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NOTE ON STRONG PRODUCT OF GRAPHS

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Dedicated to the memory of the late professor Ante Graovac

Abstract. Let G and H be graphs. The strong product $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever ($v_1 = v_2$ and u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2) or (u_1 is adjacent with u_2 and v_1 is adjacent with v_2). In this paper, we study some properties of this operation. Also, we obtain lower and upper bounds for Wiener and hyper-Wiener indices of Strong product of graphs.

1. Introduction

Throughout this paper graphs means simple connected graphs. Suppose G is a graph with vertex set $V(G)$. The distance between the vertices u and v of $V(G)$ is defined as the length of a minimal path connecting them, denoted by $d(u, v)$. The Wiener index, $W(G)$, is equal to the count of all shortest distances in a graph [20]. In other words, $W(G) = \frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u, v)$. We encourage to the interested reader to consult [6][7][9][10][16] for more information on this topic.

The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. [17], generalized Randić's definition for all connected graphs, as a generalization of the Wiener index. It is defined as $WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d^2(u, v)$. The mathematical properties and chemical meaning of this topological index are reported in [4][5][11][15][24].

The degree of a vertex v in a graph G, $deg_G(v)$, is the number of edges of G incident with v. The eccentricity $\varepsilon_G(u)$ is defined as the largest distance between u and other vertices of G. The eccentric connectivity index of a graph G is defined as

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$\xi^c(G) = \sum_{u \in V(G)} deg(u)\varepsilon(u)$ [19]. The investigation of the mathematical properties of $\xi^c(G)$ started only recently, and has so far resulted in determining the extremal values of the invariant and the extremal graphs where those values are achieved [13][22][23], and also in a number of explicit formulae for the eccentric connectivity index of several classes of graphs [1].

The Strong product $G \boxtimes H$ of graphs G and H has the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \boxtimes H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$, or $ab \in E(G)$ and $xy \in E(H)$. Occasionally one also encounters the names strong direct product or symmetric composition for the strong product [12]. It is worthy to mention here that [8] and [21] are the first two papers that considered the problem of distribution of a topological index over a graph operation.

Throughout this paper our notation is standard and taken mainly from [2] and [3].

2. Main results

For a connected graph G, the radius $r(G)$ and diameter $D(G)$ are, respectively, the minimum and maximum eccentricity among vertices of G.

Lemma 2.1. *Let G and H be graphs. Then for every vertex (a, x) of $G \boxtimes H$, we have*

$$\varepsilon_{G \boxtimes H}((a, x)) = max\{\varepsilon_G(a), \varepsilon_H(x)\}.$$

Proof. Let $(a, x) \in V(G \boxtimes H)$. By definition of the eccentricity,

$$\varepsilon_{G \boxtimes H}((a, x)) = max\{d_{G \boxtimes H}((a, x), (b, y)) \mid (b, y) \in V(G \boxtimes H)\}.$$

On the other hand, it is well-known that $d_{G \boxtimes H}((a, x), (b, y)) = max\{d_G(a, b), d_H(x, y)\}$ [12], and so

$$\begin{aligned} \varepsilon_{G \boxtimes H}((a, x)) &= max\{max\{d_G(a, b)\}, max\{d_H(x, y)\} \mid b \in V(G), y \in V(H)\} \\ &= max\{\varepsilon_G(a), \varepsilon_H(x)\}, \end{aligned}$$

which completes the proof. □

A vertex is called odd vertex if it has odd degree.

Theorem 2.1. *Let G and H be a nontrivial connected graphs. Then $G \boxtimes H$ is eulerian if and only if G and H are eulerian.*

Proof. Let G and H are eulerian then clearly $G \boxtimes H$ is eulerian. Conversely, we assume that $G \boxtimes H$ is eulerian. Suppose first that one of graphs G and H is not eulerian. Without loss of generality, we may assume that G is not eulerian. Thus, G has an odd vertex u. Let x is a vertex of H. Then, $deg_{G \boxtimes H}((u, x))$ is odd. We conclude that $G \boxtimes H$ is not eulerian if one of graphs G and H is not eulerian.

Assume next that G and H are not eulerian. Then G and H have odd vertices u and x, respectively. Therefore, the vertex (u, x) of $G \boxtimes H$ has odd degree and hence $G \boxtimes H$ is not eulerian in this case, which completes the argument. □

The complement or inverse of a graph G is a graph $\bar{G}$ on the same vertices such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G. We denote the complete graph of order n by K_n.

Theorem 2.2. *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} W(G\boxtimes H) \geqslant\ & (|V(G)|+2|E(G)|)W(H)+(|V(H)|+2|E(H)|)W(G)\\ &+|V(G)||V(H)|(|V(G)||V(H)|-|V(G)|-|V(H)|+1)\\ &-2|E(G)||V(H)|(|V(H)|-1)-2|E(H)|(|V(G)|^2-|V(G)|-|E(G)|)\end{aligned}$$

with equality if and only if $max\{D(G),D(H)\}\leqslant 2$, or $G\cong K_n$, or $H\cong K_n$.

Proof. Let G and H are nontrivial connected graphs. Suppose A is the set of vertex-pairs $(a,x),(a,y)$ that $a\in V(G)$, $x\neq y$ and $x,y\in V(H)$. Then,

$$\sum_{(u,v)\in A} d_{G\boxtimes H}(u,v)=|V(G)|W(H). \tag{2.1}$$

Similarly, for vertex-pairs $(a,x),(b,x)$ that $x\in V(H)$, $a\neq b$ and $a,b\in V(G)$, we have

$$\sum_{\{(a,x),(b,x)\}} d_{G\boxtimes H}((a,x),(b,x))=|V(H)|W(G). \tag{2.2}$$

On the other hand, the sum of all distances between vertex-pairs $(a,x),(b,y)$ that ($a\neq b\in V(G)$ and $xy\in E(H)$) or ($x\neq y\in V(H)$ and $ab\in E(G)$), is equal to

$$2|E(G)|W(H)+2|E(H)|W(G)-2|E(G)||E(H)| \tag{2.3}$$

By this fact that for every vertex-pair $(a,x),(b,y)$ of $G\boxtimes H$ such that $ab\in E(\bar{G})$ and $xy\in E(\bar{H})$, we have $d_{G\boxtimes H}((a,x),(b,y))\geqslant 2$, Equations (2.1), (2.2) and (2.3), the result is proved. ☐

Using similar arguments as Theorem 2.2 one can prove the following result.

Theorem 2.3. *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} W(G\boxtimes H) \leqslant\ & (|V(G)|+2|E(G)|)W(H)+(|V(H)|+2|E(H)|)W(G)\\ &+D\Big[\frac{|V(G)||V(H)|}{2}(|V(G)||V(H)|-|V(G)|-|V(H)|+1)\\ &-2|E(H)|\binom{|V(G)|}{2}-2|E(G)|\binom{|V(H)|}{2}\Big]+2|E(G)||E(H)|(D-1).\end{aligned}$$

where $D=max\{D(G),D(H)\}$. Moreover, the upper bound is attained if and only if $D\leqslant 2$, or $G\cong K_n$, or $H\cong K_n$.

Theorem 2.4. *Let G and H be a nontrivial connected graphs. Then*

$$\begin{aligned} WW(G \boxtimes H) \geqslant{} & (|V(G)| + 2|E(G)|)WW(H) + (|V(H)| + 2|E(H)|)WW(G) \\ & + \frac{3}{2}|V(G)||V(H)|(|V(G)||V(H)| - |V(G)| - |V(H)| + 1) \\ & - 3|E(G)||V(H)|(|V(H)| - 1) - 3|E(H)|(|V(G)|^2 - |V(G)| \\ & - \frac{4}{3}|E(G)|). \end{aligned}$$

with equality if and only if $max\{D(G), D(H)\} \leqslant 2$, *or* $G \cong K_n$, *or* $H \cong K_n$.

Proof. We split the vertex-pair set of $G \boxtimes H$ into subsets

$$\begin{aligned} A &= \{\{(a,x),(b,y)\} \mid a = b \in V(G) \text{ and } x \neq y, x, y \in V(H)\}, \\ B &= \{\{(a,x),(b,y)\} \mid a \neq b, a, b \in V(G) \text{ and } x = y \in V(H)\}, \\ C &= \{\{(a,x),(b,y)\} \mid x \neq y, x, y \in V(H) \text{ and } ab \in E(G)\}, \\ D &= \{\{(a,x),(b,y)\} \mid a \neq b, a, b \in V(G) \text{ and } xy \in E(H)\}, \\ E &= \{\{(a,x),(b,y)\} \mid ab \in E(\bar{G}) \text{ and } xy \in E(\bar{H})\}. \end{aligned}$$

It follows from the edge structure of $G \boxtimes H$ that, if $\{(a,x),(b,y)\} \in A \cup C$ then

$$d_{G\boxtimes H}((a,x),(b,y)) + d^2_{G\boxtimes H}((a,x),(b,y)) = d_H(x,y) + d^2_H(x,y),$$

if $\{(a,x),(b,y)\} \in B \cup D$ then

$$d_{G\boxtimes H}((a,x),(b,y)) + d^2_{G\boxtimes H}((a,x),(b,y)) = d_G(a,b) + d^2_G(a,b),$$

and if $\{(a,x),(b,y)\} \in E$ then

$$d_{G\boxtimes H}((a,x),(b,y)) + d^2_{G\boxtimes H}((a,x),(b,y)) \geqslant 6.$$

Therefore,

$$\begin{aligned} WW(G \boxtimes H) &= \frac{1}{2} \sum_{\{u,v\} \in A \cup D} (d_{G\boxtimes H}(u,v) + d^2_{G\boxtimes H}(u,v)) \\ &\quad + \frac{1}{2} \sum_{\{u,v\} \in B \cup C} (d_{G\boxtimes H}(u,v) + d^2_{G\boxtimes H}(u,v)) \\ &\quad + \frac{1}{2} \sum_{\{u,v\} \in E} (d_{G\boxtimes H}(u,v) + d^2_{G\boxtimes H}(u,v)) \\ &= (|V(G)| + 2|E(G)|)WW(H) \\ &\quad + (|V(H)| + 2|E(H)|)WW(G) - 2|E(G)||E(H)| \\ &\quad + \sum_{ab \in E(\bar{G})} \sum_{xy \in E(\bar{H})} (d_{G\boxtimes H}((a,x),(b,y)) + d^2_{G\boxtimes H}((a,x),(b,y))). \end{aligned}$$

On the other hand,

$$\sum_{ab\in E(\bar{G})}\sum_{xy\in E(\bar{H})}(d_{G\boxtimes H}((a,x),(b,y))+d^2_{G\boxtimes H}((a,x),(b,y)))$$
$$\begin{aligned}&\geqslant \frac{3}{2}|V(G)||V(H)|(|V(G)||V(H)|-|V(G)|-|V(H)|+1)\\&-3|E(G)||V(H)|(|V(H)|-1)-3|E(H)||V(G)|(|V(G)|-1)\\&+6|E(G)||E(H)|,\end{aligned}$$

which completes the proof. □

Theorem 2.5. *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned}WW(G\boxtimes H)\leqslant{}&(|V(G)|+2|E(G)|)WW(H)+(|V(H)|+2|E(H)|)WW(G)\\&+\frac{1}{2}D(D+1)\Big[\frac{|V(G)||V(H)|}{2}(|V(G)||V(H)|-|V(G)|-|V(H)|+1)\\&-2|E(H)|\binom{|V(G)|}{2}-2|E(G)|\binom{|V(H)|}{2}\Big]\\&+|E(G)||E(H)|(D^2+D-2),\end{aligned}$$

where $D=max\{D(G),D(H)\}$. Moreover, the upper bound is attained if and only if $D\leqslant 2$, or $G\cong K_n$, or $H\cong K_n$.

Proof. Using a similar argument as in Theorem 2.4, and the fact that for every $ab\in E(\bar{G})$ and $xy\in E(\bar{H})$, $d_{G\boxtimes H}((a,x),(b,y))\leqslant D$, we obtain the result. □

The first Zagreb index of a graph G is defined as $M_1(G)=\sum_{v\in V(G)}deg^2(v)$ and the second Zagreb of G is given by $M_2(G)=\sum_{uv\in E(G)}deg(u)deg(v)$, see [18][14] for details.

Theorem 2.6. *For every graph G and H, we have*

$$\begin{aligned}M_1(G\boxtimes H)={}&(|V(H)|+4|E(H)|)M_1(G)+(|V(G)|+4|E(G)|)M_1(H)\\&+M_1(G)M_1(H)+8|E(G)||E(H)|.\end{aligned}$$

Proof. It follows from the edge structure of $G\boxtimes H$ that, for each vertex $(a,x)\in E(G\boxtimes H)$, we have

$$deg_{G\boxtimes H}((a,x))=deg_G(a)+deg_H(x)+deg_G(a)deg_H(x),$$

as desired. □

Theorem 2.7. *For every graph G and H, we have*

$$\begin{aligned}M_2(G\boxtimes H)={}&3|E(H)|M_1(G)+3|E(G)|M_1(H)\\&+3M_1(G)M_1(H)+2M_2(G)M_2(H)\\&+(6|E(H)|+3M_1(H)+|V(H)|)M_2(G)\\&+(6|E(G)|+3M_1(G)+|V(G)|)M_2(H).\end{aligned}$$

Proof. Consider the following partition of $E(G \boxtimes H)$

$$A = \{(a,x)(b,y) \in E(G \boxtimes H) \mid ab \in E(G) \text{ and } xy \in E(H)\},$$
$$B = \{(a,x)(b,y) \in E(G \boxtimes H) \mid a = b \in V(G) \text{ and } xy \in E(H)\},$$
$$C = \{(a,x)(b,y) \in E(G \boxtimes H) \mid ab \in E(G) \text{ and } x = y \in V(H)\}.$$

The sum of $deg_{G\boxtimes H}(u)deg_{G\boxtimes H}(v)$ over all edges of A, is equal to

$$2(|E(H)|+M_1(H)+M_2(H))M_2(G)+2(|E(G)|+M_1(G))M_2(H)+M_1(G)M_1(H). \tag{2.4}$$

On the other hand, the summation of $deg_{G\boxtimes H}(u)deg_{G\boxtimes H}(v)$ over all edges of B, is equal to

$$|E(H)|M_1(G)+(2|E(G)|+M_1(G))M_1(H)+(|V(G)|+4|E(G)|+M_1(G))M_2(H), \tag{2.5}$$

and finally, summing $deg_{G\boxtimes H}(u)deg_{G\boxtimes H}(v)$ over all edges of C we arrive at

$$|E(G)|M_1(H) + (2|E(H)| + M_1(H))M_1(G) + (|V(H)| + 4|E(H)| + M_1(H))M_2(G). \tag{2.6}$$

Now, by summation of (2.4), (2.5) and (2.6), the result can be proved. □

A connected graph is called a self-centered graph if all of its vertices have the same eccentricity [3]. Then a connected graph G is self-centered if and only if $r(G) = D(G)$.

Theorem 2.8. *Let G and H be self-centered graphs that $D(H) \leqslant D(G)$. Then*

$$\xi(G \boxtimes H) = 2r(G)(|E(G)||V(H)| + |E(H)||V(G)| + 2|E(G)||E(H)|).$$

Proof. The result follows from Lemma 2.1 and the fact that

$$|E(G \boxtimes H)| = |E(G)||V(H)| + |E(H)||V(G)| + 2|E(G)||E(H)|.$$

□

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