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THE RAINBOW DOMINATION NUMBER OF A DIGRAPH

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Abstract. Let $D = (V, A)$ be a finite and simple digraph. A *II-rainbow dominating function* (2RDF) of a digraph D is a function f from the vertex set V to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N^-(v)} f(u) = \{1, 2\}$ is fulfilled, where $N^-(v)$ is the set of in-neighbors of v. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The 2-*rainbow domination number* of a digraph D, denoted by $\gamma_{r2}(D)$, is the minimum weight of a 2RDF of D. In this paper we initiate the study of rainbow domination in digraphs and we present some sharp bounds for $\gamma_{r2}(D)$.

1. Introduction

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $d^+_D(v)$ for the outdegree of a vertex v and $d^-_D(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Let $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. For $S \subseteq V(D)$, we define $N^+[S] = \bigcup_{v\in S} N^+[v]$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. Consult [2, 7] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$.

A vertex v dominates all vertices in $N^+[v]$. A subset S of vertices of D is a *dominating set* if S dominates $V(D)$. The *domination number* $\gamma(D)$ is the minimum

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cardinality of a dominating set of D. The domination number of digraphs was introduced by Chartrand, Harary and Yue [1] as the out-domination number and has been studied by several authors (see, for example [4, 8]). A *Roman dominating function* (RDF) on a digraph D is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ has an in-neighbor u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. A $\gamma_R(D)$*-function* is a Roman dominating function of D with weight $\gamma_R(D)$. The Roman domination number in digraphs was introduced by Kamaraj and Jakkammal [3] and has been studied in [5]. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a digraph D is a function f from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N^-(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *$k$-rainbow domination number* of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. A $\gamma_{rk}(D)$*-function* is a k-rainbow dominating function of D with weight $\gamma_{rk}(D)$. Note that $\gamma_{r1}(D)$ is the classical domination number $\gamma(D)$. A 2-rainbow dominating function (briefly, rainbow dominating function) $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$. Since $V_1 \cup V_2 \cup V_{1,2}$ is a dominating set when f is a 2RDF, and since assigning $\{1, 2\}$ to the vertices of a dominating set yields an 2RDF, we have

$$\gamma(D) \le \gamma_{r2}(D) \le 2\gamma(D). \tag{1.1}$$

Our purpose in this paper is to establish some bounds for the rainbow domination number of a digraph.

For $S \subseteq V(D)$, the S-out-private neighbors of a vertex v of S are the vertices of $N^+[v] \setminus N^+[S - \{v\}]$. The vertex v is its own out-private neighbor if $v \notin N^+[S - \{v\}]$. The other out-private neighbors are external, i.e., belong to $V - S$. We make use of the following result in this paper.

Proposition 1.1. [3] *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(D)$-function of a digraph D. Then*

(a) *If $w \in V_1$, then $N_D^-(w) \cap V_2 = \emptyset$.*
(b) *Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-(v) \cap V_2 \neq \emptyset$, has at least two out-private neighbors relative to V_2 in the digraph H.*

Proposition 1.2. Let $k \ge 1$ be an integer. If D is a digraph of order n, then

$$\min\{k, n\} \le \gamma_{rk}(D) \le n.$$

In particular, $\gamma_{rn}(D) = n$.

Proof. Let f be a $\gamma_{rk}(D)$-function. If there exists a vertex v such that $f(v) = \emptyset$, then the definition yields to $f(N^-(v)) = \{1, 2, \ldots, k\}$ and thus $k \leq \gamma_{rk}(D)$. If $f(v)$ is nonempty for all vertices $v \in V(D)$, then $n \leq \gamma_{rk}(D)$, and the first inequality is proved.

Next consider the function g, defined by $g(v) = \{1\}$ for each $v \in V(D)$. Then g is a k-rainbow dominating function of weight n, and so $\gamma_{rk}(D) \leq n$. □

Proposition 1.3. Let $k \geq 1$ be an integer. If D is a digraph of order n, then

$$\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1.$$

Proof. Let v be a vertex of maximum outdegree $\Delta^+(D)$. Define $f : V(D) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v) = \{1, 2, \ldots, k\}$, $f(x) = \emptyset$ if $x \in N^+(v)$ and $f(x) = \{1\}$ otherwise. It is easy to see that f is a k-rainbow dominating function of D and thus $\gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1$. □

Let $k \geq 1$ be an integer, and let D be a digraph of order $n \geq k$ and maximum outdegree $\Delta^+(D) = n - 1$. Since $n \geq k$, Proposition 1.2 leads to $\gamma_{rk}(D) \geq k$. Hence it follows from Proposition 1.3 that

$$k \leq \gamma_{rk}(D) \leq n - \Delta^+(D) + k - 1 = k$$

and therefore $\gamma_{rk}(D) = k$. This example shows that Proposition 1.3 is sharp.

2. Bounds on the rainbow domination number of digraphs

Theorem 2.1. For a digraph D, $\frac{2}{3}\gamma_R(D) \leq \gamma_{r2}(D) \leq \gamma_R(D)$.

Proof. The upper bound is immediate by definition. To prove the lower bound, let f be a $\gamma_{r2}(D)$-function and let $X_i = \{v \in V(D) \mid i \in f(v)\}$ for $i = 1, 2$. We may assume that $|X_1| \leq |X_2|$. Then $|X_1| \leq (|X_1| + |X_2|)/2 = \gamma_{r2}(D)/2$. Define $g : V(D) \to \{0, 1, 2\}$ by $g(u) = 0$ if $f(u) = \emptyset$, $g(u) = 1$ when $f(u) = \{2\}$ and $g(u) = 2$ if $1 \in f(u)$. Obviously, g is a Roman dominating function on D with $\omega(g) \leq 2|X_1| + |X_2| \leq \frac{3}{2}\gamma_{r2}(D)$ and the result follows. □

Lemma 2.1. Let $k \geq 0$ be an integer and let D be a digraph of order n.

(i) If $\gamma_R(D) = \gamma_{r2}(D) + k$, then there exists a subset U of $V(D)$ such that $|N^+[U]| = n - \gamma_R(D) + 2|U|$. In particular, there exists a subset U of $V(D)$ such that $|N^+[U]| = n - \gamma_{r2}(D) + 2|U| - k$.

(ii) For each $U \subseteq V(D)$, $\gamma_R(D) \leq n - (|N^+[U]| - 2|U|)$. In particular, if there exists a subset U of $V(D)$ such that $|N^+[U]| = n - \gamma_{r2}(D) + 2|U| - k$, then $\gamma_R(D) \leq \gamma_{r2}(D) + k$.

(iii) If U^* is a subset of $V(D)$ such that $|N^+[U^*]| - 2|U^*|$ is maximum, then $\gamma_R(D) = n - (|N^+[U^*]| - 2|U^*|)$.

Proof. (i) Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. By Proposition 1.1 (a), $N^+[V_2] = V_0 \cup V_2$. Since $\gamma_R(D) = |V_1| + 2|V_2|$, we have

$$|N^+[V_2]| = |V_0| + |V_2| = (n - |V_1| - |V_2|) + |V_2| = n - (\gamma_R(D) - 2|V_2|).$$

Hence V_2 is the desired subset of $V(D)$.

(ii) Let U be a subset of $V(D)$. Clearly, $f = (N^+[U] - U, V(D) - N^+[U], U)$ is an RDF of D of weight $n - (|N^+[U]| - 2|U|)$ and hence $\gamma_R(D) \le n - (|N^+[U]| - 2|U|)$.

(iii) It follows from (i) and the choice of U^* that

$$|N^+[U^*]| - 2|U^*| \ge n - \gamma_{r2}(D) - k = n - \gamma_R(D),$$

when $\gamma_R(D) = \gamma_{r2}(D) + k$.

By (ii), we obtain

$$\gamma_R(D) \le n - (|N^+[U^*]| - 2|U^*|) \le n - (n - \gamma_R(D)) = \gamma_R(D)$$

and the proof is complete. □

Theorem 2.2. Let $k \ge 0$ be an integer and let D be a digraph of order n. Then $\gamma_R(D) = \gamma_{r2}(D) + k$ if and only if

(i) there exists no subset U of $V(D)$ such that

$$n - \gamma_{r2}(D) + 2|U| - k + 1 \le |N^+[U]| \le n - \gamma_{r2}(D) + 2|U| \text{ and}$$

(ii) there exists a subset U of $V(D)$ such that $|N^+[U]| = n - \gamma_{r2}(D) + 2|U| - k$.

Proof. The proof is by induction on k. Let $k = 0$. If $\gamma_R(D) = \gamma_{r2}(D)$, then (i) is trivial and (ii) follows from Lemma 2.1 (i). Now assume that (i) and (ii) hold. It follows from Lemma 2.1 (ii) that $\gamma_R(D) \le \gamma_{r2}(D)$. By Theorem 2.1, we have $\gamma_R(D) = \gamma_{r2}(D)$, as desired. Therefore we may assume that $k \ge 1$ and the theorem is true for each $m \le k - 1$.

Let $\gamma_R(D) = \gamma_{r2}(D) + k$. By Lemma 2.1 (i), it suffices to show that (i) holds. Assume to the contrary that there exists an integer $0 \le i \le k - 1$ and a subset $U \subseteq V(D)$ such that $n - \gamma_{r2}(D) + 2|U| - i \le |N^+[U]|$. By choosing (U, i) so that i is as small as possible and by the inductive hypothesis we have $\gamma_R(D) \le \gamma_{r2}(D) + i < \gamma_{r2}(D) + k$ which is a contradiction. This proves the "only" part of the theorem.

Conversely, assume that (i) and (ii) hold. By Lemma 2.1 (ii) we need to show that $\gamma_R(D) \ge \gamma_{r2}(D) + k$. Suppose $i = \gamma_R(D) - \gamma_{r2}(D)$. If $0 \le i \le k - 1$, then by the inductive hypothesis there exists a set $U \subseteq V(D)$ such that $n - \gamma_{r2}(D) + 2|U| - i = |N^+[U]|$ which contradicts (i). Hence $\gamma_R(D) = \gamma_{r2}(D) + k$ and the proof is complete. □

Theorem 2.3. Let D be a digraph of order n. Then

$$\gamma_{r2}(D) \ge \left\lceil \frac{2n}{2 + \Delta^+(D)} \right\rceil.$$

Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function. Then $\gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ and $n = |V_0| + |V_1| + |V_2| + |V_{1,2}|$.

Since each vertex of V_0 has at least one in-neighbor in $V_{1,2}$ or at least one in-neighbor in V_2 and at least one in-neighbor in V_1, we deduce that $|V_0| \leq \Delta^+(|V_2| + |V_{1,2}|)$ and $|V_0| \leq \Delta^+(|V_1| + |V_{1,2}|)$. Hence, we conclude that

$$\begin{aligned}(\Delta^+ + 2)\gamma_{r2}(D) &= (\Delta^+ + 2)(|V_1| + |V_2| + 2|V_{1,2}|)\\ &= (\Delta^+ + 2)(|V_1| + |V_2|) + 4|V_{1,2}| + 2\Delta^+|V_{1,2}|\\ &\geq 2|V_1| + 2|V_2| + 2|V_0| + 4|V_{1,2}|\\ &= 2n + 2|V_{1,2}|\\ &\geq 2n,\end{aligned}$$

as desired. □

The proof of Theorem 2.3 shows that

$$\gamma_{r2}(D) \geq \left\lceil \frac{2n+2}{2+\Delta^+(D)} \right\rceil$$

if there exists a $\gamma_{2r}(D)$-function f such that $V_{1,2}^f \neq \emptyset$. Let G be a graph and $\gamma_{r2}(G)$ its 2-rainbow domination number. If G is of order n and maximum degre Δ, then Theorem 2.3 implies immediately the known bound $\gamma_{r2}(G) \geq \lceil 2n/(\Delta + 2)\rceil$, given by Sheikhoelslami and Volkmann [6]. Next we characterize the digraphs D with $\gamma_{r2}(D) = 2$.

Proposition 2.1. Let D be a digraph of order $n \geq 2$. Then $\gamma_{r2}(D) = 2$ if and only if $n = 2$ or $n \geq 3$ and $\Delta^+(D) = n - 1$ or there exist two different vertices u and v such that $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$.

Proof. If $n = 2$ or $n \geq 3$ and $\Delta^+(D) = n - 1$ or there exist two different vertices u and v such that $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$, then it is easy to see that $\gamma_{r2}(D) = 2$.

Conversely, assume that $\gamma_{r2}(D) = 2$. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function. Clearly, $2 = \gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ and thus $|V_{1,2}| \leq 1$. If $|V_{1,2}| = 1$, then $|V_1| = |V_2| = 0$ and hence $\Delta^+(D) = n - 1$. If $|V_{1,2}| = 0$, then $|V_1|, |V_2| \leq 2$. If $|V_1| = 0$ or $|V_2| = 0$, then we deduce that $n = 2$. In the remaining case that $|V_1| = |V_2| = 1$, we assume that $V_1 = \{u\}$ and $V_2 = \{v\}$. The definition of the 2-rainbow dominating function implies that each vertex of $V(D) - \{u, v\}$ has u and v as an in-neighbor. Consequently, $V(D) - \{u, v\} \subseteq N^+(u)$ and $V(D) - \{u, v\} \subseteq N^+(v)$. □

Proposition 2.2. Let D be a digraph of order n. Then $\gamma_{r2}(D) < n$ if and only if $\Delta^+(D) \geq 2$ or $\Delta^-(D) \geq 2$.

Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(D)$-function of D. The hypothesis $|V_0| + |V_1| + |V_2| + |V_{1,2}| = n > \gamma_{r2}(D) = |V_1| + |V_2| + 2|V_{1,2}|$ implies $|V_0| \geq |V_{1,2}| + 1$. If some vertex $w \in V_0$ has no in-neighbor in $V_{1,2}$, then $|N^-(w) \cap V_1| \geq 1$ and $|N^-(w) \cap V_2| \geq 1$

which implies that $\Delta^-(D) \geq d^-(w) \geq 2$. So we may assume each vertex $w \in V_0$ has at least one in-neighbor in $V_{1,2}$. Then we have

$$\sum_{u \in V_{1,2}} d_D^+(u) \geq |V_0| \geq |V_{1,2}| + 1.$$

If we suppose on the contrary that $\Delta^+(D) \leq 1$, then we arrive at the contradiction

$$|V_{1,2}| \geq \sum_{u \in V_{1,2}} d_D^+(u) \geq |V_{1,2}| + 1.$$

If $\Delta^+(D) \geq 2$, then Proposition 1.3 implies that $\gamma_{r2}(D) \leq n - \Delta^+(D) + 1 < n$. If $\Delta^-(D) \geq 2$, then assume that u is a vertex with in-degree $\Delta^-(D)$ and let $v, w \in N^-(u)$ be two distinct vertices. Then $(\{u\}, V(D) - \{u, v\}, \{v\}, \emptyset)$ is a rainbow dominating function of D implying that $\gamma_{r2}(D) < n$, and the proof is complete. □

Corollary 2.1. If D is a directed path or directed cycle of order n, then $\gamma_{r2}(D) = n$.

Next we characterize the digraphs which attain the lower bound in (1.1).

Proposition 2.3. Let D be a digraph on n vertices. Then $\gamma(D) = \gamma_{r2}(D)$ if and only if D has a $\gamma(D)$-set S that partitions into two nonempty subsets S_1 and S_2 such that $N^+(S_1) = V(D) - (S_1 \cup S_2)$ and $N^+(S_2) = V(D) - (S_1 \cup S_2)$.

Proof. Assume that $\gamma(D) = \gamma_{r2}(D)$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a γ_{r2}-function of D. If $\gamma(D) = \gamma_{r2}(D) = n$, then clearly D is empty and the result is immediate. Let $\gamma(D) = \gamma_{r2}(D) < n$. Then the assumption implies that we have equality in $\gamma(D) \leq |V_1| + |V_2| + |V_{1,2}| \leq |V_1| + |V_2| + 2|V_{1,2}| = \gamma_{r2}(D)$. This implies that $|V_{1,2}| = 0$ and hence we deduce that each vertex in V_0 has at least one in-neighbor in V_1 and one in-neighbor in V_2. Therefore, $V(D) - (V_1 \cup V_2) \subseteq N^+(V_1)$ and $V(D) - (V_1 \cup V_2) \subseteq N^+(V_2)$. If there is an arc (a, b) in $D[V_1 \cup V_2]$, then obviously $(V_1 \cup V_2) - \{b\}$ is a dominating set of D which is a contradiction. Hence $V_1 \cup V_2$ is dominating set of D with $V(D) - (V_1 \cup V_2) = N^+(V_1)$ and $V(D) - (V_1 \cup V_2) = N^+(V_2)$.
Conversely, assume that D has a minimum dominating set S that partitions into two nonempty subsets S_1 and S_2 such that $N^+(S_1) = V(D) - (S_1 \cup S_2)$ and $N^+(S_2) = V(D) - (S_1 \cup S_2)$. It is straightforward to verify that the function $(V(D) - (S_1 \cup S_2), S_1, S_2, \emptyset)$ is a rainbow dominating function of D of weight $\gamma(D)$ and hence $\gamma(D) = \gamma_{r2}(D)$. This completes the proof. □

3. Cartesian product of directed cycles

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs which have disjoint vertex sets V_1 and V_2 and disjoint arc sets A_1 and A_2, respectively. The Cartesian product $D_1 \square D_2$ is the digraph with vertex set $V_1 \times V_2$ and for any two vertices (x_1, x_2) and (y_1, y_2) of $D_1 \square D_2$, $(x_1, x_2)(y_1, y_2) \in A(D_1 \square D_2)$ if one of the following holds:

(i) $x_1 = y_1$ and $x_2 y_2 \in A(D_2)$;
(ii) $x_1 y_1 \in A(D_1)$ and $x_2 = y_2$.

We denote the vertices of a directed cycle C_n by the integers $\{1,2,\ldots,n\}$ considered modulo n. Note that $N^+((i,j)) = \{(i,j+1),(i+1,j)\}$ for any vertex $(i,j) \in V(C_m \Box C_n)$, the first and second digit are considered modulo m and n, respectively. For any $k \in \{1,2,\ldots,n\}$, we will denote by C_m^k the subdigraph of $C_m \Box C_n$ induced by the vertices $\{(j,k) | j \in \{1,2,\ldots,m\}\}$. Note that C_m^k is isomorphic to C_m. Let f be a $\gamma_{r2}(C_m \Box C_n)$-function and set $a_k = \sum_{x \in V(C_m^k)} |f(x)|$ for any $k \in \{1,2,\ldots,n\}$. Then $\gamma_{r2}(C_m \Box C_n) = \sum_{k=1}^n a_k$. It is easy to see that $C_m \Box C_n \cong C_n \Box C_m$ for any directed cycles of length $m,n \geq 2$, and so $\gamma_{r2}(C_m \Box C_n) = \gamma_{r2}(C_n \Box C_m)$.

Theorem 3.1. For $n \geq 2$, $\gamma_{r2}(C_2 \Box C_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$

Proof. First let n be even. Define $f : V(C_2 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $f((1,2i-1)) = \{1\}$ for $1 \leq i \leq \frac{n}{2}$, $f((2,2i)) = \{2\}$ for $1 \leq i \leq \frac{n}{2}$ and $f(x) = \emptyset$ otherwise. Obviously, f is a 2RDF of $C_2 \Box C_n$ of weight n and hence $\gamma_{r2}(C_2 \Box C_n) \leq n$. On the other hand, since $\Delta^+(C_2 \Box C_n) = 2$, it follows from Theorem 2.3 that $\gamma_{r2}(C_2 \Box C_n) = n$.

Now let n be odd. We claim that $\gamma_{r2}(C_2 \Box C_n) \geq n+1$. Assume to the contrary that $\gamma_{r2}(C_2 \Box C_n) \leq n$. Let f be a $\gamma_{r2}(C_2 \Box C_n)$-function and set $a_k = \sum_{x \in V(C_2^k)} |f(x)|$ for any $k \in \{1,2,\ldots,n\}$. If $a_k = 0$ for some k, say $k = 3$, then $f((1,3)) = f((2,3)) = \emptyset$ and to dominate the vertices $(1,3)$ and $(2,3)$ we must have $f((1,2)) = \{1,2\}$ and $f((2,2)) = \{1,2\}$, respectively. Then the function $g : V(C_2 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $g((1,2)) = \{1\}, g((2,2)) = \emptyset, g((2,1)) = g((2,3)) = \{2\}$ and $g(x) = f(x)$ for $x \in V(C_2 \Box C_n) - \{(1,2),(2,2),(2,1),(2,3)\}$ is a 2RDF of $C_2 \Box C_n$ of weight less than $\omega(f)$ which is a contradiction. Thus $a_k \geq 1$ for each k. By assumption $a_k = 1$ for each k. We may assume, without loss of generality, that $f((1,2)) = \{1\}$. To dominate $(2,2)$, we must have $f((2,1)) = \{2\}$. Since $a_3 = 1$ and $f((2,2)) = \emptyset$, to dominate $((1,3))$, we must have $f((2,3)) = \{2\}$. Repeating this process we obtain $f((1,2i)) = \{1\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $f((2,2i-1)) = \{2\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(x) = \emptyset$ otherwise. But then the vertex $(1,1)$ is not dominated, a contradiction. Thus $\gamma_{r2}(C_2 \Box C_n) \geq n+1$. Define $g : V(C_2 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $g((1,1)) = \{1\}, g((1,2i)) = \{1\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $g((2,2i-1)) = \{2\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $g(x) = \emptyset$ otherwise. Clearly, g is a 2RDF of $C_2 \Box C_n$ of weight $n+1$ and hence $\gamma_{r2}(C_2 \Box C_n) = n+1$. $\Box$

Theorem 3.2. For $n \geq 2$, $\gamma_{r2}(C_3 \Box C_n) = 2n$.

Proof. First we prove $\gamma_{r2}(C_3 \Box C_n) \leq 2n$. Define $g : V(C_3 \Box C_n) \to \mathcal{P}(\{1,2\})$ as follows:

If $n \equiv 0 \pmod 3$, then $g((1,3i+1)) = g((2,3i+2)) = g((3,3i+3)) = \{1\}, g((1,3i+3)) = g((2,3i+1)) = g((3,3i+2)) = \{2\}$ for $0 \leq i \leq \frac{n}{3} - 1$ and $g(x) = \emptyset$ otherwise,

if $n \equiv 1 \pmod 3$, then $g((3,n))) = \{1\}, g((2,n)) = \{2\}, g((1,3i+1)) = g((2,3i+2)) = g((3,3i+3)) = \{1\}, g((1,3i+3)) = g((2,3i+1)) = g((3,3i+2)) = \{2\}$ for $0 \leq i \leq \frac{n-1}{3} - 1$ and $g(x) = \emptyset$ otherwise,

if $n \equiv 2 \pmod 3$, then $g((1,n)) = g((1,n-1)) = g((3,n)) = \{1\}, g((2,n-1)) = \{2\}, g((1,3i+1)) = g((2,3i+2)) = g((3,3i+3)) = \{1\}, g((1,3i+3)) = g((2,3i+1)) =$

$g((3,3i+2)) = \{2\}$ for $0 \le i \le \frac{n-2}{3} - 1$ and $g(x) = \emptyset$ otherwise. It is easy to see that in each case, g is a 2RDF of $C_3 \Box C_n$ of weight $2n$ and hence $\gamma_{r2}(C_3 \Box C_n) \le 2n$.

Now we show that $\gamma_{r2}(C_3 \Box C_n) \ge 2n$. First we prove that for any $\gamma_{r2}(C_3 \Box C_n)$-function f, $\sum_{x \in V(C_m^k)} |f(x)| \ge 1$ for each k. Let to the contrary that $\sum_{x \in V(C_m^k)} |f(x)| = 0$ for some k, say $k = n$. Then we must have $f((1,n-1)) = f((2,n-1)) = f((3,n-1)) = \{1,2\}$. Then the function f_1 defined by $f_1((1,n-1)) = f_1((2,n-1)) = f_1((3,n-1)) = \{1\}$, $f_1((1,n)) = f_1((2,n)) = \{2\}$ and $f_1(x) = f(x)$ otherwise, is a 2RDF of G of weight less than $\omega(f)$, a contradiction.

Let g be a $\gamma_{r2}(C_3 \Box C_n)$-function such that the size of the set $S = \{k \mid a_k = \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \le i \le n\}$ is minimum. We claim that $|S| = 0$ implying that $\gamma_{r2}(C_3 \Box C_n) = \omega(g) \ge 2n$. Assume to the contrary that $|S| \ge 1$. Suppose, without loss of generality, that $a_n = 1$ and that $g((1,n)) = \{1\}$ and $g((2,n)) = g((3,n)) = \emptyset$. To dominate $(2,n)$ and $(3,n)$ we must have $2 \in g((2,n-1))$ and $g((3,n-1)) = \{1,2\}$, respectively. If $n = 3$, then clearly $a_1 \ge 2$ and so $\gamma_{r2}(C_3 \Box C_n) \ge 2n$. Let $n \ge 4$. Consider two cases.

Case 1. $a_{n-2} = 1$.
As above we have $a_{n-3} \ge 3$. Thus $a_{n-3} + a_{n-2} + a_{n-1} + a_n \ge 8$. If $n = 4$, we are done. Suppose $n \ge 5$. Since $a_{n-4} \ge 1$, $1 \in \cup_{i=1}^{3} g((i,n-4))$ or $2 \in \cup_{i=1}^{3} g((i,n-4))$. Let $1 \in \cup_{i=1}^{3} g((i,n-4))$ (the case $2 \in \cup_{i=1}^{3} g((i,n-4))$ is similar).

If $1 \in g((3,n-4))$, then define $h : V(C_3 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $h((1,n-3)) = h((1,n-1)) = h((2,n-3)) = h((3,n-1)) = \{2\}, h((1,n)) = h((2,n)) = h((2,n-2)) = h((3,n-2)) = \{1\}, h((1,n-2)) = h((2,n-1)) = h((3,n-3)) = h((3,n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((2,n-4))$, then define $h : V(C_3 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $h((1,n-3)) = h((2,n-1)) = h((3,n-1)) = h((3,n-3)) = \{2\}, h((1,n)) = h((1,n-2)) = h((2,n-2)) = h((3,n)) = \{1\}, h((2,n-2)) = h((1,n-1)) = h((2,n-3)) = h((2,n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((1,n-4))$, then define $h : V(C_3 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $h((2,n-3)) = h((2,n-1)) = h((3,n-3)) = h((3,n-1)) = \{2\}, h((1,n)) = h((1,n-2)) = h((3,n-2)) = h((3,n)) = \{1\}, h((2,n-2)) = h((1,n-1)) = h((1,n-3)) = h((2,n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

Clearly, h is a 2RDF of $C_3 \Box C_n$ for which $|\{k \mid \sum_{x \in V(C_m^k)} |h(x)| = 1 \text{ and } 1 \le i \le n\}| < |\{k \mid \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \le i \le n\}|$ which contradicts the choice of g.

Case 2. $a_{n-2} \ge 2$.
Then $a_{n-2} + a_{n-1} + a_n \ge 6$. Since $a_{n-3} \ge 1$, $1 \in \cup_{i=1}^{3} g((i,n-3))$ or $2 \in \cup_{i=1}^{3} g((i,n-3))$. Assume $1 \in \cup_{i=1}^{3} g((i,n-3))$ (the case $2 \in \cup_{i=1}^{3} g((i,n-3))$ is similar).

If $1 \in g((3,n-3))$, then define $h : V(C_3 \Box C_n) \to \mathcal{P}(\{1,2\})$ by $h((1,n-2)) = h((2,n-2)) = h((2,n-1)) = \{2\}, h((1,n)) = h((3,n)) = h((3,n-1)) = \{1\}, h((1,n-1)) = h((2,n)) = h((3,n-2)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((2, n-3))$, then define $h : V(C_3 \square C_n) \to \mathcal{P}(\{1,2\})$ by $h((1, n-2)) = h((2, n-1)) = h((3, n-2)) = \{2\}, h((1, n)) = h((3, n)) = h((3, n-1)) = \{1\}, h((1, n-1)) = h((2, n-2)) = h((2, n)) = \emptyset$ and $h(x) = g(x)$ otherwise.

If $1 \in g((1, n-3))$, then define $h : V(C_3 \square C_n) \to \mathcal{P}(\{1,2\})$ by $h((2, n-2)) = h((3, n-2)) = h((3, n-1)) = \{2\}, h((1, n)) = h((1, n-1)) = h((2, n)) = \{1\}, h((3, n)) = h((1, n-2)) = h((2, n-1)) = \emptyset$ and $h(x) = g(x)$ otherwise.

Clearly, h is a 2RDF of $C_3 \square C_n$ for which $|\{k \mid \sum_{x \in V(C_m^k)} |h(x)| = 1 \text{ and } 1 \leq i \leq n\}| < |\{k \mid \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}|$ which is a contradiction again.

Therefore, $|\{k \mid a_k = \sum_{x \in V(C_m^k)} |g(x)| = 1 \text{ and } 1 \leq i \leq n\}| = 0$ and hence $\gamma_{r2}(C_3 \square C_n) = \omega(g) \geq 2n$. Thus $\gamma_{r2}(C_3 \square C_n) = 2n$ and the proof is complete. □

Proposition 3.1. If $m = 2r$ and $n = 2s$ for some positive integers r, s, then $\gamma_{r2}(C_m \square C_n) = \frac{mn}{2}$.

Proof. Define $f : V(C_m \square C_n) \to \mathcal{P}(\{1,2\})$ by $f((2i-1, 2j-1)) = \{1\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$, $f((2i, 2j)) = \{2\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is a 2RDF of $C_m \square C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r2}(C_m \square C_n) \leq \frac{mn}{2}$. Now the results follows from Theorem 2.3. □

4. Strong product of directed cycles

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs which have disjoint vertex sets V_1 and V_2 and disjoint arc sets A_1 and A_2, respectively. The strong product $D_1 \otimes D_2$ is the digraph with vertex set $V_1 \times V_2$ and for any two vertices (x_1, x_2) and (y_1, y_2) of $D_1 \otimes D_2$, $(x_1, x_2)(y_1, y_2) \in A(D_1 \otimes D_2)$ if one of the following holds:

(i) $x_1 y_1 \in A(D_1)$ and $x_2 y_2 \in A(D_2)$;
(ii) $x_1 = y_1$ and $x_2 y_2 \in A(D_2)$;
(iii) $x_1 y_1 \in A(D_1)$ and $x_2 = y_2$.

We denote the vertices of a directed cycle C_n by the integers $\{1, 2, \ldots, n\}$ considered modulo n. There is an arc xy from x to y in C_n if and only if $y = x + 1 \pmod n$ and $N^+((i,j)) = \{(i, j+1), (i+1, j), (i+1, j+1)\}$ for any vertex $(i,j) \in V(C_m \otimes C_n)$, the first and second digit are considered modulo m and n, respectively. We use the notation defined in Section 3.

Lemma 4.1. For positive integers $m, n \geq 2$, $\gamma_{r2}(C_m \otimes C_n) \geq \lceil \frac{mn}{2} \rceil$.

Proof. Observe that the vertices of C_m^k are dominated by vertices of C_m^{k-1} or C_m^k, $k = 1, 2, \ldots, n$. Especially, the vertices of C_m^1 are dominated by C_m^1 and C_m^n. We show that $\sum_{k=1}^{n} a_k \geq \lceil \frac{mn}{2} \rceil$. In order to this, we show that $a_k + a_{k+1} \geq m$ for each $k = 1, 2, \ldots, n$, where $a_{n+1} = a_1$. First let $a_{k+1} = 0$. Then to rainbowly dominate $(i, k+1)$ for each $1 \leq i \leq m$, we must have $|f((i-1, k))| + |f((i, k))| \geq 2$. Then $2a_k = \sum_{i=1}^{m} (|f((i-1, k))| + |f((i, k))|) \geq 2m$ and hence $a_k + a_{k+1} \geq m$. If $a_{k+1} = t$, then it is not hard to see that $a_k \geq m - t$ and so $a_k + a_{k+1} \geq m$. Therefore,

$$2\gamma_{r2}(C_m \otimes C_n) = 2\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_k + a_{k+1}) \geq nm$$

where $a_{n+1} = a_1$. This implies that $\gamma_{r2}(C_m \otimes C_n) \geq \lceil \frac{mn}{2} \rceil$. □

Proposition 4.1. If $m = 2r$ and $n = 2s$ for some positive integers r, s, then $\gamma_{r2}(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

Proof. Define $f : V(D) \to \mathcal{P}(\{1,2\})$ by $f((2i-1, 2j-1)) = \{1,2\}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$, and $f(x) = \emptyset$ otherwise. It is easy to see that f is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r2}(C_m \otimes C_n) \leq \frac{mn}{2}$. Now the results follows from Lemma 4.1. □

Proposition 4.2. If $m = 4r$ and $n = 2s + 1$ for some positive integers r, s, then $\gamma_{r2}(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

Proof. Let

$$\begin{aligned}
W_1 &= \{(4i+1, 1) \mid 0 \leq i \leq r-1\},\\
W_2 &= \{(4i+3, 1) \mid 0 \leq i \leq r-1\},\\
Z_1^1 &= \{(4i+2, 2j) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\},\\
Z_1^2 &= \{(4i+4, 2j) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\},\\
Z_2^1 &= \{(4i+4, 2j+1) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\},\\
Z_2^2 &= \{(4i+2, 2j+1) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\}.
\end{aligned}$$

Define $f : V(C_m \otimes C_n) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1\}$ for $x \in W_1 \cup Z_1^1 \cup Z_2^1$, $f(x) = \{2\}$ for $x \in W_2 \cup Z_1^2 \cup Z_2^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that f is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2}$ and so $\gamma_{r2}(C_m \otimes C_n) \leq \frac{mn}{2}$. It follows from Lemma 4.1 that $\gamma_{r2}(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$. □

Proposition 4.3. If $m = 4r + 2$ and $n = 2s + 1$ for some positive integers r, s, then $\lceil \frac{mn}{2} \rceil \leq \gamma_{r2}(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil + 1$.

Proof. Let

$$\begin{aligned}
C_1 &= \{(4r+2, 2s+1)\} \cup \{(4i+4, 2s+1) \mid 0 \leq i \leq r-1\},\\
C_2 &= \{(4r+1, 2s+1)\} \cup \{(4i+2, 2s+1) \mid 0 \leq i \leq r-1\},\\
C_{1,2} &= \{(4r+1, 2j+1) \mid 0 \leq j \leq s-1\},\\
Z_1^1 &= \{(4i+1, 2j+1) \mid 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq s-1\},\\
Z_1^2 &= \{(4i+1, 2j) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\},\\
Z_2^1 &= \{(4i+3, 2j) \mid 0 \leq i \leq r-1 \text{ and } 1 \leq j \leq s\},\\
Z_2^2 &= \{(4i+3, 2j+1) \mid 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq s-1\}.
\end{aligned}$$

Define $f : V(C_m \otimes C_n) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1,2\}$ for $x \in C_{1,2}$, $f(x) = \{1\}$ for $x \in C_1 \cup Z_1^1 \cup Z_2^1$, $f(x) = \{2\}$ for $x \in C_2 \cup Z_1^2 \cup Z_2^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that f is a 2RDF of $C_m \otimes C_n$ with weight $\frac{mn}{2} + 1$ and so $\gamma_{r2}(C_m \otimes C_n) \leq \frac{mn}{2} + 1$. It follows from Lemma 4.1 that $\lceil \frac{mn}{2} \rceil \leq \gamma_{r2}(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil + 1$. □

Lemma 4.2. If $m = 4r + 3$ and $n = 4s + 3$ for some non-negative integers r, s, then $\gamma_{r2}(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$.

Proof. Let

$$\begin{aligned}
R_1^1 &= \{(4i+1, 4j+1) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\},\\
R_1^2 &= \{(4i+1, 4j+3) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\},\\
R_2^1 &= \{(4i+2, 4j+2) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\},\\
R_2^2 &= \{(4i+2, 4j+4) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s-1\},\\
R_3^1 &= \{(4i+3, 4j+3) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\},\\
R_3^2 &= \{(4i+3, 4j+1) \mid 0 \leq i \leq r \text{ and } 0 \leq j \leq s\},\\
R_4^1 &= \{(4i+4, 4j+4) \mid 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq s-1\},\\
R_4^2 &= \{(4i+4, 4j+2) \mid 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq s\}.
\end{aligned}$$

Define $f : V(C_m \otimes C_n) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1\}$ for $x \in R_1^1 \cup R_2^1 \cup R_3^1 \cup R_4^1$, $f(x) = \{2\}$ for $x \in R_1^2 \cup R_2^2 \cup R_3^2 \cup R_4^2$, and $f(x) = \emptyset$ otherwise. It is easy to see that f is a 2RDF of $C_m \otimes C_n$ with weight $\lfloor \frac{mn}{2} \rfloor + 1$ and so $\gamma_{r2}(C_m \otimes C_n) \leq \lceil \frac{mn}{2} \rceil$. By Lemma 4.1 we obtain $\gamma_{r2}(C_m \otimes C_n) = \lceil \frac{mn}{2} \rceil$. □

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