in [a,b] : x+x^2 \ge \frac{1}{b-a}\int_a^b (x+x^2)dx\right\};$$

$$A^c = \left\{x \in [a,b] : x+x^2 < \frac{1}{b-a}\int_a^b (x+x^2)dx\right\}.$$

Then we have

$$\int_a^b \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} f''(x)dx$$
$$\leq \Gamma \int_A \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx$$
$$+ \gamma \int_{A^c} \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx$$

and

$$\int_a^b \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} f''(x)dx$$
$$\geq \gamma \int_A \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx$$
$$+ \Gamma \int_{A^c} \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx.$$

Since

$$\int_A \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx$$
$$= -\int_{A^c} \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx,$$

it follows that

$$\frac{1}{b-a}\left|\int_a^b \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} f''(x)dx\right|$$
$$\leq \frac{\Gamma-\gamma}{b-a} \int_A \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx$$
$$= -\frac{\Gamma-\gamma}{b-a} \int_{A^c} \left\{(x+x^2) - \frac{1}{b-a}\int_a^b (x+x^2)dx\right\} dx. \tag{2.2}$$

Putting

$$C := \frac{1}{b-a}\int_a^b (x+x^2)dx = \frac{1}{2}(b+a) + \frac{1}{3}(b^2+ab+a^2).$$

Therefore, from (2.2), it is enough to discuss the following integral,

$$\int_{A^c} (x^2+x-C)dx. \tag{2.3}$$

It is easy to see that $1+4C<0$ dose not hold in $x \in [a,b]$, thus we only consider the case $1+4C \geq 0$. Denote the solutions of the equation $x^2+x-C=0$ by x_1 and x_2, then

$$x_1 = \frac{-1-\sqrt{1+4C}}{2}, \quad x_2 = \frac{-1+\sqrt{1+4C}}{2}.$$

Case 1. $-\frac{3}{2} - 2b \leq a \leq -\frac{3}{4} - \frac{b}{2}$**.**

This case implies $b \geq -\frac{1}{2}$ and the claim

$$a \leq x_1 \leq x_2 \leq b. \tag{2.4}$$

Since $-\frac{3}{2} - 2b \leq a \leq -\frac{3}{4} - \frac{b}{2}$, we have

$$b \geq -\frac{1}{2}, \quad (b+2a)(b-a) \leq \frac{3}{2}(b-a), \quad (a+2b)(a-b) \leq \frac{3}{2}(b-a),$$

which yield

$$\sqrt{1+4C} \leq -2a-1, \quad \sqrt{1+4C} \leq 2b+1.$$

The claim (2.4) can be obtained. Then we have

$$\int_{A^c} (x^2+x-C)dx = \int_{x_1}^{x_2} (x^2+x-C)dx = -\left(\frac{1}{6}+\frac{2}{3}C\right)\sqrt{1+4C}.$$

Case 2. $-\frac{3}{4} - \frac{b}{2} \leq a \leq b$**.**

With the same proof of Case 1, it is easy to check

$$x_1 \leq a \leq x_2 \leq b.$$

Then we have

$$\begin{aligned}\int_{A^c} (x^2+x-C)dx &= \int_{a}^{x_2} (x^2+x-C)dx \\ &= -\frac{1}{6}\left[x_2(4C+1)+2a^3+3a^2-2aC+C\right].\end{aligned}$$

Case 3. $a \leq -\frac{3}{2} - 2b$**.**

With the same proof of Case 1, it is easy to check

$$a \leq x_1 \leq b \leq x_2.$$

Then we have

$$\begin{aligned}\int_{A^c} (x^2+x-C)dx &= \int_{x_1}^{b} (x^2+x-C)dx \\ &= \frac{1}{6}\left[x_1(4C+1)+2b^3+3b^2-6bC-C\right].\end{aligned}$$

From (2.1), (2.2) and above discussion, the desired result can be obtained. □

Next we shall show the following similar Ostrowski-Grüss type inequality.

Theorem 2.2. *Let $f : [a,b] \to (-\infty,\infty)$ be a function such that the derivative f' is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in (-\infty,\infty)$*

such that $\gamma \le f''(x) \le \Gamma$ *a.e. on* $[a,b]$. *Then for* $t \in [a,b]$, *we have*

$$\begin{aligned} &\Big| f(t) + \frac{[f'(b) - f'(a)](t-a)(b-t)}{2(b-a)} + \frac{(b+a)(f'(b)+f'(a))}{2} \\ &+ \frac{(t+a)f(a) - (b+t)f(b)}{b-a} \Big| \le \frac{\Gamma - \gamma}{b-a} G(t,a,b) \end{aligned} \tag{2.5}$$

where if $t + a > 0$, *then*

$$G(t,a,b) = \frac{(t+a)}{2}(t^2 - C_1^2) - (at+C)(t-C_1) + \frac{(b+t)}{2}(b^2 - C_2^2) - (bt+C)(b-C_2);$$

if $t+a<0$, $t+b<0$, *then*

$$G(t,a,b) = \frac{(t+a)}{2}(C_3^2 - a^2) - (at+C)(C_3-a) + \frac{(b+t)}{2}(C_4^2 - t^2) - (bt+C)(C_4-t);$$

if $t+a<0$, $t+b>0$, *then*

$$G(t,a,b) = \frac{(t+a)}{2}(C_3^2 - a^2) - (at+C)(C_3-a) + \frac{(b+t)}{2}(b^2 - C_2^2) - (bt+C)(b-C_2);$$

if $t+a=0$, $t \ne b$, *then*

$$G(t,a,b) = \frac{(b+t)}{2}(b^2 - C_2^2) - (bt+C)(b-C_2);$$

if $t+b=0$, $t \ne a$, *then*

$$G(t,a,b) = \frac{(t+a)}{2}(C_3^2 - a^2) - (at+C)(C_3-a);$$

if $t+a=0$, $t=b$ *or* $t+b=0$, $t=a$, *then*

$$G(t,a,b) = 0,$$

and where

$$C := \frac{1}{2}[t^2 - (b+a)t + (b^2+ab+a^2)],$$

$$C_1 = \left(\frac{at+C}{t+a} \vee a\right) \wedge t, \quad C_2 = \left(\frac{bt+C}{t+b} \vee t\right) \wedge b,$$

$$C_3 = a \vee \left(\frac{at+C}{t+a} \wedge t\right), \quad C_4 = t \vee \left(\frac{bt+C}{t+b} \wedge b\right).$$

Proof. Let

$$K(t,x) = \begin{cases} (t-x)(x-a), & a \le x \le t; \\ (x-t)(b-x), & t \le x \le b. \end{cases} \tag{2.6}$$

Then it is easy to see that

$$\frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx = \frac{1}{2}[t^2 - (b+a)t + (b^2+ab+a^2)], \tag{2.7}$$

and

$$\frac{1}{b-a}\int_a^b (K(t,x)+x^2)f''(x)dx$$
$$= f(t) + \frac{f'(b)b^2 - f'(a)a^2}{b-a} + \frac{(t+a)f(a)-(b+t)f(b)}{b-a}. \tag{2.8}$$

Therefore, from (2.7) and (2.8), we get

$$f(t) + \frac{[f'(b)-f'(a)](t-a)(b-t)}{2(b-a)} + \frac{(b+a)(f'(b)+f'(a))}{2}$$
$$+ \frac{(t+a)f(a)-(b+t)f(b)}{b-a}$$
$$=\frac{1}{b-a}\int_a^b \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] f''(x)dx. \tag{2.9}$$

Let

$$B = \left\{x \in [a,b], (K(t,x)+x^2) \geq \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right\};$$
$$B^c = \left\{x \in [a,b], (K(t,x)+x^2) < \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right\}.$$

As the proof of Theorem 2.1, we only need to estimate the bound of the following inequality

$$\frac{1}{b-a}\left|\int_a^b \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] f''(x)dx\right|$$
$$\leq \frac{\Gamma-\gamma}{b-a}\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] dx. \tag{2.10}$$

Next we shall divide our proof into five cases.

Case 1: $t+a>0$. Under this case, it follows that $b+t>0$. Then if $a \leq x \leq t$, we have

$$K(t,x)+x^2 \geq C \Longleftrightarrow x \geq \frac{at+C}{t+a},$$

and if $t \leq x \leq b$, we obtain

$$K(t,x)+x^2 \geq C \Longleftrightarrow x \geq \frac{bt+C}{t+b}.$$

Therefore

$$\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] dx$$
$$= \frac{(t+a)}{2}(t^2 - C_1^2) - (at+C)(t-C_1) + \frac{(b+t)}{2}(b^2 - C_2^2) - (bt+C)(b-C_2),$$

where

$$C_1 = \left(\frac{at+C}{t+a} \vee a\right) \wedge t, \quad C_2 = \left(\frac{bt+C}{t+b} \vee t\right) \wedge b.$$

Case 2: $t+a<0$, $t+b<0$. Then if $a \le x \le t$, we have

$$K(t,x)+x^2 \ge C \Longleftrightarrow x \le \frac{at+C}{t+a},$$

and if $t \le x \le b$, we obtain

$$K(t,x)+x^2 \ge C \Longleftrightarrow x \le \frac{bt+C}{t+b},$$

Therefore

$$\begin{aligned}&\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] dx\\ &= \frac{(t+a)}{2}(C_3^2-a^2) - (at+C)(C_3-a) + \frac{(b+t)}{2}(C_4^2-t^2) - (bt+C)(C_4-t),\end{aligned}$$

where

$$C_3 = a \vee \left(\frac{at+C}{t+a} \wedge t\right), \quad C_4 = t \vee \left(\frac{bt+C}{t+b} \wedge b\right).$$

Case 3: $t+a<0$, $t+b>0$. Then if $a \le x \le t$, we have

$$K(t,x)+x^2 \ge C \Longleftrightarrow x \le \frac{at+C}{t+a},$$

and if $t \le x \le b$, we obtain

$$K(t,x)+x^2 \ge C \Longleftrightarrow x \ge \frac{bt+C}{t+b}.$$

Therefore

$$\begin{aligned}&\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx\right] dx\\ &= \frac{(t+a)}{2}(C_3^2-a^2) - (at+C)(C_3-a) + \frac{(b+t)}{2}(b^2-C_2^2) - (bt+C)(b-C_2).\end{aligned}$$

Case 4: $t+a=0$. This case implies $a \le 0$ and $t=-a \ge 0$. Furthermore, it is easy to see that

$$at + C = \frac{1}{2}[t^2 - (b-a)t + (b^2+ab+a^2)] \ge 0, \quad \forall\, t \in [a,b].$$

Then if $a \le x \le t$, we have

$$\begin{aligned}\{x : K(t,x)+x^2 \ge C\} &= \emptyset, \quad \text{for } \; t \ne b\\ \{x : K(t,x)+x^2 = C\} &= [a,t], \quad \text{for } \; t = b\end{aligned}$$

and if $t \le x \le b$, we obtain

$$K(t,x)+x^2 \ge C \Longleftrightarrow x \ge \frac{bt+C}{t+b} \quad \text{for } \; t \ne b.$$

Therefore,

$$\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx \right] dx$$
$$= \begin{cases} \frac{(b+t)}{2}(b^2 - C_2^2) - (bt+C)(b-C_2), & \text{for } \ t \neq b; \\ 0, & \text{for } \ t = b. \end{cases}$$

Case 5: $t+b=0$**.** This case implies $b \geq 0$ and $t = -b \leq 0$. Furthermore, it is easy to see that

$$bt + C = \frac{1}{2}[t^2 - (a-b)t + (b^2 + ab + a^2)] \geq 0, \quad \forall\, t \in [a,b].$$

Then if $a \leq x \leq t$, we have

$$K(t,x) + x^2 \geq C \Longleftrightarrow x \leq \frac{at+C}{t+a} \quad \text{for} \quad t \neq a,$$

and if $t \leq x \leq b$, we obtain

$$\{x : K(t,x) + x^2 \geq C\} = \emptyset \quad \text{for} \quad t \neq a,$$
$$\{x : K(t,x) + x^2 = C\} = [a,b] \quad \text{for} \quad t = a.$$

Therefore,

$$\int_B \left[(K(t,x)+x^2) - \frac{1}{b-a}\int_a^b (K(t,x)+x^2)dx \right] dx$$
$$= \begin{cases} \frac{(t+a)}{2}(C_3^2 - a^2) - (at+C)(C_3-a), & \text{for } \ t \neq a; \\ 0 & \text{for } \ t = a. \end{cases}$$

□

Remark 2.1. We give an example for the case $t=a$ to show a perturbed two points inequality. Let $a \geq 0$, then $b > 0$. It is easy to get

$$C = \frac{1}{2}(a^2+b^2), \quad C_1 = a, \quad C_2 = \frac{1}{2}(a+b), \quad G(t,a,b) = \frac{1}{8}(b-a)^2(b+a).$$

Similarly, if $a < 0$ and $a+b > 0$, then

$$C = \frac{1}{2}(a^2+b^2), \quad C_3 = a, \quad C_2 = \frac{1}{2}(a+b), \quad G(t,a,b) = \frac{1}{8}(b-a)^2(b+a);$$

if $a < 0$ and $a+b < 0$, then

$$C = \frac{1}{2}(a^2+b^2), \quad C_3 = a, \quad C_4 = \frac{1}{2}(a+b), \quad G(t,a,b) = \frac{1}{8}(b-a)^2(b+a);$$

Hence, we have

$$\left| \frac{1}{b-a}(f(b)-f(a)) - \frac{1}{2}(f^{'}(b)+f^{'}(a)) \right| \leq \frac{1}{8}(\Gamma - \gamma)(b-a).$$

If we rewrite the above inequality as follows

$$\left|\frac{1}{b-a}\int_a^b f^{'}(x)dx - \frac{1}{2}(f^{'}(b)+f^{'}(a))\right| \leq \frac{1}{8}(\Gamma-\gamma)(b-a),$$

then, the bound is formally better than the inequality (1.1).

Remark 2.2. If we take $b > 0$, $a = -b$ and $t = 0$, then we have

$$C = \frac{1}{2}b^2, \quad C_3 = -\frac{1}{2}b, \quad C_2 = \frac{1}{2}b, \quad G(t,a,b) = \frac{1}{4}b^3,$$

which implies

$$\left|f(0) + \frac{b}{4}(f^{'}(b)-f^{'}(a)) - \frac{1}{2}(f(b)+f(a))\right| \leq \frac{b^2(\Gamma-\gamma)}{8}.$$

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