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ONE TRIPLED FIXED POINT RESULT IN PARTIALLY ORDERED 0-COMPLETE PARTIAL METRIC SPACES

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Abstract. In this paper, using the setting of partially ordered 0-complete partial metric spaces, one tripled fixed point result is obtained. Our established result generalizes and improves the existing tripled fixed point results in the literature in the sense that weaker conditions are used.

1. Introduction and preliminaries

Matthews [14] generalized the concept of a metric space by introducing partial metric spaces. Based on the notion of partial metric spaces, Matthews [13][14], Oltra and Valero [19], Ilić et al. [12], obtained some fixed point theorems for mappings satisfying different contractive conditions. Recently, Berinde and Borcut introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces ([10][11]). Aydi et al. presented tripled coincidence theorem for weak $\varphi-$ contractions in partially ordered metric space ([9]). For some new results on partial metric spaces see [1]–[8].

The aim of this paper is to continue the study of tripled fixed points in partially ordered 0-complete partial metric spaces.

Consistent with Matthews [13][14] O'Neill [16][17] and Oltra et al.[18] the following definitions and results will be needed throughout this paper.

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

($\mathbf{p}_1$) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
($\mathbf{p}_2$) $p(x,x) \leq p(x,y)$,

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($\mathbf{p}_3$) $p(x,y) = p(y,x)$,
($\mathbf{p}_4$) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. ([14][15])

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x_n,x)$;
(ii) a sequence $\{x_n\}$ in a partial metric space (X,p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$;
(iii) a partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$. In this case, p is a 0-complete partial metric on X.

Remark 1.1. (**1**) ([15]) Clearly, a limit of a sequence in a partial metric space does not need to be unique. Moreover, the function $p(\cdot,\cdot)$ does not need to be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n,y_n) \to p(x,y)$. For example, if $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$ for $x,y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n,x) = x = p(x,x)$ for each $x \geq 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

(**2**) ([3]) However, if $p(x_n,x) \to p(x,x) = 0$ then $p(x_n,y) \to p(x,y)$ for all $y \in X$.

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers [2][15][20].

Lemma 1.1. *Let (X,p) be a partial metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to+\infty} p(x_n,x_{n+1}) = 0.$$

If $\{x_n\}$ is not a 0-Cauchy sequence in (X,p), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k \geq k$ and the following four sequences

$$p(x_{m_k},x_{n_k}), p(x_{m_k},x_{n_k+1}), p(x_{m_k-1},x_{n_k}), p(x_{m_k-1},x_{n_k+1}),$$

tend to ε when $k \to +\infty$.

Definition 1.3. ([9][10]) Let $(X,\preceq)$ be a partially ordered set. The mapping $F : X^3 \to X$ is said to have the mixed monotone property if F is non-decreasing in first and third variable and non-increasing in second variable. An element $(x,y,x) \in X^3$ is called a tripled fixed point of F if

$$F(x,y,z) = x, F(y,x,y) = y \text{ and } F(z,y,x) = z.$$

Definition 1.4. ([9]) Let $(X,\preceq)$ be a partially ordered set and p be a partial metric on X. We say that $(X,p,\preceq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $p(x_n, x) \to p(x, x)$, then $x_n \preceq x$ for all n;
(ii) if a non-increasing sequence $\{y_n\}$ is such that $p(y_n, y) \to p(y, y)$, then $y \preceq y_n$ for all n.

Definition 1.5. The following class of mappings is defined as

$$\Phi = \left\{ \varphi \mid \varphi : [0, \infty) \to [0, \infty),\ \varphi(t) < t \text{ and } \lim_{r \to t^+} \varphi(r) < t,\ t > 0 \right\}.$$

It is clear that for $\varphi \in \Phi$ holds $\varphi(0) = 0$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$.

2. Main result

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a 0-complete partial metric space. Suppose $F : X^3 \to X$ is such that F has a mixed monotone property. Assume there is a function $\varphi \in \Phi$ such that*

$$p(F(x, y, z), F(u, v, w)) \le \varphi(\max\{p(x, u), p(y, v), p(z, w)\}) \tag{2.1}$$

for any $x, y, z, u, v, w \in X$ for which (x, y, z) and (u, v, w) are comparable. Suppose that either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that (x_0, y_0, z_0) and $(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$ are comparable, then there exists $(x, y, z) \in X$ such that

$$x = F(x, y, z),\ \ y = F(y, x, y) \ \text{ and } z = F(z, y, x),$$

that is, F has a tripled fixed point.

Remark 2.1. It is worth to mention that $(x, y, z), (uv, w) \in X^3$ are comparable if and only if $x \preceq u, y \succeq v$ and $z \preceq w$ or $x \succeq u, y \preceq v$ and $z \succeq w$.

Proof. Let us consider sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n)\, y_{n+1} = F(y_n, x_n, y_n), z_{n+1} = F(z_n, y_n, x_n)$$

for $n = 0, 1, 2, \ldots$. It is easy proved by induction that $\{x_n\}, \{z_n\}$ are non-decreasing and $\{y_n\}$ is a non-increasing sequence. Therefore, consider the two possible cases:
1) Let $x_n = x_{n+1}$, $y_n = y_{n+1}$ and $z_n = z_{n+1}$ for some $n \in \mathbb{N}$. In this case $x_n = F(x_n, y_n, z_n)$, $y_n = F(y_n, x_n, y_n)$, $z_n = F(z_n, y_n, x_n)$ and (x_n, y_n, z_n) is a tripled fixed point of F.
So, we will consider the case when
2) at least one of the conditions $x_n \ne x_{n+1}$, $y_n \ne y_{n+1}$ and $z_n \ne z_{n+1}$ is satisfied for every $n \in \mathbb{N}$ (and so $p(x_{n+1}, x_n) > 0$, $p(y_{n+1}, y_n) > 0$ and $p(z_{n+1}, z_n) > 0$ for every $n \in \mathbb{N}$). Applying (2.1) with $x = x_n$, $y = y_n$, $z = z_n$, $u = x_{n+1}$, $v = y_{n+1}$ and $w = z_{n+1}$ we have

$$\begin{aligned} &p(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})) \\ &\quad \le \varphi(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1}), p(z_n, z_{n+1})\}), \end{aligned}$$

that is,

$$p\left(x_{n+1}, x_{n+2}\right) \leq \varphi\left(\max\left\{p\left(x_n, x_{n+1}\right), p\left(y_n, y_{n+1}\right), p\left(z_n, z_{n+1}\right)\right\}\right).$$

Similarly, we have

$$p\left(y_{n+1}, y_{n+2}\right) \leq \varphi\left(\max\left\{p\left(y_n, y_{n+1}\right), p\left(x_n, x_{n+1}\right), p\left(y_n, y_{n+1}\right)\right\}\right)$$

and

$$p\left(z_{n+1}, z_{n+2}\right) \leq \varphi\left(\max\left\{p\left(z_n, z_{n+1}\right), p\left(y_n, y_{n+1}\right), p\left(x_n, x_{n+1}\right)\right\}\right).$$

It follows

$$\begin{aligned} &\max\left\{p\left(x_{n+1}, x_{n+2}\right), p\left(y_{n+1}, y_{n+2}\right), p\left(y_{n+1}, y_{n+2}\right)\right\} \\ &\leq \varphi\left(\max\left\{p\left(x_n, x_{n+1}\right), p\left(y_n, y_{n+1}\right), p\left(z_n, z_{n+1}\right)\right\}\right). \end{aligned} \tag{2.2}$$

If we denote by $\delta_n = \left(\max\left\{p\left(x_n, x_{n+1}\right), p\left(y_n, y_{n+1}\right), p\left(z_n, z_{n+1}\right)\right\}\right)$, we can conclude that $\{\delta_n\}$ is monotone decreasing. Therefore, $\delta_n \to \delta^* \geq 0$ when $n \to \infty$.

We now prove that $\delta^* = 0$. Assume, on contrary, that $\delta^* > 0$. If we write (2.2) as $\delta_{n+1} \leq \varphi(\delta_n)$ and if we pass to the limit when $n \to \infty$, we obtain that $\delta^* \leq \varphi(\delta^*) < \delta^*$, which is a contradiction. Hence,

$$\lim_{n\to\infty} \max\left\{p\left(x_n, x_{n+1}\right), p\left(y_n, y_{n+1}\right), p\left(z_n, z_{n+1}\right)\right\} = 0,$$

and it follows $\lim_{n\to\infty} p\left(x_{n+1}, x_n\right) = \lim_{n\to\infty} p\left(y_{n+1}, y_n\right) = \lim_{n\to\infty} p\left(z_{n+1}, z_n\right) = 0$.

We next prove that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are 0-Cauchy sequences in the space (X, p). If at least one of sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ is not a 0-Cauchy sequence, it means that $\max\left\{p\left(x_n, x_m\right), p\left(y_n, y_m\right), p\left(z_n, z_m\right)\right\}$ does not tend to 0 when $n, m \to \infty$. It means that there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) \geq k$ and the following four sequences

$$\begin{aligned} &\max\left\{p\left(x_{n(k)}, x_{m(k)}\right), p\left(y_{n(k)}, y_{m(k)}\right), p\left(z_{n(k)}, z_{m(k)}\right)\right\}, \\ &\max\left\{p\left(x_{n(k)+1}, x_{m(k)}\right), p\left(y_{n(k)+1}, y_{m(k)}\right), p\left(z_{n(k)+1}, z_{m(k)}\right)\right\}, \\ &\max\left\{p\left(x_{n(k)}, x_{m(k)-1}\right), p\left(y_{n(k)}, y_{m(k)-1}\right), p\left(z_{n(k)}, z_{m(k)-1}\right)\right\}, \\ &\max\left\{p\left(x_{n(k)+1}, x_{m(k)-1}\right), p\left(y_{n(k)+1}, y_{m(k)-1}\right), p\left(z_{n(k)+1}, z_{m(k)-1}\right)\right\}, \end{aligned}$$

all tend to ε^+ when $k \to \infty$. The proof is identical as in [15][20]

Applying the condition (2.1) to elements $x = x_{n(k)}$, $y = y_{n(k)}$, $z = z_{n(k)}$, $u = x_{m(k)-1}$, $v = y_{m(k)-1}$ and $w = z_{m(k)-1}$, we get that

$$\begin{aligned} &p\left(x_{n(k)+1}, x_{m(k)}\right) = p\left(F\left(x_{n(k)}, y_{n(k)}, z_{n(k)}\right), F\left(x_{m(k)-1}, y_{m(k)-1}, z_{m(k)-1}\right)\right) \\ &\leq \varphi\left(\max\left\{p\left(x_{n(k)}, x_{m(k)-1}\right), p\left(y_{n(k)}, y_{m(k)-1}\right), p\left(z_{n(k)}, z_{m(k)-1}\right)\right\}\right). \end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned} &p\left(y_{n(k)+1}, y_{m(k)}\right) = p\left(F\left(y_{n(k)}, x_{n(k)}, y_{n(k)}\right), F\left(y_{m(k)-1}, x_{m(k)-1}, y_{m(k)-1}\right)\right) \\ &\leq \varphi\left(\max\left\{p\left(y_{n(k)}, y_{m(k)-1}\right), p\left(x_{n(k)}, x_{m(k)-1}\right), p\left(y_{n(k)}, y_{m(k)-1}\right)\right\}\right) \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
&p\left(z_{n(k)+1}, z_{m(k)}\right) = p\left(F\left(z_{n(k)}, y_{n(k)}, x_{n(k)}\right), F\left(z_{m(k)-1}, y_{m(k)-1}, x_{m(k)-1}\right)\right) \\
&\leq \varphi\left(\max\left\{p\left(z_{n(k)}, z_{m(k)-1}\right), p\left(y_{n(k)}, y_{m(k)-1}\right), p\left(x_{n(k)}, x_{m(k)-1}\right)\right\}\right). \qquad (2.5)
\end{aligned}
$$

From (2.3), (2.4) and (2.5) we have

$$
\begin{aligned}
&\max\left\{p\left(x_{n(k)+1}, x_{m(k)}\right), p\left(y_{n(k)+1}, y_{m(k)}\right), p\left(z_{n(k)+1}, z_{m(k)}\right)\right\} \\
&\leq \varphi\left(\max\left\{p\left(x_{n(k)}, x_{m(k)-1}\right), p\left(y_{n(k)}, y_{m(k)-1}\right), p\left(z_{n(k)}, z_{m(k)-1}\right)\right\}\right).
\end{aligned}
$$

If we pass to limit when $k \to \infty$ we get $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, which is a contradiction since $\varepsilon > 0$.

This shows that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are 0-Cauchy sequences in the spaces (X, p). Since (X, p) is a 0-complete, there exist $x, y, z \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, $\lim_{n\to\infty} p(y_n, y) = p(y, y) = 0$ and $\lim_{n\to\infty} p(z_n, z) = p(z, z) = 0$.

Suppose that F is continuous ([21]). We have

$$
p(x, F(x, y, z)) \leq p(x, x_{n+1}) + p(x_{n+1}, F(x, y, z)). \qquad (2.6)
$$

It holds that $p(x, x_{n+1}) \to p(x, x) = 0$ when $n \to \infty$ and

$$
p(x_{n+1}, F(x, y, z)) = p(F(x_n, y_n, z_n), F(x, y, z)) \to p(F(x, y, z), F(x, y, z))
$$

when $n \to \infty$.

Hence, when $n \to \infty$ from (2.6) follows

$$
p(x, F(x, y, z)) \leq 0 + p(F(x, y, z), F(x, y, z)) = p(F(x, y, z), F(x, y, z)).
$$

According to the property $(\mathbf{p}_2)$ of partial metric space, we have

$$
p(x, F(x, y, z)) = p(F(x, y, z), F(x, y, z)).
$$

Similarly, we have

$$
p(y, F(y, x, y)) = p(F(y, x, y), F(y, x, y))
$$

and

$$
p(z, F(z, y, x)) = p(F(z, y, x), F(z, y, x)).
$$

Since (x, y, z) is comparable to (u, v, w) according to the condition (2.1) with $x = u$, $y = v$, $z = w$ we obtain $p(F(x, y, z), F(x, y, z)) = 0$. Similarly, we have $p(F(y, x, y), F(y, x, y)) = 0$ and $p(F(z, y, x), F(z, y, x)) = 0$. It follows that $x = F(x, y, z)$, $y = F(y, x, y)$ and $z = F(z, y, x)$.

Suppose that $(X, p, \preceq)$ is regular. Then, since (x_n, y_n, z_n) is comparable with (x, y, z) we have, according to (2.1), that

$$
\begin{aligned}
&p(F(x_n, y_n, z_n), F(x, y, z)) \leq \varphi(\max\{p(x_n, x), p(y_n, y), p(z_n, z)\}), \\
&p(F(y_n, x_n, y_n), F(y, x, y)) \leq \varphi(\max\{p(y_n, y), p(x_n, x), p(y_n, y)\}), \\
&p(F(z_n, y_n, x_n), F(z, y, x)) \leq \varphi(\max\{p(z_n, z), p(y_n, y), p(x_n, x)\}),
\end{aligned}
$$

or

$$p(x_{n+1}, F(x, y, z)) + p(y_{n+1}, F(y, x, y)) + p(z_{n+1}, F(z, y, x)) \tag{2.7}$$
$$\leq 3\varphi(\max\{p(x_n, x), p(y_n, y), p(z_n, z)\}).$$

Now, taking limit as $n \to \infty$ in (2.7) and by Remark 1.1 (2) it follows that

$$p(x, F(x, y, z)) + p(y, F(y, x, y)) + p(z, F(z, y, x)) = 0,$$

that is, $x = F(x, y, z)$, $y = F(y, x, y)$ and $z = F(z, y, x)$. □

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