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A NOTE ON SIGNED CYCLE DOMINATION IN GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple graph. A function $f : E \to \{-1, 1\}$ is said to be a signed cycle dominating function (SCDF) of G if $\sum_{e\in E(C)} f(e) \geq 1$ holds for every induced cycle C of G. The signed cycle domination number of G is defined as $\gamma'_{sc}(G) = \min\{\sum_{e\in E(G)} f(e) \mid f$ is an $SCDF$ of $G\}$. B. Xu [4] conjectured that for any maximal planar graph G of order $n \geq 3$, $\gamma'_{sc}(G) = n - 2$. In this paper, we first prove that the conjecture is true and then we show that if G is a connected cubic claw-free graph of order n, then $\gamma'_{sc}(G) \leq n$.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [3] for terminology and notation which are not defined here. For a vertex $v \in V(G)$, $N_G(v)$ will denote the *open neighborhood* of v in G and $N_G[v] = N_G(v) \cup \{v\}$ will denote its *closed neighborhood*. For every nonempty subset V' of $V(G)$ the subgraph of G whose vertex set is V' and whose edge set is the set of edges of G with both ends in V' is called the *induced subgraph* of G by V' and denoted by $G[V']$. A cycle C of G is said to be an induced cycle if $G[V(C)] = C$.

In 2009, Xu [4] introduced a new kind of the edge domination as follows. A function $f : E \to \{-1, 1\}$ is said to be a *signed cycle dominating function* (SCDF) of G if $\sum_{e\in E(C)} f(e) \geq 1$ holds for any induced cycle C of G. The *signed cycle domination number* of G is defined as $\gamma'_{sc}(G) = \min\{\sum_{e\in E(G)} f(e) \mid f$ is an $SCDF$ of $G\}$. If f is an SCDF of G, then for any $H \subseteq G$ we write $f(H) = \sum_{e\in E(H)} f(e)$. A $\gamma'_{sc}(G)$-function is a signed cycle dominating function with $f(G) = \gamma'_{sc}(G)$.

A planar graph G is said to be *maximal* if $G + e$ is not a planar graph for any new edge e. Clearly, every face of a maximal planar graph is a triangle. Xu in [4] posed the following conjecture.

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Conjecture 1. For every maximal planar graph G of order $n \geq 3$, $\gamma'_{sc}(G) = n - 2$.

In this paper, we first prove that Conjecture 1 is true and then we prove that if G is a connected cubic claw-free graph of order n, then $\gamma'_{sc}(G) \leq n$.

We make use of the following results (see [3]) in this paper.

Theorem A. *Every maximal planar graph G of order $n \geq 4$, is 3-connected.*

Theorem B. *The dual graph of a simple 3-connected plane graph is both simple and 3-connected.*

Theorem C. *Let G be a maximal planar graph of order $n \geq 3$, then $|E(G)| = 3n-6$.*

Theorem D. *Every 3-regular graph with no cut-edge has a 1-factor.*

Theorem E. (Xu [4]) For any maximal planar graph G of order $n \geq 3$, $\gamma'_{sc}(G) \geq n-2$.

2. Main results

Theorem 2.1. *For every maximal planar graph G of order $n \geq 3$, $\gamma'_{sc}(G) = n - 2$.*

Proof. The statement is obviously true for $n = 3, 4$. Let $n \geq 5$. Then clearly $G \neq K_n$. Let $M_1, \ldots, M_r$ be the faces of G. Obviously, every edge is in exactly two faces of G and

$$3r = 2|E(G)|. \tag{2.1}$$

Since faces of G are triangles, each face can be represented by a unique 3-subset of $V(G)$. Let G' be the graph with vertex set $\{M_1, \ldots, M_r\}$ and $M_iM_j \in E(G')$ if and only if M_i and M_j share an edge. By Theorems A and B, G' has no cut-edges.

Since G' is 3-regular with no cut-edges, G' has a perfect matching by Theorem D. Let $\{e_i = M_iM'_i \mid 1 \leq i \leq \frac{r}{2}\}$ be a perfect matching of G'. Then obviously $|M_i \cap M'_i| = 2$, that is, M_i and M'_i share an edge. Assume that $M_i \cap M'_i = \{x_i, x'_i\}$, which forces $x_ix'_i \in E(G)$. Define $f : E(G) \to \{-1, +1\}$ by

$$f(e) = \begin{cases} -1 & \text{if } e \in \{x_1x'_1, \ldots, x_{\frac{r}{2}}x'_{\frac{r}{2}}\} \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, f is an SCDF for G and by (2.1) and Theorem C,

$$\gamma'_{sc}(G) \leq f(G) = |E(G)| - r = |E(G)| - \frac{2}{3}|E(G)| = \frac{|E(G)|}{3} = \frac{3n-6}{3} = n - 2.$$

Using Theorem E, the proof is complete. $\square$

A claw-free graph is a graph with no induced subgraph isomorphic to $K_{1,3}$. In this section we present an upper bound for the signed cycle domination number of connected 3-regular claw-free graphs. A subset S of $V(G)$ is said to be a 2-*packing set* of G if for every pair of distinct vertices s and s' in S, the distance $d(s, s')$ between s and s' is at least 3. Let $\rho(G) = \max\{|S| : S \text{ is a 2-packing set of } G\}$ and call it the 2-packing number of G. A $\rho(G)$-set is a 2-packing set of G with cardinality $\rho(G)$. Moo Young Sohn et al. in [2] proved the following theorem.

Theorem F. *Every connected cubic graph G of order n whose each vertex is contained in at least one triangle satisfies $\rho(G) \geq \frac{n}{6}$.*

Theorem 2.2. *For every connected cubic claw-free graph G of order n, $\gamma'_{sc}(G) \leq n$.*

Proof. Let $S = \{x_1, \ldots, x_{\rho(G)}\}$ be a $\rho(G)$-set. Suppose that $N(x_i) = \{y_1^i, y_2^i, y_3^i\}$ for each $1 \leq i \leq \rho(G)$. Since G is a claw-free graph, we may assume $y_1^i y_2^i \in E(G)$ for each $1 \leq i \leq \rho(G)$. Let $M = \{y_1^i y_2^i \mid 1 \leq i \leq \rho(G)\}$ and let G_1 be the subgraph induced by $\{y_1^i, y_2^i \mid 1 \leq i \leq \rho(G)\}$. Obviously, $\Delta(G_1) \leq 2$ and so G_1 is the union of disjoint paths and cycles. Let $C_1, \ldots, C_r$ be the even cycles of G_1. Obviously, each C_i contains at least two edges in M, hence $r \leq \frac{\rho(G)}{2}$. Without loss of generality, we may assume $y_1^i y_2^i \in E(C_i)$ for $i = 1, \ldots, r$. Define $f : E(G) \to \{-1, +1\}$ by

$$f(e) = \begin{cases} -1 & \text{if } e \in (M \setminus \{y_1^i y_2^i \mid 1 \leq i \leq r\}) \cup \{x_i y_3^i \mid 1 \leq i \leq \rho(G)\} \\ 1 & \text{otherwise.} \end{cases}$$

We claim that f is a signed cycle dominating function of G. Let, to the contrary, C be an induced cycle of G such that $f(E(C)) \leq 0$. Since $\{e \in E(G) \mid f(e) = -1\}$ is a matching of G, we have $f(E(C)) = 0$. Therefore, C is an even cycle of G and f assigns alternatively $-1, +1$ to the edges of C. If $x_i y_3^i \in E(C)$ for some i, then obviously $y_1^i y_2^i \notin E(C)$ because C is an induced cycle. This forces f assigns $+1$ to two consecutive edges of C, which is a contradiction. Hence, C is an induced cycle in G_1. Now by the definition of f and the fact that $y_1^i y_2^i \in E(C_i)$ for $i = 1, \ldots, r$ we see that f assigns $+1$ to three consecutive edges of C, a contradiction. Thus f is a signed cycle dominating function of G. Let $T = \{e \in E(G) \mid f(e) = -1\}$. By Theorem F and the fact that $r \leq \frac{\rho(G)}{2}$ we have

$$\begin{aligned} \gamma'_{sc}(G) &\leq f(E(G)) = |E(G)| - 2|T| \\ &= \frac{3n}{2} - 2(2\rho(G) - r) \leq \frac{3n}{2} - 3\rho(G) \leq n, \end{aligned}$$

as desired. □

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