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EXTREMELY IRREGULAR GRAPHS

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Abstract. The *irregularity* of a graph G is defined as $irr(G) = \sum |d(x) - d(y)|$ where $d(x)$ is the degree of vertex x and the summation embraces all pairs of adjacent vertices of G. We characterize the graphs minimum and maximum values of irr.

1. Introduction

In this paper we are concerned with simple graphs, namely graphs without directed, multiple, or weighted edges, and without loops. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. An edge of G, connecting the vertices u and v will be denoted by uv. The degree of a vertex v of the graph G will be denoted by $d(v)$ or, when misunderstanding is possible, by $d(v|G)$.

As well known, a graph whose all vertices have mutually equal degrees is said to be regular. Then, a graph in which not all vertices have equal degrees can be viewed as somehow deviating from regularity. In the mathematical literature several measures of such irregularity were proposed [3] [9] [8] [4] [5] [6]. One of these was put forward by Albertson [2], who considered the quantity

$$irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)| \ . \tag{1.1}$$

The graph invariant $irr(G)$ was sometimes referred to as the *Albertson index* [9] or the *third Zagreb index* [7]. In this work, we use the terminology accepted by the majority of contemporary researchers [8] [10] [1] [11], according to which $irr(G)$ is the *irregularity* of the graph G.

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2. Graphs with smallest irregularity

From the definition (1.1) it is evident that the irregularity of a (simple) graph is a positive integer or zero. It can be easily shown [2] that irr must be an even integer.

Evidently, $irr(G) = 0$ if every component of the graph G is regular. Thus G itself needs not be regular. If G is connected, then $irr(G) = 0$ holds if and only if G is a regular graph.

The next-smallest possible value of irregularity is 2. If $irr(G) = 2$, then the irregularity of one component of G must be equal to 2, and all other components of G must be regular graphs. Therefore, in the following examples we may restrict the considerations to connected graphs. All graphs mentioned in these examples are assumed to have disjoint vertex sets.

Example 2.1. For $n \geq 3$, the path graph P_n is the only tree of order n with irregularity 2 [8].

Example 2.2. Let R_3 be a regular graph of degree 3, and let e be its arbitrary edge. Insert a vertex x of degree two on e. The graph thus obtained has irregularity 2.

Example 2.3. In fact, arbitrarily many vertices of degree two may be inserted on the edge e of R_3 .

Example 2.4. Connect the vertex x, specified in Example 2.2, with a vertex of the cycle C_k , $k \geq 3$. The graph thus obtained has irregularity 2.

Example 2.5. Let R_4 be a regular graph of degree 4, and let f be its arbitrary edge. Insert a vertex y of degree two on f. Let R_3 be a regular graph of degree 3, and let e be its arbitrary edge. Insert a vertex x of degree two on e. Connect the vertices x and y by a new edge. The graph thus obtained has irregularity 2.

Example 2.6. Example 2.5 can be generalized. Let R_k be a regular graph of degree k , $k \geq 5$, and let f be its arbitrary edge. Insert a vertex y of degree two on f. For $i = 1, 2, \ldots, k-3$, let $R_{3,i}$ be regular graphs of degree 3. Let e_i be an arbitrary edge of $R_{3,i}$. Insert a vertex x_i of degree two on e_i. Connect the vertex y with the vertices $x_1, x_2, \ldots, x_{k-3}$ by $k-3$ new edges. The graph thus obtained has irregularity 2.

It would be interesting to see if other examples of graphs with irregularity 2 could be constructed.

3. Graphs with greatest irregularity

The problem of characterizing graphs with greatest irregularity was studied already by Albertson [2]. He was able to demonstrate that for graphs of order n vertices, $\frac{4}{27}\, n^3$ is an asymptotically tight upper bound for irr. Recently [1] this bound was improved. In what follows we arrive at an equivalent result, using a reasoning different from that in [1].

Denote by $\Upsilon_{n,m}$ the set of all graphs with n vertices and m edges. Let $H_{n,m} \in \Upsilon_{n,m}$ be a graph containing at least one vertex of degree $n-1$, but $H_{n,m} \ncong K_n$. In addition, if $S \subset V(H_{n,m})$ denotes the set of vertices of degree $n-1$, then $H_{n,m} - S$ is either trivial or is a forest with at most one component that is a star, with all other components being trivial.

Theorem 3.1. *If* $G \in \Upsilon_{n,m}$ *, then* $irr(G) \leq irr(H_{n,m})$ *.*

Proof. We first prove that if G is a graph with maximum irregularity among connected graphs with n vertices, then it has at least one vertex of degree $n-1$. To do this, assume that G has no vertex of degree $n-1$. Suppose that u is the vertex of G with the maximum degree and that the vertex v is not adjacent to it. It is clear that if H is obtained from G by adding the edge uv, then $irr(H) \geq irr(G)$, as desired.

Next, suppose that S is the set of vertices of degree $n-1$ in G and $|S| = \xi$. Then

$$irr(G) = (n-\xi-1)(n-\xi)\xi - 2\,|E(G-S)| + irr(G-S)\ . \tag{3.1}$$

So, if the irregularity of $G-S$ is maximum, then the irregularity of G is maximum. We notice that for a given degree sequence $d_1, d_2, \ldots, d_n$, the irregularity is maximum if each vertex with greater degree is adjacent to a vertex with smaller degree.

Suppose that $y_1, \ldots, y_t$ are vertices of $G-S$ such that $0 < d(y_1|G-S) \leq \ldots \leq d(y_t|G-S)$. Thus $y_1, \ldots, y_{d_t}$ are adjacent to y_t so that $d_t = d(y_t|G-S)$. Let y_k be not adjacent to y_t, but be is adjacent to y_i , $1 \leq i \leq d_t$ and $d_t < k < t$. So, if H is obtained from $G-S$ by adding an edge $y_k y_t$ and deleting the edge $y_i y_k$, then

$$irr(H) \geq irr(G-S) + 2\big[d(y_t|G-S) - d(y_k|G-S)\big] + 1 > irr(G-S)\ .$$

Suppose that the vertex y_k , $d_t < k < t$, is not in the same component as y_t and it has the maximum degree in its component. In addition, suppose that y_i and y_j are vertices such that $y_i y_j \in E(G-S)$ and $1 \leq i < j \leq d_t$. So, if H is obtained from $G-S$ by adding an edge $y_k y_t$ and deleting the edge $y_i y_j$, then

$$irr(H) \geq irr(G-S) + 2\big[d(y_t|G-S) - d(y_j|G-S)\big] + 1 > irr(G-S)\ . \tag{3.2}$$

Finally, suppose that the vertex y_k , $d_t < k < t$, is not in the same component as y_t and it has the maximum degree in its component, and that y_i and y_j are vertices in the same component as y_k such that $y_i y_j \in E(G-S)$, $d_t < i < j < t$ and $i \neq k \neq j$. Then if H is obtained from $G-S$ by adding an edge $y_k y_t$ and deleting the edge $y_i y_j$, then relations (3.2) also hold.

In order to complete the proof, it is enough to notice that $irr(S_p) + irr(S_q) \leq irr(S_{p+q})$, where S_n denotes the star of order n. ☐

Let G_1 and G_2 be graphs with disjoint vertex sets. By $G_1 + G_2$ we denote the graph obtained from G_1 and G_2 by connecting all vertices of G_1 with all vertices of G_2.

The complement of a graph G is denoted by $\overline{G}$.

Define $T_{p,q} = K_p + \overline{K}_q$. It is not difficult to see that:

$$irr(T_{\lfloor n/3\rfloor,n-\lfloor n/3\rfloor}) = \left\lfloor \frac{n}{3} \right\rfloor \left(n - \left\lfloor \frac{n}{3} \right\rfloor\right) \left(n - \left\lfloor \frac{n}{3} \right\rfloor - 1\right).$$

We claim that $T_{\lfloor n/3\rfloor,n-\lfloor n/3\rfloor}$ is the graph with maximal irregularity among the connected graphs with n vertices, a result that independently was obtained by Abdo, Cohen and Dimitrov [1].

Theorem 3.2. *Let G be a graph with n vertices. Then*

$$irr(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \left(n - \left\lfloor \frac{n}{3} \right\rfloor\right) \left(n - \left\lfloor \frac{n}{3} \right\rfloor - 1\right).$$

Proof. As before, let ξ be the number of vertices of degree $n-1$. Then by Eq. (3.1) in Theorem 3.1, the maximum value of the function $f(\xi) = (n-\xi-1)(n-\xi)\xi$ is equal to the maximum of the irregularity, implying the result. □

Corollary 3.1. [2] *Let G be a graph with n vertices. Then for sufficiently large n,*

$$irr(G) \leq \frac{4}{27}n^3$$

i. e.,

$$\lim_{n\to\infty} \max\left\{irr(G)\right\} = \frac{4}{27}n^3 .$$

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