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STRONG CONVERGENCE THEOREM FOR SEMIGROUP OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS USING VISCOSITY APPROXIMATION

BALWANT SINGH THAKUR

Abstract. Viscosity method also called elliptic regularization, provide an efficient approach to a large number of problems coming from different branches of mathematical and physical sciences. One of the recent trend in the iterative construction of fixed point of nonlinear mappings is to use viscosity approximation method. In this paper we propose implicit and explicit viscosity method (VAM) for strongly continuous semigroup of asymptotically nonexpansive mappings and prove strong convergence of proposed VAM which converges strongly to a solution of a variational inequality.

1. Introduction

Viscosity method provide an efficient approach to a large number of problems coming from different branches of mathematical analysis: Mathematical programming, variational problem, partial differential equations, control theory, ill-posed problems... A major feature of these methods is to provide as a limit of the solution of the approximate problems, a particular (possibly relaxed or generalized) solution of the original problem, called a viscosity solution, which has remarkable properties. The viscosity method is also called elliptic regularization, it has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition. It plays a central role too in the study of degenerated elliptic and parabolic second order equations [15][16][17]. Various applications of the viscosity methods can be found in optimal control theory, singular perturbations,

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minimal cost problem [1][18][19] and in stochastic control theory [12].

First abstract formulation of the properties of the viscosity approximation have been given by Tykhonov [26] in 1963 when studying ill-posed problems (see [10] for details). The concept of viscosity solution for Hamilton-Jacobi equations, which plays a crucial role in control theory, game theory and partial differential equations has been introduced by Crandall and Lions [9].

Let us now make precise the mathematical abstract setting. Given $f : X \to \mathbb{R} \cup \{+\infty\}$ an extended real valued function, its definition may include some constraints, let us consider the minimization problem

$$\min\{f(x) \ : \ x \in X\} \tag{$\mathcal{P}$}$$

which is assumed to be ill-posed (lack of existence, or uniqueness, or stability of a solution). Let $g : X \to \mathbb{R}_+ \cup \{+\infty\}$, be a nonnegative real-valued function (the viscosity function), and, for any $\varepsilon > 0$ a small parameter intended to go to zero, let us consider the approximate minimization problem

$$\min\{f(x) + \varepsilon g(x) \ : \ x \in X\} \tag{$\mathcal{P}_\varepsilon$}.$$

The viscosity function g usually enjoys nice properties, which make the approximate minimization problem $(\mathcal{P}_\varepsilon)$ well posed (existence, uniqueness and stability). So, it is assumed that, for all $\varepsilon > 0$, there exists a solution u_ε of $(\mathcal{P}_\varepsilon)$. The central question is to study the convergence of the sequence $\{u_\varepsilon\}$, $\varepsilon \to 0$ and the characterization of its limit.

On the otherhand numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. Due to practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be a flourishing topic of interest in fixed point theory.

Let K be a closed convex subset of Hilbert space E and let $T : K \to K$ be a nonexpansive mapping ($\|Tx - Ty\| \leq \|x - y\|\,, \ \forall x, y \in K$). The most straightforward attempt to solve the fixed point problem for nonexpansive mappings is by Picard iteration :

$$K \ni x_0 \mapsto x_1 = Tx_0 \mapsto x_2 = Tx_1 \mapsto \cdots \mapsto x_n = Tx_{n-1} \mapsto x_{n+1} = Tx_n \mapsto \cdots$$

or more compactly,

$$x_{n+1} = Tx_n\,, \quad \forall\, n \geq 0 \ \ (x_0 \in K)\,. \tag{1.1}$$

Unfortunately, algorithm (1.1) may fail to produce a norm convergence sequence $\{x_n\}$.

In view of celebrated Banach contraction principle, the attempt to approximate fixed point of nonexpansive self mappings seems very promising.

For given $u \in K$ and each $t \in (0,1)$ define a contraction $T_t : K \to K$ by

$$T_t x = tu + (1-t)Tx\,, \quad \forall x \in K\,.$$

Clearly T_t is $(1-t)$ contraction, so by Banach contraction principle, it has a unique fixed point $z_t \in K$, i.e. z_t is the unique solution of equation

$$z_t = tu + (1-t)Tz_t\,, \tag{1.2}$$

here z_t is defined implicitly.

In 1967, Browder [2] proved that z_t defined by (1.2) converges strongly to a fixed point of T as $t \to 0$.

In the same year, Halpern [13] device an explicit iteration method which converges in norm to a fixed point of T, the iteration process is known as Halpern iterative method and defined as below.
For a sequence $\{\alpha_n\}$ in $(0,1)$, obtain the modified version of (1.1)

$$\begin{aligned} K \ni x_0 \mapsto x_1 = \alpha_0 u + (1-\alpha_0)Tx_0 \mapsto x_2 = \alpha_1 u + (1-\alpha_1)Tx_1 \mapsto \cdots \\ \mapsto x_n = \alpha_{n-1}u + (1-\alpha_{n-1})Tx_{n-1} \mapsto x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \mapsto \cdots \end{aligned}$$

or more compactly

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n\;, \quad n \geq 0\,. \tag{1.3}$$

Further, it is proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies certain conditions.

Another iteration process which is widely used to approximate fixed point of nonexpansive mappings is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n\,, \quad \forall\, n \geq 0 \;\; (x_0 \in K \text{ arbitrary}) \tag{1.4}$$

and the sequence $\{\alpha_n\}$ is in the interval $[0,1]$. Iteration (1.4) is known as Mann iteration process [20].

The advantage that process (1.4) over the process (1.3) (though the former has only weak convergence in general) is the use of average mapping $\alpha I + (1-\alpha)T$ in each iteration step. This averaged mapping behaves more regularly than the nonexpansive mapping T itself (see Bruck [4]). The weakness of process (1.4) is however, its weak convergence.

Given a real number $t \in (0,1)$ and a contraction mapping $f : K \to K$ with contraction constant $\alpha \in [0,1)$. Define a mapping $T_t : K \to K$ by

$$T_t x = tf(x) + (1-t)Tx, \quad x \in K\,.$$

Clearly T_t is a $(1-t(1-\alpha))$ contraction, and so has a unique fixed point $x_t \in K$. Thus x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t\,. \tag{1.5}$$

In 2000, Moudafi [21] proposed viscosity approximation method for nonexpansive mapping and proved following theorems.

Theorem M1. Let K be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of K into itself such that $F(T) \neq \emptyset$. Let f be a contraction of K into itself and define a sequence $\{x_n\}$ in implicit way by

$$x_n = \frac{1}{1+\varepsilon_n} Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)\,,$$

for all $n \in \mathbb{N}$, where $\{\varepsilon_n\} \subset (0,1)$ tending to zero. Then $\{x_n\}$ converges strongly to the unique solution $\bar{x} \in K$ of the variational inequality

$$\langle (I-f)\bar{x}, \bar{x}-x\rangle \leq 0\,.$$

In other words, $\bar{x}$ is the unique fixed point of $P_{F(T)}f$.
Here I is identity mapping and $F(T)$ the set of fixed points of T.

Theorem M2. Let K be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of K into itself such that $F(T) \neq \emptyset$. Let f be a contraction of K into itself and let $\{x_n\}$ be a sequence defined by $x_0 \in K$ arbitrary, and

$$x_{n+1} = \frac{1}{1+\varepsilon_n} Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)\,,$$

for all $n \in \mathbb{N}$, where $\{\varepsilon_n\} \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0\,, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0\,.$$

Then $\{x_n\}$ converges strongly to the unique solution $\bar{x} \in K$ of the variational inequality

$$\langle (I-f)\bar{x}, \bar{x}-x\rangle \leq 0\,.$$

Xu [28] studied the strong convergence of x_t defined by (1.5) as $t \to 0$ in either a Hilbert space or a uniformly smooth Banach space and showed that the strong $\lim_{t\to 0} x_t$ is the unique solution of certain variational inequality.
He also introduced the following iterative algorithm

$$\begin{aligned} &x_0 \in K \\ &x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n)\,, \quad \forall n \geq 0\,, \end{aligned} \tag{1.6}$$

further, he extended Theorem M2 for iterative algorithm (1.6) to a Banach space setting where $\{\alpha_n\} \subset (0,1)$ satisfying

$$\alpha_n \to 0; \ \sum_{n=0}^{\infty} \alpha_n = \infty; \ \text{either} \ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ or } \lim_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1\,.$$

Before proceeding further, let us recall some definitions.
Let K be a closed convex subset of a Banach space E. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$. $\mathfrak{T} := \{T(t) : t \in \mathbb{R}_+\}$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers, is said to be *strongly continuous semigroup of asymptotically nonexpansive mappings* from K in to K if the following conditions are satisfied [7][25]

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s + t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;
(3) for each $t \in \mathbb{R}_+$, $T(t)$ be an asymptotically nonexpansive mapping on K,i.e.

$$\|(T(t))^n x - (T(t))^n y\| \leq k_n^{(t)} \|x - y\|$$

where $\{k_n^{(t)}\}$ is a sequence with $k_n^{(t)} \geq 1$ and $\lim_{n\to\infty} k_n^{(t)} = 1$.
(4) for each $x \in K$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into K is continuous.

If in the above definition, condition (3) is replaced by the following condition

(3)* for each $t \in \mathbb{R}_+$, $T(t)$ be an nonexpansive mapping on K,

then $\mathfrak{T}$ is called strongly continuous semi-group of nonexpansive mapping on K.

Shioji and Takahashi [24] proved following theorem for continuous semigroup of nonexpansive mappings.

Theorem ST. Let K be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on K such that $F = \cap_{t\in\mathbb{R}_+} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_n \alpha_n = 0$, $t_n > 0$ and $\lim_n t_n = \infty$. Fix $u \in K$ and define a sequence $\{u_n\}$ in K by

$$u_n = (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) u_n ds + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of F nearest to u.

Suzuki [25] proved the following result.

Theorem S. Let K be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on K such that $F = \cap_{t\in\mathbb{R}_+} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in K$ and define a sequence $\{u_n\}$ in K by

$$u_n = (1 - \alpha_n) T(t_n) u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of F nearest to u.

Xu [27] established a Banach space version of the Theorem S.

Recently Chen and He [6] studied in Banach space, following implicit and explicit viscosity approximation process for a nonexpansive semigroup

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}\,;$$
$$y_{n+1} = \beta_n f(y_n) + (1-\beta_n)T(t_n)y_n, \quad \forall n \in \mathbb{N}\,;$$

and proved that these sequences convergence strongly to $q \in F = \cap_{t\geq 0}F(T(t))$, which is the unique solution in F to some variational inequality, where $\{\alpha_n\}$, $\{\beta_n\}$,$\{t_n\}$ satisfy some appropriate conditions, and $f : K \to K$ is a contraction.

It is important to note that, when we study approximation schemes for asymptotically nonexpansive mappings, it looks more complicated than the schemes for nonexpansive mappings. Because T is not always nonexpansive (mapping which do not increase distance), i.e. T may increase distances. In order to overcome difficulties caused by increasingness of T, one need to adjust, the defining mapping at each iteration step in the iteration schemes, i.e., one has to use T^n (instead of T) at step n as the defining mapping. Though T^n may still increase distance (as $k_n \geq 1$), however since $k_n \to 1$ as $n \to \infty$, eventually T^n would increase distance marginally. Schu [23] was first to use the above idea and defined iteration scheme for asymptotically nonexpansive mapping which is known as modified Mann iteration.

Motivated by the above results and a recent work of Ceng et al. [5], in this paper we propose implicit and explicit viscosity approximation method (VAM) for strongly continuous semigroup of asymptotically nonexpansive mappings and prove strong convergence theorem for proposed VAM.

2. Preliminaries

Let E be a real Banach space and E^* be its dual space. Let K be a nonempty subset of E and let $T : K \to K$ be a self mapping of K. The fixed point set of T is denoted by $F(T) := \{x \in K : Tx = x\}$. The notation $\rightharpoonup$ denotes weak convergence and the notation $\to$ denotes strong convergence. Also, $I : K \to K$ denotes the identity mapping of K.

A Banach space E is said to satisfy Opial's condition [22], if for any sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$ as $n \to \infty$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|\,, \quad \forall\, y \in E \text{ with } x \neq y\,.$$

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}_+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{x^* \in E^*, \langle x, x^* \rangle = \|x\| \, \|x^*\| \, , \|x^*\| = \varphi(\|x\|)\}, \ \ \forall x \in E\,,$$

is called the duality mapping with gauge function φ, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair.

In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J is referred to as the normalized duality mapping. Browder [3] initiated the study of certain class of nonlinear operators by means of the duality mapping J_φ. Set

$$\Phi(t) = \int_0^t \varphi(r)dr\,, \quad \text{for every } t \in \mathbb{R}_+\,.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|.\|)$ at x. Now recall that E is said to have a weakly continuous duality map if there exists a gauge φ such that the duality map J_φ is single valued and continuous from E with the weak topology to E^* with the weak* topology. A space with a weakly continuous duality map is easily seen to satisfy Opial's condition (cf. [3][11]). Every $l^p(1 < p < \infty)$ space has a weakly continuous duality map with the gauge $\varphi(t) = t^{p-1}$.

We will use the following properties of duality mappings.

Lemma 2.1 ([5][29]). *Let E be a real Banach space. Let J_φ be the duality map associated with the gauge φ.*

(i) *For all $x, y \in E$ and $j \in J_\varphi(x + y)$,*

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, j \rangle \ .$$

In particular, for $x, y \in E$ and $j \in J(x + y)$

$$\|x + y\|^2 \le \|x\|^2 + 2 \langle y, j \rangle \ .$$

(ii) *For $\lambda \in \mathbb{R}$ and for nonzero $x \in E$,*

$$J_\varphi(\lambda x) = \operatorname{sgn}(\lambda) \, \frac{\varphi\left(|\lambda| \cdot \|x\|\right)}{\|x\|} \, J(x)\,.$$

(iii) *Assume J_φ is weakly continuous. Then for any sequence $\{x_n\}$ in E which converges weakly to a point x^*, we have for all $y \in E$,*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x^*\|) + \Phi(\|y - x^*\|)\,.$$

In particular, E satisfies Opial's condition.

(iv) *J is surjective if and only if E is reflexive.*

The following lemma will be needed in the sequel.

Lemma 2.2. [8][28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\lambda_n)a_n + \lambda_n\mu_n\,, \quad n = 0,1,2,\cdots,$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ and $\{\mu_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \mu_n \le 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n\mu_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Through out this section, E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ associated with a gauge φ, K is a nonempty closed convex subset of E, $f : K \to K$ is an α-contraction, i.e. $\|fx - fy\| \le \alpha \|x-y\|$, for some $\alpha \in [0,1)$ and for all $x, y \in K$. $\{T(t) : t \in \mathbb{R}_+\}$ is a strongly continuous semigroup of asymptotically nonexpansive mappings on K such that $F = \cap_{t>0} F(T(t)) \neq \emptyset$.

For a fixed $n \ge 1$, let $k_n = \max\{k_n^{(t)} : t \in \mathbb{R}_+\}$, then $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$.

We now propose an implicit VAM for strongly continuous semigroup of asymptotically nonexpansive mappings, which generates a sequence $\{x_n\}$ implicitly by.

$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{1-\alpha_n}{k_n} f x_n + \frac{\alpha_n}{k_n} (T(t_n))^n x_n \tag{3.1}$$

and the explicit VAM, which generates a sequence $\{z_n\}$ explicitly ($x_0 \in K$ arbitrary) given by

$$z_{n+1} = \left(1 - \frac{1}{k_n}\right) z_n + \frac{1-\alpha_n}{k_n} f z_n + \frac{\alpha_n}{k_n} (T(t_n))^n z_n \tag{3.2}$$

where $\{\alpha_n\}$ and $\{t_n\}$ be sequence of real numbers satisfying satisfying

$$\begin{cases} 0 < \alpha_n < \dfrac{1-\alpha}{k_n - \alpha} \quad \text{and} \quad \displaystyle\lim_{n\to\infty} \frac{k_n - 1}{1-\alpha_n} = 0\,, \\ t_n > 0 \ (\forall n) \quad \text{and} \quad \displaystyle\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \frac{1-\alpha_n}{t_n}. \end{cases} \tag{3.3}$$

For each n, consider a mapping $H : K \to K$ defined by

$$Hx = \left(1 - \frac{1}{k_n}\right) x + \frac{1-\alpha_n}{k_n} f x + \frac{\alpha_n}{k_n} (T(t_n))^n x\,, \quad x \in K\,.$$

For each $x, y \in K$, we have

$$\begin{aligned} \|Hx - Hy\| &\le \left(1 - \frac{1}{k_n}\right) \|x-y\| + \frac{1-\alpha_n}{k_n} \|fx - fy\| + \frac{\alpha_n}{k_n} \|(T(t_n))^n x - (T(t_n))^n y\| \\ &\le \left(1 - \frac{1}{k_n} + \frac{\alpha(1-\alpha_n)}{k_n} + \alpha_n\right) \|x-y\|\,, \end{aligned}$$

as a consequence of (3.3)

$$1-\frac{1}{k_n}+\frac{\alpha(1-\alpha_n)}{k_n}+\alpha_n<1\,,$$

hence H is a contraction, and the iteration is well defined.

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by the implicit scheme (3.1). If (3.3) holds, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality*

$$\langle (f-I)x^*, J(x-x^*)\rangle \le 0\,, \quad \forall\, x \in F\,. \tag{3.4}$$

Proof. First we show that $\{x_n\}$ is bounded. Notice that Φ is convex with $\Phi(0)=0$, so that $\Phi(\lambda\mu) \le \Phi(\mu)$ for all $\lambda \in [0,1]$. For any fixed $p \in F$, from (3.1) and Lemma 2.1 (i), we have

$$\begin{aligned}
\Phi\left(\|x_n-p\|\right) &= \Phi\left\|\left(1-\frac{1}{k_n}\right)(x_n-p)+\frac{1-\alpha_n}{k_n}(fx_n-fp)\right.\\
&\qquad \left.+\frac{1-\alpha_n}{k_n}(fp-p)+\frac{\alpha_n}{k_n}\left((T(t_n))^n x_n-p\right)\right\|\Big)\\
&\le \Phi\left\|\left(1-\frac{1}{k_n}\right)(x_n-p)+\frac{1-\alpha_n}{k_n}(fx_n-fp)+\frac{\alpha_n}{k_n}\left((T(t_n))^n x_n-p\right)\right\|\\
&\qquad +\frac{1-\alpha_n}{k_n}\left\langle fp-p, J_\varphi(x_n-p)\right\rangle\\
&\le \left[1-\frac{1}{k_n}+\frac{(1-\alpha_n)\alpha}{k_n}+\alpha_n\right]\Phi\left(\|x_n-p\|\right)\\
&\qquad +\frac{1-\alpha_n}{k_n}\left\langle fp-p, J_\varphi(x_n-p)\right\rangle\\
&= \left[1-\frac{1-\alpha(1-\alpha_n)-\alpha_n k_n}{k_n}\right]\Phi\left(\|x_n-p\|\right)\\
&\qquad +\frac{1-\alpha_n}{k_n}\left\langle fp-p, J_\varphi(x_n-p)\right\rangle\\
&= (1-\eta_n)\Phi\left(\|x_n-p\|\right)+\frac{1-\alpha_n}{k_n}\left\langle fp-p, J_\varphi(x_n-p)\right\rangle\,,
\end{aligned}$$

where $\eta_n = \frac{1-\alpha(1-\alpha_n)-\alpha_n k_n}{k_n}$.

Therefore,

$$\Phi\left(\|x_n-p\|\right) \le \frac{1-\alpha_n}{k_n\eta_n}\left\langle fp-p, J_\varphi(x_n-p)\right\rangle\,, \tag{3.5}$$

inequality (3.5) hods for all duality mapping J_φ; in particular, if we take normalized duality mapping J, then we get

$$\|x_n - p\|^2 \leq \frac{2(1-\alpha_n)}{k_n \eta_n} \langle fp - p, J(x_n - p) \rangle \ ,$$

which implies that

$$\|x_n - p\| \leq \frac{2(1-\alpha_n)}{k_n \eta_n} \|fp - p\| \ . \tag{3.6}$$

Using (3.3), we have

$$\frac{1-\alpha_n}{k_n \eta_n} = \left(1 - \frac{k_n - 1}{1-\alpha_n}\alpha_n - \alpha\right)^{-1} \to \frac{1}{1-\alpha} \ . \tag{3.7}$$

In view of (3.6) and (3.7), we get that $\{x_n\}$ is bounded and so $\{f(x_n)\}$ and $\{(T(t_n))^n x_n\}$ are bounded. Since E is reflexive and $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $x_{n_j} \rightharpoonup x^* \in K$ as $j \to \infty$.

Put $u_j = x_{n_j}\ ,\ \ \beta_j = \alpha_{n_j}\ ,\ \ s_j = t_{n_j}$ for $j \in \mathbb{N}$ and fix $t > 0$.
Now,

$$\begin{aligned}
\left\|u_j - (T(t))^j x^*\right\| &\leq \sum_{k=0}^{[t/s_j]-1} \left\|(T((k+1)s_j))^j u_j - (T(ks_j))^j u_j\right\| \\
&\quad + \left\|(T([t/s_j]\, s_j))^j u_j - (T([t/s_j]\, s_j))^j x^*\right\| \\
&\quad + \left\|(T([t/s_j]\, s_j))^j x^* - (T(t))^j x^*\right\| \\
&\leq [t/s_j]\, k_j \left\|(T(s_j))^j u_j - u_j\right\| + k_j \|u_j - x^*\| \\
&\quad + k_j \left\|(T(t - [t/s_j]\, s_j))^j x^* - x^*\right\| \\
&= t k_j \left(\frac{1-\beta_j}{s_j}\right) \left\|(T(s_j))^j u_j - f(u_j)\right\| + k_j \|u_j - x^*\| \\
&\quad + k_j \max\left\{\left\|(T(s))^j x^* - x^*\right\| : 0 \leq s \leq s_j\right\} \ ,
\end{aligned}$$

for all $j \in \mathbb{N}$, we have

$$\limsup_{j\to\infty} \left\|u_j - (T(t))^j x^*\right\| \leq \limsup_{j\to\infty} \|u_j - x^*\| \ .$$

By Lemma 2.1 (iii), E satisfies Opial's condition, above inequality implies that $(T(t))^j x^* \to x^*$ as $j \to \infty$. This gives

$$x^* = \lim_{j\to\infty} (T(t))^j x^* = \lim_{j\to\infty} (T(t))^{j+1} x^* = T(t)\left(\lim_{j\to\infty} (T(t))^j x^*\right) = T(t)x^* \ .$$

Therefore $x^* \in F$.

Now in (3.5), replacing x_n with x_{n_j}, we have

$$\Phi\left(\|x_{n_j} - p\|\right) \leq \frac{1-\alpha_{n_j}}{k_{n_j}\eta_{n_j}} \left\langle f(x^*) - x^*, J_\varphi(x_{n_j} - x^*)\right\rangle .$$

Since the duality mapping J_φ is weak continuous and $x_{n_j} \rightharpoonup x^*$, taking the limit as $j \to \infty$, we obtain

$$\lim_{j\to\infty} \Phi\left(\|x_{n_j} - p\|\right) \leq 0 .$$

that is $x_{n_j} \to x^*$ as $j \to \infty$.

We further show that x^* solve the variational inequality (3.4).
For any $p \in F$, we have

$$\begin{aligned}\left\langle (T(t_n))^n x_n - p \, , J(x_n - p)\right\rangle &\leq \|(T(t_n))^n x_n - p\| \, \|x_n - p\| \\ &\leq k_n \|x_n - p\|^2 \ .\end{aligned}$$

So by (3.1), we have

$$\begin{aligned}\Phi\left(\|x_n - p\|\right) &= \Phi\left(\left\|\left(1 - \frac{1}{k_n}\right)(x_n - p) + \frac{1-\alpha_n}{k_n}\ \left[(f(x_n) - x_n) + (x_n - p)\right]\right.\right.\\ &\qquad \left.\left. + \frac{\alpha_n}{k_n}\ \left((T(t_n))^n x_n - p\right)\right\|\right) \\ &\leq \Phi\left(\left\|\left(1 - \frac{\alpha_n}{k_n}\right)(x_n - p) + \frac{\alpha_n}{k_n}\ \left((T(t_n))^n x_n - p\right)\right\|\right) \\ &\qquad + \frac{1-\alpha_n}{k_n}\ \left\langle f(x_n) - x_n, J_\varphi(x_n - p)\right\rangle \\ &\leq \left(1 - \frac{\alpha_n}{k_n} + \alpha_n\right)\Phi\left(\|x_n - p\|\right) + \frac{1-\alpha_n}{k_n}\ \left\langle f(x_n) - x_n \, , J_\varphi(x_n - p)\right\rangle \\ &= \left(1 + \alpha_n\left(\frac{k_n - 1}{k_n}\right)\right)\Phi\left(\|x_n - p\|\right) + \frac{1-\alpha_n}{k_n}\ \left\langle f(x_n) - x_n \, , J_\varphi(x_n - p)\right\rangle \ ,\end{aligned}$$

after simplification, we have

$$\frac{1-\alpha_n}{k_n}\ \left\langle f(x_n) - x_n \, , J_\varphi(p - x_n)\right\rangle \leq \alpha_n\left(\frac{k_n - 1}{k_n}\right)\Phi\left(\|x_n - p\|\right) .$$

This implies that

$$\left\langle f(x_n) - x_n \, , J_\varphi(p - x_n)\right\rangle \leq \left(\frac{k_n - 1}{1-\alpha_n}\right)\Phi\left(\|x_n - p\|\right) . \tag{3.8}$$

Replacing x_n with x_{n_j} in (3.8), passing through the limit $j \to \infty$, by using (3.3), we get

$$\begin{aligned}\langle f(x^*) - x^*\,, J_\varphi(p - x^*)\rangle &= \lim_{j\to\infty} \left\langle f(x_{n_j}) - x_{n_j}\,, J_\varphi(p - x_{n_j})\right\rangle \\ &\leq \limsup_{j\to\infty} \left(\frac{k_{n_j} - 1}{1 - \alpha_{n_j}}\right) \Phi\left(\left\|x_{n_j} - p\right\|\right) = 0\,.\end{aligned}$$

By Lemma 2.1(ii), $J(x^* - p)$ is a positive scalar multiple of $J_\varphi(x^* - p)$, hence, x^* is a solution to (3.4).

In summary, we have proved that every weak limit point of $\{x_n\}$ is a strong limit point of $\{x_n\}$ and this limit point solves the variational inequality (3.4). So, to see that the full sequence $\{x_n\}$ actually converges strongly to p, it is suffices to prove that the variational inequality (3.4) can have only one solution this is an easy consequence of the contractivity of f.

Uniqueness of the solution

Assume that both $u \in F$ and $v \in F$ are solutions of (3.4), then we have

$$\langle (f - I)u, J(v - u)\rangle \leq 0 \quad \text{and} \quad \langle (f - I)v, J(u - v)\rangle \leq 0\,.$$

Adding them yields,

$$\langle (f - I)u - (f - I)v, J(u - u)\rangle \leq 0\,. \tag{3.9}$$

However, the α-contractivity of f and (3.9), gives

$$(1 - \alpha)\,\|u - v\|^2 \leq \langle (I - f)u - (I - f)v, J(u - v)\rangle \leq 0\,,$$

so, we must have $u = v$. Hence (3.4) can have at most one solution.
This completes the proof. □

In the next theorem, we study strong convergence of explicit scheme (3.2)

Theorem 3.2. *Let $\{z_n\}$ be the sequence generated by the explicit scheme* (3.2)*. If* (3.3) *holds and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Then $\{z_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality*

$$\langle (f - I)x^*, J(x - x^*)\rangle \leq 0\,, \quad \forall\, x \in F\,. \tag{3.10}$$

Proof. For any fixed $p \in F$,

$$\begin{aligned}
\|z_{n+1} - p\| &\leq \left(1 - \frac{1}{k_n}\right) \|z_n - p\| + \frac{1-\alpha_n}{k_n} \|f(z_n) - p\| + \frac{\alpha_n}{k_n} \|(T(t_n))^n z_n - p\| \\
&\leq \left(1 - \frac{1}{k_n} + \alpha_n\right) \|z_n - p\| + \frac{1-\alpha_n}{k_n} (\|f(z_n) - f(p)\| + \|f(p) - p\|) \\
&\leq \left(1 - \frac{1}{k_n} + \frac{(1-\alpha_n)\alpha}{k_n} + \alpha_n\right) \|z_n - p\| + \frac{1-\alpha_n}{k_n} \|f(p) - p\| \\
&\leq (1 - \eta_n) \|z_n - p\| + \eta_n(\gamma \|f(p) - p\|) \\
&\leq \max \{\|z_n - p\|, \gamma \|f(p) - p\|\} ,
\end{aligned}$$

where

$$\eta_n = \frac{1 - (\alpha + (k_n - \alpha)\alpha_n)}{k_n}, \quad \gamma \geq \frac{1-\alpha_n}{k_n \eta_n} .$$

By induction

$$\|z_n - p\| \leq \max \{\|z_0 - p\|, \gamma \|f(p) - p\|\} .$$

Hence $\{z_n\}$ is bounded and so are $\{f(z_n)\}$ and $\{(T(t_n))^n z_n\}$. It follows from the proof of Theorem 3.1(Uniqueness of solution), that there is a unique solution $q \in F$ of variational inequality

$$\langle (f - I)q, J(u - q)\rangle \leq 0, \quad \forall\, u \in F . \tag{3.11}$$

Next, we show that

$$\limsup_{n\to\infty} \langle (f - I)q, J(z_{n+1} - q)\rangle \leq 0 . \tag{3.12}$$

Since $\{z_n\}$ is bounded and E is reflexive, we can take a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\{z_{n_j}\} \rightharpoonup x^* \in K$. Weak continuity of J_φ imply that,

$$J(z_{n_j} - q) = \frac{\|z_{n_j} - q\|}{\varphi(\|z_{n_j} - q\|)} J_\varphi(z_{n_j} - q) \rightharpoonup \frac{\|x^* - q\|}{\varphi(\|x^* - q\|)} J_\varphi(x^* - q) = J(x^* - q) .$$

Hence,

$$\begin{aligned}
\limsup_{n\to\infty} \langle (f - I)q, J(z_{n+1} - q)\rangle &= \limsup_{j\to\infty} \left\langle (f - I)q, J(z_{n_j+1} - q)\right\rangle \\
&= \langle (f - I)q, J(x^* - q)\rangle .
\end{aligned} \tag{3.13}$$

Now we show that $x^* \in F$.
Put $u_j = z_{n_j}, \beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$, $k_j = k_{n_j}$ for $j \in \mathbb{N}$, fix $t > 0$, we have

$$\begin{aligned}\left\|u_{j+1} - (T(t))^j x^*\right\| &\leq \sum_{k=0}^{[t/s_j]-1} \left\|(T((k+1)s_j))^j u_j - (T(ks_j))^j u_{j+1}\right\| \\ &\quad + \left\|(T([t/s_j]\, s_j))^j u_{j+1} - (T([t/s_j]\, s_j))^j x^*\right\| \\ &\quad + \left\|(T([t/s_j]\, s_j))^j x^* - (T(t))^j x^*\right\| \\ &\leq [t/s_j] \left\|(T(s_j))^j u_j - u_{j+1}\right\| + k_j \left\|u_{j+1} - x^*\right\| . \end{aligned} \tag{3.14}$$

From (3.2), we have

$$\begin{aligned} u_{j+1} - (T(s_j))^j u_j &= \left(1 - \frac{1}{k_j}\right) \left[u_j - (T(s_j))^j u_j\right] \\ &\quad + \frac{1-\alpha_j}{k_j} \left[f u_j - (T(s_j))^j u_j\right] . \end{aligned} \tag{3.15}$$

Using (3.14) and (3.15), we get

$$\begin{aligned}\left\|u_{j+1} - (T(t))^j x^*\right\| &\leq [t/s_j] \cdot \frac{1-\beta_j}{k_j} \left\|(T(s_j))^j u_j - f(u_j)\right\| + k_j \left\|u_{j+1} - x^*\right\| \\ &\quad + [t/s_j] \left(1 - \frac{1}{k_j}\right) \left\|(T(s_j))^j u_j - u_j\right\| \\ &\quad + k_j \max\left\{\left\|(T(s))^j x^* - x^*\right\| : 0 \leq s \leq s_j\right\} . \end{aligned}$$

So, for all $j \in \mathbb{N}$, we have

$$\limsup_{j\to\infty} \left\|u_{j+1} - (T(t))^j x^*\right\| \leq \limsup_{j\to\infty} \left\|u_{j+1} - x^*\right\| .$$

By Lemma 2.1(iii), E satisfies Opial condition, this implies that $(T(t))^j x^* \to x^*$ as $j \to \infty$. This gives

$$x^* = \lim_{j\to\infty} (T(t))^j x^* = \lim_{j\to\infty} (T(t))^{j+1} x^* = T(t) \left(\lim_{j\to\infty} (T(t))^j x^*\right) = T(t) x^* ,$$

therefore $x^* \in F$. Hence, from (3.13) and (3.11), we have

$$\limsup_{n\to\infty} \langle (f-I)q, J(z_{n+1} - q)\rangle = \langle (f-I)q, J(x^* - q)\rangle \leq 0 ,$$

and (3.12) is proved.

Finally, we show that $z_n \to q$. Now

$$
\begin{aligned}
&\left\| \left(1-\frac{1}{k_n}\right)(z_n-q)+\frac{1-\alpha_n}{k_n}\left(f(z_n)-f(q)\right)+\frac{\alpha_n}{k_n}\left(T(t)\right)^n z_n - q\right\| \\
&\qquad \le \left(1-\frac{1}{k_n}\right)\|z_n-q\|+\frac{1-\alpha_n}{k_n}\|f(z_n)-f(q)\|+\frac{\alpha_n}{k_n}\|(T(t))^n z_n-q\| \\
&\qquad \le \left(1-\frac{1}{k_n}\right)\|z_n-q\|+\frac{(1-\alpha_n)\alpha}{k_n}\|z_n-q\|+\alpha_n\|z_n-q\| \\
&\qquad = \left[1-\frac{1}{k_n}+\frac{(1-\alpha_n)\alpha}{k_n}+\alpha_n\right]\|z_n-q\| \\
&\qquad = \left[1-\left(\frac{1-(\alpha+(k_n-\alpha)\alpha_n)}{k_n}\right)\right]\|z_n-q\| \, . \qquad (3.16)
\end{aligned}
$$

Using Lemma 2.1(i) and (3.16), we have

$$
\begin{aligned}
\|z_{n+1}-q\|^2 &= \left\| \left(1-\frac{1}{k_n}\right)(z_n-q)+\frac{1-\alpha_n}{k_n}(f(z_n)-f(q)) \right. \\
&\qquad \left. +\frac{\alpha_n}{k_n}\left((T(t))^n z_n-q\right)+\frac{1-\alpha_n}{k_n}(f(q)-q)\right\|^2 \\
&\le \left\| \left(1-\frac{1}{k_n}\right)(z_n-q)+\frac{1-\alpha_n}{k_n}(f(z_n)-f(q))+\frac{\alpha_n}{k_n}\left((T(t))^n z_n-q\right)\right\|^2 \\
&\qquad +\frac{2(1-\alpha_n)}{k_n}\langle f(q)-q, J(z_{n+1}-q)\rangle \\
&\le \left[1-\left(\frac{1-(\alpha+(k_n-\alpha)\alpha_n)}{k_n}\right)\right]^2\|z_n-q\|^2 \\
&\qquad +\frac{2(1-\alpha_n)}{k_n}\langle f(q)-q, J(z_{n+1}-q)\rangle \\
&\le \left[1-\left(\frac{1-(\alpha+(k_n-\alpha)\alpha_n)}{k_n}\right)\right]\|z_n-q\|^2 \\
&\qquad +\frac{2(1-\alpha_n)}{k_n}\langle f(q)-q, J(z_{n+1}-q)\rangle \\
&= (1-\lambda_n)\|z_n-q\|^2+\lambda_n\mu_n \, , \qquad (3.17)
\end{aligned}
$$

where

$$
\lambda_n=\frac{1-(\alpha+(k_n-\alpha)\alpha_n)}{k_n} \quad \text{and} \quad \mu_n=\frac{2(1-\alpha_n)}{k_n\lambda_n}\langle f(q)-q, J(z_{n+1}-q)\rangle \, ,
$$

after simplification

$$
\lambda_n=\frac{(1-\alpha_n)}{k_n}\left[(k_n-\alpha)-\frac{k_n-1}{1-\alpha_n}\right] \, ,
$$

hence
$$\lim_{n\to\infty} \frac{\lambda_n}{1-\alpha_n} = 1-\alpha > 0\,.$$
In other words $\lambda_n = \circ(1-\alpha_n)$, since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, we also have $\sum_{n=1}^{\infty} \lambda_n = \infty$ and using (3.7), by (3.12), we obtain

$$\limsup_{n\to\infty} \mu_n = \frac{2}{1-\alpha} \limsup_{n\to\infty} \langle f(q) - q, J(z_{n+1} - q)\rangle \le 0\,, \tag{3.18}$$

applying Lemma 2.2 to (3.17) and noticing (3.18), we conclude that $\|z_n - q\| \to 0$, that is, $z_n \to q$, as required.
This completes the proof.

□

If we take contraction f to be constant, then we have the following corollaries from Theorem 3.1 and Theorem 3.2.

Corollary 3.1. *Let $u \in K$ and $\{x_n\}$ be the sequence generated by the implicit scheme*
$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{1-\alpha_n}{k_n}\, u + \frac{\alpha_n}{k_n}\,(T(t_n))^n\, x_n\,.$$
If (3.3) holds, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality
$$\langle x^* - u, J(x - x^*)\rangle \le 0 \quad \forall\, x \in F\,.$$

Corollary 3.2. *Let $u \in K$ and $\{x_n\}$ be the sequence generated by the explicit scheme*
$$z_{n+1} = \left(1 - \frac{1}{k_n}\right) z_n + \frac{1-\alpha_n}{k_n}\, u + \frac{\alpha_n}{k_n}\,(T(t_n))^n\, z_n\,.$$
If (3.3) holds and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality
$$\langle x^* - u, J(x - x^*)\rangle \le 0 \quad \forall\, x \in F\,.$$

If K is a compact subset of a real smooth Banach space E, then weak sequential continuity of duality mapping may not be need. In this case we have the following corollaries from Theorem 3.1 and Theorem 3.2.

Corollary 3.3. *Let K be a nonempty convex, compact subset of a real smooth Banach space E and let $\{x_n\}$ be the sequence generated by the implicit scheme* (3.1)*. If* (3.3) *holds, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality* (3.4).

Corollary 3.4. *Let K be a nonempty convex, compact subset of a real smooth Banach space E and let $\{x_n\}$ be the sequence generated by the explicit scheme* (3.2)*. If* (3.3) *holds and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality* (3.10).

Since every Hilbert space is reflexive and satisfies Opial's condition, we have following corollaries.

Corollary 3.5. *Let K be a nonempty closed,convex subset of a Hilbert space E and let $\{x_n\}$ be the sequence generated by the implicit scheme* (3.1)*. If* (3.3) *holds, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality* (3.4)*.*

Corollary 3.6. *Let K be a nonempty closed,convex subset of a Hilbert space E and let $\{x_n\}$ be the sequence generated by the explicit scheme* (3.2)*. If* (3.3) *holds and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{x_n\}$ converges strongly to $x^* \in F$ as $n \to \infty$ such that x^* is the unique solution in F to variational inequality* (3.10)*.*

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School of Studies in Mathematics,
Pt. Ravishankar Shukla University, Raipur (C.G.) 492010,
India
E-mail address: balwantst@gmail.com