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A COUPLED COINCIDENCE POINT THEOREM IN PARTIALLY ORDERED METRIC SPACES

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Abstract. We prove a coupled coincidence point theorem in partially ordered metric spaces for mappings $F : X \times X \to X$ having the g-mixed monotone property. The main result of this paper extends and improves the corresponding results in [6][10][8][4]. Some examples are given to illustrate our work.

1. Introduction and preliminaries

Existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered recently in [1]–[19] and reference therein. Some existence results of solutions for matrix equations, ordinary differential equations or integral equations by applying fixed point theorems are presented in [2][6][7][10] and [15]–[17].

In [6], Bhaskar and Lakshmikantham introduced the notions of mixed monotone mapping and of coupled fixed point and proved some coupled fixed point theorems for the mixed monotone mappings and also discussed the existence and uniqueness of solution for a periodic boundary value problem. These concepts are defined as follows.

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

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An element $(x, y) \in X \times X$ is called a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.

Let $(X, \preceq)$ be a partially ordered set for which there exists a metric d on X such that (X, d) is a complete metric space. The main results of Bhaskar and Lakshmikantham in [6] are some coupled fixed point theorems for mixed monotone mappings $F : X \times X \to X$, satisfying a contractive condition of the form

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \text{ for each } x \succeq u \text{ and } y \preceq v, \tag{1.1}$$

where $k \in [0, 1)$.

Luong and Thuan in [10] and Harjani, Lopez and Sadarangani in [8] proved some generalizations of the main results in [6] and discussed the existence and uniqueness of the solution of nonlinear integral equations.

The main result of Luong and Thuan [10] refers to mappings F satisfying the more general contractive condition

$$\phi\left(d\left(F(x,y), F(u,v)\right)\right) \leq \frac{1}{2}\phi\left(d(x,u) + d(y,v)\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \tag{1.2}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, with $\phi \in \Phi$ and $\psi \in \Psi$, where Φ denotes the set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(i) ϕ is continuous and non-decreasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$,
(iii) $\phi(t + s) \leq \phi(t) + \phi(s)$, for all $t, s \in [0, \infty)$,

while Ψ denotes the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{t \to r}\psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0+}\psi(t) = 0$.

The main result of Harjani, Lopez and Sadarangani [8] is obtained for mappings F satisfying the contractive condition

$$\varphi\left(d\left(F(x,y), F(u,v)\right)\right) \leq \varphi\left(\max\{d(x,u), d(y,v)\}\right) - \psi\left(\max\{d(x,u), d(y,v)\}\right), \tag{1.3}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where φ, ψ are altering distance functions. An altering distance function is a function $\varphi : [0, \infty) \to [0, \infty)$ which satisfies

(i) φ is continuous and non-decreasing;
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

On the other hand, Lakshmikantham and Ciric [9] established coupled coincidence and coupled fixed point theorems for two mappings $F : X \times X \to X$ and $g : X \to X$, where F has the mixed g-monotone property and the functions F and g commute, as an extension of the fixed point results in [6]. Choudhury and Kundu in [3] introduced the concept of compatibility and proved the result established in [9] under a different set of conditions. Precisely, they established their result by assuming that F and g are compatible mappings.

Definition 1.1. ([9]) Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ are two mappings. We say that F has the mixed g-monotone property if $F(x,y)$ is g-nondecreasing in its first argument and is g- non increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 1.2. ([9]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) \text{ and } gy = F(y, x)$$

Definition 1.3. ([3]) The mappings F and g where $F : X \times X \to X$, $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for all $x, y \in X$ are satisfied.

Afterwards, using the concept of compatible mappings, Choudhury, Metiya and Kundu [4] proved coupled coincidence point theorem for two compatible mappings $F : X \times X \to X$ and $g : X \to X$ satisfying the following inequality

$$\begin{aligned} &\varphi(d(F(x, y), F(u, v))) \\ &\leq \varphi(\max\{d(gx, gu), d(gy, gv)\}) - \psi(\max\{d(gx, gu), d(gy, gv)\}), \end{aligned} \tag{1.4}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where φ is an altering distance function and $\psi : [0, \infty) \to [0, \infty)$ is continuous and $\psi(t) = 0$ if and only if $t = 0$.

Note that the result of Choudhury, Metiya and Kundu [4] is a generalization of the result of Harjani, Lopez and Sadarangani [8].

In this paper, we first slightly extend the concept of compatible mappings into the context of partially ordered metric spaces and then prove a coupled coincidence theorem for such mappings in partially ordered complete metric spaces. Our result is a generalization of the results of Bhaskar and Lakshmikantham [6], Luong and Thuan [10], Harjani, Lopez, Sadarangani [8] and Choudhury, Metiya, Kundu [4].

2. Main result

Definition 2.1. (see, e.g. [13]) Let $(X, \preceq, d)$ be a partially ordered metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be O-compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\{gx_n\}$, $\{gy_n\}$ are monotone and

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$$

for all $x, y \in X$ are satisfied.

Let $(X, \preceq, d)$ be a partially ordered metric space. If $F : X\times X \to X$ and $g : X \to X$ are compatible then they are O-compatible. However, the converse is not true. The following example shows that there exist mappings which are O-compatible but not compatible.

Example 2.1. (see, e.g. [13]) Let $X = \{0\} \cup [1/2, 2]$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. We consider the following order relation on X

$$x, y \in X \quad x \preceq y \quad \Leftrightarrow \quad x = y \text{ or } (x, y) \in \{(0, 0), (0, 1), (1, 1)\}.$$

Let $F : X \times X \to X$ be given by

$$F(x, y) = \begin{cases} 0 & \text{if } x, y \in \{0\} \cup [1/2, 1] \\ 1 & \text{otherwise} \end{cases}$$

and $g : X \to X$ be defined by

$$gx = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 1/2 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 3/2 \\ 1/2 & \text{if } 3/2 < x \le 2 \end{cases}$$

Then F and g are O-compatible. Indeed, let $\{x_n\}, \{y_n\}$ in X such that $\{gx_n\}, \{gy_n\}$ are monotone and

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$$

for some $x, y \in X$. Since $F(x_n, y_n) = F(y_n, x_n) \in \{0, 1\}$ for all n, $x = y \in \{0, 1\}$. The case $x = y = 1$ is impossible. In fact, if $x = y = 1$, then since $\{gx_n\}, \{gy_n\}$ are monotone, $gx_n = gy_n = 1$, for all $n \ge n_1$, for some n_1. That is, $x_n, y_n \in [1/2, 1]$, for all $n \ge n_1$. This implies $F(x_n, y_n) = F(y_n, x_n) = 0$, for all $n \ge n_1$, which is a contradiction. Thus $x = y = 0$. That implies $gx_n = gy_n = 0$, for all $n \ge n_2$, for some n_2, that is, $x_n = y_n = 0$, for all $n \ge n_2$. Thus, for all $n \ge n_2$,

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = d(gF(y_n, x_n), F(gy_n, gx_n)) = 0.$$

Hence

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

hold. Therefore F and g are O-compatible.
However, F and g are not compatible. Indeed, let $\{x_n\}, \{y_n\}$ in X be defined by

$$x_n = y_n = 1 + \frac{1}{n+1}, \quad n = 1, 2, 3, \ldots$$

We have

$$F(x_n, y_n) = F(y_n, x_n) = F\left(1 + \frac{1}{n+1}, 1 + \frac{1}{n+1}\right) = 1$$

and

$$gx_n = gy_n = g\left(1 + \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1 \ \textit{as} \ n \to \infty,$$

but

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = d\left(F\left(1 - \frac{1}{n+1}, 1 - \frac{1}{n+1}\right), g1\right) = d(0,1) = 1 \nrightarrow 0,$$

as $n \to \infty$. Thus, F and g are not compatible.

To prove our main results in the next section, we shall need the following Lemma.

Lemma 2.1. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of positive real numbers such that*

$$\lim_{n\to\infty} \max\{x_n, y_n\} = \alpha > 0$$

Then there exist subsequences $\{x_{n_{k_j}}\}$ of $\{x_n\}$ and $\{y_{n_{k_j}}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty} x_{n_{k_j}} = \alpha_1, \lim_{j\to\infty} y_{n_{k_j}} = \alpha_2 \ \textit{and} \ \max\{\alpha_1, \alpha_2\} = \alpha.$$

Proof. Since the sequence $\max\{x_n, y_n\}$ is convergent, it is bounded. On other hand, due to the inequalities $0 \leq x_n, y_n \leq \max\{x_n, y_n\}$, the sequences $\{x_n\}$ and $\{y_n\}$ are also bounded. Since $\{x_n\}$ is bounded, by Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$. Assume that $\lim_{k\to\infty} x_{n_k} = \alpha_1$. Also, due to the fact that $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $\lim_{j\to\infty} y_{n_{k_j}} = \alpha_2$. Since $\lim_{k\to\infty} x_{n_k} = \alpha_1$, we have $\lim_{j\to\infty} x_{n_{k_j}} = \alpha_1$. Finally, we have

$$\alpha = \lim_{j\to\infty} \max\left\{x_{n_{k_j}}, y_{n_{k_j}}\right\} = \max\left\{\lim_{j\to\infty} x_{n_{k_j}}, \lim_{j\to\infty} y_{n_{k_j}}\right\} = \max\{\alpha_1, \alpha_2\}.$$

□

Let Θ denote the class of all functions $\theta : [0,\infty)^2 \to [0,\infty)$ satisfying

$$\lim_{\substack{t_1\to r_1\\ t_2\to r_2}} \theta\left(t_1,t_2\right) > 0 \quad \text{for all } (r_1,r_2)\in[0,\infty)^2,$$

with $\max\{r_1,r_2\} > 0$ and $\theta(t_1,t_2) = 0$ if and only if $t_1 = t_2 = 0$. For example, $\theta(t_1,t_2) = k\max\{t_1,t_2\}, k > 0$; $\theta(t_1,t_2) = at_1^p + bt_2^q, a,b,p,q > 0$ for all $(t_1,t_2) \in [0,\infty)^2$ are in Θ.

Now we are going to prove our main result.

Theorem 2.1. *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $F : X\times X \to X$ and $g : X \to X$ be two mappings such F has the mixed g-monotone property. Assume that*

$$\begin{aligned}&\varphi\left(d(F(x,y),F(u,v))\right)\\ &\qquad\le \varphi\left(\max\{d(gx,gu),d(gy,gv)\}\right) - \theta\left(d(gx,gu),d(gy,gv)\right)\end{aligned} \tag{2.1}$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where φ is an altering distance function and $\theta\in\Theta$. Let $F(X\times X)\subseteq g(X)$, g is continuous and F and g are O-compatible mappings. Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\}\to x$, then $gx_n \preceq gx$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}\to y$, then $gy \preceq gy_n$ for all n.*

If there exist $x_0,y_0\in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$, then there exist $x,y\in X$ such that $gx = F(x,y)$ and $gy = F(y,x)$, that is, F and g have a coupled coincidence point in X.

Proof. Let $x_0,y_0\in X$ be such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$. Since $F(X\times X)\subseteq g(X))$, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$gx_{n+1} = F\left(x_n,y_n\right) \text{ and } gy_{n+1} = F(y_n,x_n), \text{ for all } n\ge 0. \tag{2.2}$$

By the mixed g-monotone property of F, using the mathematical induction, it is easy to show that

$$gx_n \preceq gx_{n+1}, \tag{2.3}$$

and

$$gy_n \succeq gy_{n+1}, \tag{2.4}$$

for all $n\ge 0$.

If there is $n_0\ge 1$ such that $d(gx_{n_0},gx_{n_0-1}) = d(gy_{n_0},gy_{n_0-1}) = 0$ then $gx_{n_0-1} = gx_{n_0} = F(x_{n_0-1},y_{n_0-1})$ and $gy_{n_0-1} = gy_{n_0} = F(y_{n_0-1},x_{n_0-1})$, that is, (x_{n_0-1},y_{n_0-1}) is a coupled coincidence point of F and g. Now, we may assume that $d\left(gx_n,gx_{n-1}\right) + d\left(gy_n,gy_{n-1}\right) > 0$ for all $n\ge 1$.

Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, from (2.1) and (2.2), we have

$$\begin{aligned}\varphi\left(d\left(gx_{n+1}, gx_n\right)\right) &= \varphi\left(d\left(F\left(x_n, y_n\right), F\left(x_{n-1}, y_{n-1}\right)\right)\right)\\ &\leq \varphi\left(\max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}\right)\\ &\quad - \theta\left(d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right). \end{aligned} \tag{2.5}$$

As $\theta(t_1, t_2) > 0$, for all $(t_1, t_2) \in [0, \infty)^2, t_1 + t_2 > 0$, we have

$$\varphi\left(d\left(gx_{n+1}, gx_n\right)\right) < \varphi\left(\max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}\right).$$

Since φ is non decreasing, we get

$$d\left(gx_{n+1}, gx_n\right) \leq \max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}. \tag{2.6}$$

Similarly, since $gy_{n-1} \succeq gy_n$ and $gx_{n-1} \preceq gx_n$, from (2.1), (2.2) and the properties of θ, we also have

$$\begin{aligned}\varphi\left(d\left(gy_n, gy_{n+1}\right)\right) &= \varphi\left(d\left(F\left(y_{n-1}, x_{n-1}\right), F\left(y_n, x_n\right)\right)\right)\\ &\leq \varphi\left(\max\left\{d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right\}\right)\\ &\quad - \theta\left(d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right) \end{aligned} \tag{2.7}$$

and, consequently,

$$d\left(gy_{n+1}, gy_n\right) \leq \max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}. \tag{2.8}$$

From (2.6) and (2.8), we have

$$\max\left\{d\left(gx_{n+1}, gx_n\right), d\left(gy_{n+1}, gy_n\right)\right\} \leq \max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}.$$

If we set $\delta_n = \max\left\{d\left(gx_{n+1}, gx_n\right), d\left(gy_{n+1}, gy_n\right)\right\}$, then the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\left\{d\left(gx_{n+1}, gx_n\right), d\left(gy_{n+1}, gy_n\right)\right\} = \delta. \tag{2.9}$$

We shall show that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. By Lemma 2.1, the sequences $\{d(g_{n+1}, gx_n)\}$ and $\{d(gy_{n+1}, gy_n)\}$ have convergent sequences that be still denoted $\{d(g_{n+1}, gx_n)\}$ and $\{d(gy_{n+1}, gy_n)\}$, respectively, with $\lim\limits_{n\to\infty} d\left(gx_{n+1}, gx_n\right) = \delta_1$ and $\lim\limits_{n\to\infty} d\left(gy_{n+1}, gy_n\right) = \delta_2$ and $\max\{\delta_1, \delta_2\} = \delta > 0$. Since $\varphi : [0, \infty) \to [0, \infty)$ is non decreasing, we have $\varphi\left(\max\{a, b\}\right) = \max\{\varphi(a), \varphi(b)\}$ for $a, b \in [0, \infty)$. Thus, from (2.5) and (2.7), we have

$$\begin{aligned}&\varphi\left(\max\left\{d\left(gx_{n+1}, gx_n\right), d\left(gy_{n+1}, gy_n\right)\right\}\right)\\ &\qquad= \max\left\{\varphi\left(d\left(gx_{n+1}, gx_n\right)\right), \varphi\left(d\left(gy_{n+1}, gy_n\right)\right)\right\}\\ &\qquad\leq \varphi\left(\max\left\{d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right\}\right)\\ &\qquad\quad - \min\left\{\theta\left(d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right), \theta\left(d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right)\right\}.\end{aligned}$$

Then taking the limit as $n \to \infty$ in both sides of the previous enequality, we have

$$\begin{aligned}
\varphi(\delta) &= \lim_{n\to\infty} \varphi(\delta_n) \\
&\leq \lim_{n\to\infty} \left[\varphi(\delta_{n-1}) - \min \left\{ \begin{array}{l} \theta\left(d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right), \\ \theta\left(d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right) \end{array} \right\} \right] \\
&= \varphi(\delta) - \lim_{\substack{d(x_n, x_{n-1})\to\delta_1 \\ d(y_n, y_{n-1})\to\delta_2}} \min \left\{ \begin{array}{l} \theta\left(d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right), \\ \theta\left(d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right) \end{array} \right\} \\
&= \varphi(\delta) - \min \left\{ \begin{array}{l} \lim\limits_{\substack{d(x_n, x_{n-1})\to\delta_1 \\ d(y_n, y_{n-1})\to\delta_2}} \theta\left(d\left(gx_n, gx_{n-1}\right), d\left(gy_n, gy_{n-1}\right)\right), \\ \lim\limits_{\substack{d(x_n, x_{n-1})\to\delta_1 \\ d(y_n, y_{n-1})\to\delta_2}} \theta\left(d\left(gy_n, gy_{n-1}\right), d\left(gx_n, gx_{n-1}\right)\right) \end{array} \right\} \\
&< \varphi(\delta),
\end{aligned}$$

which is a contradiction. Thus, $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\left\{d\left(gx_{n+1}, gx_n\right), d\left(gy_{n+1}, gy_n\right)\right\} = 0. \tag{2.10}$$

In what follows, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. This means that there exists an $\varepsilon > 0$ for wich we can find subsequences $\left\{gx_{n(k)}\right\}, \left\{gx_{m(k)}\right\}$ of $\{gx_n\}$ and $\left\{gy_{n(k)}\right\}, \left\{gy_{m(k)}\right\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \geq \varepsilon \tag{2.11}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.11). Then

$$\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)}\right), d\left(gy_{n(k)-1}, gy_{m(k)}\right)\right\} < \varepsilon \tag{2.12}$$

Using the triangle inequality and (2.12), we have

$$\begin{aligned}
d\left(gx_{n(k)}, gx_{m(k)}\right) &\leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gx_{n(k)-1}, gx_{m(k)}\right) \\
&< d\left(gx_{n(k)}, gx_{n(k)-1}\right) + \varepsilon
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
d\left(gy_{n(k)}, gy_{m(k)}\right) &\leq d\left(gy_{n(k)}, gy_{n(k)-1}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) \\
&< d\left(gy_{n(k)}, gy_{n(k)-1}\right) + \varepsilon
\end{aligned} \tag{2.14}$$

From (2.11), (2.13) and (2.14), we have

$$\begin{aligned}
\varepsilon &\leq \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \\
&< \max\left\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right\} + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the inequalities above and using (2.10) we get

$$\lim_{k\to\infty} \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} = \varepsilon. \tag{2.15}$$

By the triangle inequality we get

$$d\left(gx_{n(k)}, gx_{m(k)}\right) \leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gx_{n(k)-1}, gx_{m(k)-1}\right) + d\left(gx_{m(k)-1}, gx_{m(k)}\right)$$

and

$$d\left(gy_{n(k)}, gy_{m(k)}\right) \leq d\left(gy_{n(k)}, gy_{n(k)-1}\right) + d\left(gy_{n(k)-1}, gy_{m(k)-1}\right) + d\left(gy_{m(k-1}, gy_{m(k)}\right).$$

From the last two inequalities and (2.11), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\} \\ &\leq \max\left\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right\} \\ &\quad + \max\left\{d\left(gx_{m(k)-1}, gx_{m(k)}\right), d\left(gy_{m(k)-1}, gy_{m(k)}\right)\right\} \\ &\quad + \max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\}. \end{aligned} \tag{2.16}$$

Again, by the triangle inequality,

$$\begin{aligned} d\left(gx_{n(k)-1}, gx_{m(k)-1}\right) &\leq d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gx_{m(k)}, gx_{m(k)-1}\right) \\ &< d\left(gx_{m(k)}, gx_{m(k)-1}\right) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} d\left(gy_{n(k)-1}, gy_{m(k)-1}\right) &\leq d\left(gy_{n(k)-1}, gy_{m(k)}\right) + d\left(gy_{m(k)}, gy_{m(k)-1}\right) \\ &< d\left(gy_{m(k)}, gy_{m(k)-1}\right) + \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\} \\ &\quad < \max\left\{d\left(gx_{m(k)}, gx_{m(k)-1}\right), d\left(gy_{m(k)}, gy_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we have

$$\begin{aligned} &\varepsilon - \max\left\{d\left(gx_{n(k)}, gx_{n(k)-1}\right), d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right\} \\ &\quad - \max\left\{d\left(gx_{m(k)-1}, gx_{m(k)}\right), d\left(gy_{m(k)-1}, gy_{m(k)}\right)\right\} \\ &\qquad \leq \max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\} \\ &\qquad < \max\left\{d\left(gx_{m(k)}, gx_{m(k)-1}\right), d\left(gy_{m(k)}, gy_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ in the inequalities above and using (2.10), we get

$$\lim_{k\to\infty} \max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\} = \varepsilon. \tag{2.18}$$

Therefore, $\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right)\right\}$ and $\left\{d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\}$ have subsequences converging to ε_1 and ε_2, respectively. From (2.18), we have $\max\{\varepsilon_1, \varepsilon_2\} = \varepsilon > 0$. We may assume that $\lim_{k\to\infty} d\left(gx_{n(k)-1}, gx_{m(k)-1}\right) = \varepsilon_1$ and $\lim_{k\to\infty} d\left(gy_{n(k)-1}, gy_{m(k)-1}\right) = \varepsilon_2$.

Since $n(k) > m(k)$, $gx_{n(k)-1} \succeq gx_{m(k)-1}$ and $gy_{n(k)-1} \preceq gy_{m(k)-1}$, from (2.1) and (2.2),

$$\begin{aligned}\varphi\left(d\left(gx_{n(k)}, gx_{m(k)}\right)\right) &= \varphi\left(d\left(F\left(x_{n(k)-1}, y_{n(k)-1}\right), F\left(x_{m(k)-1}, y_{m(k)-1}\right)\right)\right)\\ &\leq \varphi\left(\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\}\right)\\ &\quad - \theta\left(d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right). \qquad (2.19)\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi\left(d\left(gy_{m(k)}, gy_{n(k)}\right)\right) &= \varphi\left(d\left(F\left(y_{m(k)-1}, x_{m(k)-1}\right), F\left(y_{n(k)-1}, x_{n(k)-1}\right)\right)\right)\\ &\leq \varphi\left(\max\left\{d\left(gy_{n(k)-1}, gy_{m(k)-1}\right), d\left(gx_{n(k)-1}, gx_{m(k)-1}\right)\right\}\right)\\ &\quad - \theta\left(d\left(gy_{n(k)-1}, gy_{m(k)-1}\right), d\left(gx_{n(k)-1}, gx_{m(k)-1}\right)\right). \qquad (2.20)\end{aligned}$$

From (2.19) and (2.20), we have

$$\begin{aligned}&\varphi\left(\max\left\{d\left(gx_{n(k)}, gx_{m(k)}\right), d\left(gy_{n(k)}, gy_{m(k)}\right)\right\}\right)\\ &\quad= \max\left\{\varphi\left(d\left(gx_{n(k)}, gx_{m(k)}\right)\right), \varphi\left(d\left(gy_{n(k)}, gy_{m(k)}\right)\right)\right\}\\ &\quad\leq \varphi\left(\max\left\{d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right\}\right)\\ &\qquad - \min\left\{\begin{array}{l}\theta\left(d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right),\\ \theta\left(d\left(gy_{n(k)-1}, gy_{m(k)-1}\right), d\left(gx_{n(k)-1}, gx_{m(k)-1}\right)\right)\end{array}\right\}.\end{aligned}$$

By passing to subsequences, we obtain

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \lim_{k\to\infty}\min\left\{\begin{array}{l}\theta\left(d\left(gx_{n(k)-1}, gx_{m(k)-1}\right), d\left(gy_{n(k)-1}, gy_{m(k)-1}\right)\right),\\ \theta\left(d\left(gy_{n(k)-1}, gy_{m(k)-1}\right), d\left(gx_{n(k)-1}, gx_{m(k)-1}\right)\right)\end{array}\right\} < \varphi(\varepsilon),$$

which is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y. \qquad (2.21)$$

Thus

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \qquad (2.22)$$

Since F and g are O-compatible, from (2.22), we have

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \qquad (2.23)$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0. \qquad (2.24)$$

Now, suppose that assumption (a) holds. We have

$$d\left(gx, F\left(g\left(x_n\right), g\left(y_n\right)\right)\right) \leq d\left(gx, gF\left(x_n, y_n\right)\right) + d\left(gF\left(x_n, y_n\right), F\left(gx_n, gy_n\right)\right). \qquad (2.25)$$

Taking the limit as $n \to \infty$ in (2.25) and by using (2.21), (2.23) and the continuity of F and g we get $d(gx, F(x, y)) = 0$. Similarly, we can show that $d(gy, F(y, x)) = 0$. Therefore, $gx = F(x, y)$ and $gy = F(y, x)$. Finally, suppose that assumption (b) holds. Since $\{gx_n\}$ is a non decreasing sequence and $gx_n \to x$ and $\{gy_n\}$ is a non increasing sequence and $gy_n \to y$, we have $ggx_n \preceq gx$ and $ggy_n \succeq gy$, for all n.

Since F and g are compatible and g is continuous, from (2.21), (2.23) and (2.24) we have

$$\lim_{n\to\infty} ggx_n = gx = \lim_{n\to\infty} gF(x_n, y_n) = \lim_{n\to\infty} F(gx_n, gy_n) \tag{2.26}$$

and

$$\lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} gF(y_n, x_n) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{2.27}$$

We have

$$\begin{aligned}\varphi(d(F(x,y), F(gx_n, gy_n))) &\leq \varphi(\max\{d(gx, ggx_n), d(gy, ggy_n)\}) \\ &\quad - \theta(d(gx, ggx_n), d(gy, ggy_n)) \\ &\leq \varphi(\max\{d(gx, ggx_n), d(gy, ggy_n)\}).\end{aligned}$$

We also have

$$\begin{aligned}d(gx, F(x,y)) &\leq d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(x,y)) \\ &= d(gx, ggx_{n+1}) + d(gF(x_n, y_n), F(x,y)) \\ &= d(gx, ggx_{n+1}) + d(F(x,y), F(gx_n, gy_n)).\end{aligned}$$

Since φ is non-decreasing, we have

$$\begin{aligned}\varphi(d(gx, F(x,y)) - d(gx, ggx_{n+1}) &\leq \varphi(d(F(x,y), F(gx_n, gy_n))) \\ &\leq \varphi(\max\{d(ggx_n, gx), d(ggy_n, gy)\})\end{aligned}$$

Taking the limit as $n \to \infty$ in the previous inequalities and using (2.26), (2.27) and the continuity of φ, we obtain

$$\begin{aligned}\varphi(d(gx, F(x,y))) &= \varphi(d(gx, F(x,y)) - \lim_{n\to\infty} d(gx, ggx_{n+1})) \\ &\leq \varphi(\lim_{n\to\infty} \max\{d(ggx_n, gx), d(ggy_n, gy)\}) \\ &= \varphi(0) = 0\end{aligned}$$

which implies that $d(gx, F(x,y) = 0$. Hence $gx = F(x,y)$. Similarly, one can show that $gy = F(y,x)$. □

Corollary 2.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such F has the mixed g-monotone property. Assume that*

$$\begin{aligned}&\varphi(d(F(x,y), F(u,v))) \\ &\quad \leq \varphi(\max\{d(gx, gu), d(gy, gv)\}) - \psi(\max\{d(gx, gu), d(gy, gv)\})\end{aligned} \tag{2.28}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where φ is an altering distance function and $\psi : [0,\infty) \to [0,\infty)$ is such that $\lim_{t\to r} \psi(t) > 0$ for each $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$. Assume also that $F(X \times X) \subseteq g(X)$, g is continuous and F and *g are O-compatible mappings. Suppose either the assumption (a) or (b) in Theorem 2.1 holds. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.*

Proof. In Theorem 2.1, taking $\theta(t_1, t_2) = \psi(\max\{t_1, t_2\})$ for all $t_1, t_2 \in [0, \infty)$, we get Corollary 2.1. □

Corollary 2.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such F has the mixed g-monotone property. Assume that*

$$\begin{aligned}(2.29)\qquad &\varphi\left(d(F(x, y), F(u, v))\right)\\ &\quad\leq \varphi\left(\max\{d(gx, gu), d(gy, gv)\}\right) - \psi\left(d(gx, gu) + d(gy, gv)\right)\end{aligned}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where φ is an altering distance function and $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to r} \psi(t) > 0$ for each $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$. Assume also that $F(X \times X) \subseteq g(X)$, g is continuous and F and g are O-compatible mappings. Suppose either the assumption (a) or (b) in Theorem 2.1 holds. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.

Proof. In Theorem 2.1, taking $\theta(t_1, t_2) = \psi(t_1 + t_2)$ for all $t_1, t_2 \in [0, \infty)$, we get Corollary 2.2. □

In Theorem 2.1, taking $gx = x$ for all $x \in X$, we get the following corollary.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

Assume that

$$(2.30)\qquad \varphi\left(d(F(x, y), F(u, v))\right) \leq \varphi\left(\max\{d(x, u), d(y, v)\}\right) - \theta\left(d(x, u), d(y, v)\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where φ is an altering distance function and $\theta \in \Theta$. Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

Then F has a coupled fixed point in X.

Remark 2.1. 1) In Corollary 2.1, the function ψ need not be continuous as was assumed in Theorem 3.1 in [4], so Corollary 2.1 is an improvement of the result of Choudhury, Metiya and Kundu [4].

2) Notice that Theorem 2.1 of Luong and Thuan [10] is also a consequence of Corollary 2.3. In fact, the condition (1.2) appearing in Theorem 2.1 in [10]

$$\phi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\frac{1}{2}\phi\left(d\left(x,u\right)+d\left(y,v\right)\right)-\psi\left(\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\right)$$

can be written as follows

$$2\phi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\phi\left(d\left(x,u\right)+d\left(y,v\right)\right)-2\psi\left(\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\right).$$

Since ϕ is subadditive, $\phi\left(2d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq2\phi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)$. Thus,

(2.31)
$$\phi\left(2d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\phi\left(2\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\right)-2\psi\left(\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\right).$$

Taking $\varphi(t)=\phi(2t)$ for all $t\in[0,\infty)$ and $\theta(t_1,t_2)=2\psi\left(\frac{t_1+t_2}{2}\right)$ for all $(t_1,t_2)\in[0,\infty)^2$ then φ is an altering distance function and $\theta\in\Theta$ and (2.31) can be written as

$$\varphi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\varphi\left(\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\right)-\theta\left(d\left(x,u\right),d\left(y,v\right)\right).$$

Since φ is non decreasing and

$$\frac{d\left(x,u\right)+d\left(y,v\right)}{2}\leq\max\left\{d\left(x,u\right),d\left(y,v\right)\right\},$$

we have

$$\varphi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right)\leq\varphi\left(\max\left\{d\left(x,u\right),d\left(y,v\right)\right\}\right)-\theta\left(d\left(x,u\right),d\left(y,v\right)\right).$$

Therefore, by applying Corollary 2.3 we obtain the desired result.

3. Examples

In this section, we give some examples to show that our results are effective.

Example 3.1. Let $X=[0,\infty)$. Then $(X,\leq)$ is a totally ordered set with the usual ordering of real numbers. Let $d(x,y)=|x-y|$, for all $x,y\in X$. Then (X,d) is a complete metric space and X has the property as in Theorem 2.1. Let $F:X\times X\to X$ be defined by

$$F(x,y)=\begin{cases}\frac{x^3-y^3}{4} & \text{if } x,y\in X, x\geq y,\\ 0 & \text{if } x<y.\end{cases}$$

and $g:X\to X$ be defined by

$$gx=x^3,\ \text{ for all } x\in X$$

Then F has the mixed g-monotone property, $F(X\times X\subseteq g(X))$, g is continuous. Let $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that

$$\lim_{n\to\infty}F\left(x_n,y_n\right)=x,\ \ \lim_{n\to\infty}gx_n=x,$$

and
$$\lim_{n\to\infty} F\left(y_n, x_n\right) = y, \quad \lim_{n\to\infty} gy_n = y$$
Then, obviously, $x = y = 0$.
For $n \geq 0$, we have
$$F(x_n, y_n) = \begin{cases} \frac{x_n^3 - y_n^3}{4} & \text{if } x_n \geq y_n, \\ 0 & \text{if } x_n < y_n. \end{cases}$$
and
$$F(y_n, x_n) = \begin{cases} \frac{y_n^3 - x_n^3}{4} & \text{if } y_n \geq x_n, \\ 0 & \text{if } y_n < x_n. \end{cases}$$
and $gx_n = x_n^3, gy_n = y_n^3$.
Then it easy to see that
$$\lim_{n\to\infty} d\left(gF(x_n, y_n), F(gx_n, gy_n)\right) = 0$$
and
$$\lim_{n\to\infty} d\left(gF(y_n, x_n), F(gy_n, gx_n)\right) = 0,$$
that is, F and g are compatible and thus are O-compatible. Let $\varphi : [0, \infty) \to [0, \infty)$ be given by
$$\varphi(t) = t, \quad \text{for all } t \in [0, \infty)$$
and $\theta : [0, \infty)^2 \to [0, \infty)$ be given by
$$\theta(t_1, t_2) = \frac{t_1 + t_2}{4}, \quad \text{for all } (t_1, t_2) \in [0, \infty)^2$$
Then φ is an altering distance function and $\theta \in \Theta$.
We next show that the inequality (2.1) of Theorem 2.1 holds.
We take $x, y, u, v \in X$ such that $gx \geq gu$ and $gy \leq gv$, that is, $x^3 \geq u^3$ and $y^3 \leq v^3$.
Let $A = d\left(gx, gu\right) + d\left(gy, gv\right) = |x^3 - u^3| + |y^3 - v^3|$. Then
$$\max\left\{d\left(gx, gu\right), d\left(gy, gv\right)\right\} = \max\left\{\left|x^3 - u^3\right|, \left|y^3 - v^3\right|\right\} \geq \frac{1}{2}A.$$
We have the following possible cases.
Case 1. $x \geq y$ and $u \geq v$. Then
$$d\left(F\left(x, y\right), F\left(u, v\right)\right) = d\left(\frac{x^3 - y^3}{4}, \frac{u^3 - v^3}{4}\right) = \left|\frac{x^3 - y^3}{4} - \frac{u^3 - v^3}{4}\right|$$
$$= \left|\frac{x^3 - u^3}{4} + \frac{v^3 - y^3}{4}\right| \leq \frac{1}{4}A$$
Case 2. $x \geq y$ and $u < v$. Then
$$d\left(F\left(x, y\right), F\left(u, v\right)\right) = d\left(\frac{x^3 - y^3}{4}, 0\right) = \frac{x^3 - y^3}{4} = \frac{x^3 - u^3 + u^3 - y^3}{4}$$
$$\leq \frac{x^3 - u^3}{4} + \frac{v^3 - y^3}{4} \leq \frac{1}{4}A.$$

Case 3. $x < y$ and $u \geq v$. Then

$$\begin{aligned} d\left(F\left(x,y\right),F\left(u,v\right)\right) &= d\left(0,\frac{u^3-v^3}{4}\right) = \frac{u^3-v^3}{4} = \frac{u^3-x^3+x^3-v^3}{4} \\ &= \frac{x^3-v^3}{4} - \frac{x^3-u^3}{4} \leq \frac{x^3-v^3}{4} + \frac{x^3-u^3}{4} \\ &\leq \frac{x^3-u^3}{4} + \frac{y^3-v^3}{4} \leq \frac{1}{4}A. \end{aligned}$$

Case 4. $x < y$ and $u < v$. Then

$$d\left(F\left(x,y\right),F\left(u,v\right)\right) = d\left(0,0\right) = 0 \leq \frac{1}{4}A$$

In all above cases, we have

$$\begin{aligned} \varphi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right) &\leq \frac{1}{4}A = \frac{1}{2}A - \frac{1}{4}A \\ &\leq \max\left\{d\left(gx,gu\right),d\left(gy,gv\right)\right\} - \frac{d\left(gx,gu\right)+d\left(gy,gv\right)}{4} \\ &= \varphi\left(\max\left\{d\left(gx,gu\right),d\left(gy,gv\right)\right\}\right) - \theta\left(d\left(gx,gu\right),d\left(gy,gv\right)\right) \end{aligned}$$

Hence all the conditions of Theorem 2.1 are satisfied and it is seen that $(0,0)$ is a coupled coincidence point of F and g in X.

Example 3.2. Let $(X,d,\preceq)$, F and g be defined as in Example 2.1. Then

(i) X is complete and X has the property
- if a non-decreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all n,
- if a non-increasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all n.

(ii) $F(X \times X) = \{0,1\} \subset \{0\} \cup [1/2,1] = g(X)$

(iii) g is continuous and g and F are O-compatible.

(iv) There exist $x_0 = 0, y_0 = 1$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$.

(v) F has the mixed g-monotone property. Indeed, for every $y \in X$, let $x_1,x_2 \in X$ such that $gx_1 \preceq gx_2$
- if $gx_1 = gx_2$ then $x_1,x_2 = 0$ or $x_1,x_2 \in [1/2,1]$ or $x_1,x_2 \in (1,3/2]$ or $x_1,x_2 \in (3/2,2]$. Thus, $F(x_1,y) = 0 = F(x_2,y)$ if $y \in \{0\} \cup [1/2,1]$ and $x_1,x_2 = 0$ or $x_1,x_2 \in [1/2,1]$, otherwise $F(x_1,y) = 1 = F(x_2,y)$.
- if $gx_1 \prec gx_2$, then $gx_1 = 0$ and $gx_2 = 1$, i.e., $x_1 = 0$ and $x_2 \in [1/2,1]$. Thus $F(x_1,y) = 0 = F(x_2,y)$ if $y \in \{0\} \cup [1/2,1]$ and $F(x_1,y) = 1 = F(x_2,y)$ if $y \in (1,2]$

Therefore, F is g-non-decreasing in its first argument. Similarly, F is g-non-increasing in its second argument.

(vi) For $x,y,u,v \in X$, if $gx \succeq gu$ and $gy \preceq gv$ then $d(F(x,y),F(u,v)) = 0$. Indeed,
- if $gx \succ gu$ and $gy \prec gv$ then $y = u = 0$ and $x,v \in [1/2,1]$. Thus $d(F(x,y),F(u,v)) = d(0,0) = 0$.

- if $gx = gu$ and $gy \prec gv$ then $y = 0$ and $v \in [1/2, 1]$. Thus if $x = u = 0$ or $x, u \in [1/2, 1]$ then $d(F(x, y), F(u, v)) = d(0, 0) = 0$, otherwise $d(F(x, y), F(u, v) = d(1, 1) = 0$. Similarly, if $gx \succ gu$ and $gy = gv$ then $d(F(x, y), F(u, v)) = 0$.
- if $gx = gu$ and $gy = gv$ then both x, u are in one of the sets $\{0\}$, $[1/2, 1]$, $(1, 3/2]$ or $(3/2, 2]$ and both y, v are also in one of the sets $\{0\}$, $[1/2, 1]$, $(1, 3/2]$ or $(3/2, 2]$. Thus $d(F(x, y), F(u, v)) = d(0, 0) = 0$ if $x = u = 0$ or $x, u \in [1/2, 1]$ and $y = v = 0$ or $y, v \in [1/2, 1]$, otherwise, $d(F(x, y), F(u, v)) = d(1, 1) = 0$

Therefore, all the conditions of Theorem 2.1 are satisfied with $\phi(t) = t$ and $\theta(t_1, t_2) = \max\{t_1, t_2\}/2$. Applying Theorem 2.1, we conclude that F and g have a coupled coincidence point. Note that, we cannot apply the result of Choudhury and Kundu [3], the result of Choudhury, Metiya and Kundu [4] as well as the result of Lakshmikantham and Ćirić [9] to the mappings in this example.

References

[1] R. P. Agarwal, M. A. El-Gebeily, D. O. Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008) 1-8.

[2] I. Altun, H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. 2010 (2010) 17 pages. Article ID 621469.

[3] B. S. Choudhury, A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Analysis **73** (2010) 2524-2531.

[4] B. S. Choudhury, N. Metiya, A. Kundu, *Coupled coincidence point theorems in ordered metric spaces*, Ann. Univ. Ferrara **57** (2011) 1-16.

[5] Lj. Ćirić, N. Cakić, M. Rajović, J. S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl. 2008 (2008) 11 pages. Article ID 131294.

[6] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006) 1379-1393.

[7] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, NonlinearAnal. **72** (2010) 1188-1197.

[8] J. Harjani, B. Lopez, K. Sadarangani, *Fixed point theorems for mixed monotone operators and applications to integral equations*, Nonlinear Anal. **74** (2011) 1749-1760.

[9] V. Lakshmikantham, L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009) 4341-4349.

[10] N. V. Luong, N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011) 983-992.

[11] N. V. Luong, N. X. Thuan, *Coupled fixed point theorems in partially ordered metric spaces*, Bull. Math. Anal. Appl. Vol **2** (4)(2010), 16-24.

[12] N. V. Luong, N. X. Thuan, *Coupled fixed point theorems for mixed monotone mappings and an application to integral equations*, Compt. Math. Appl. **62** (2011) 4238-4248.

[13] N. V. Luong, N. X. Thuan, *Coupled points in ordered generalized metric spaces and application to integro-differential equations*, An. St. Univ. Ovidius Constanta (accepted).

[14] H. K. Nashine, B. Samet, *Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal. **74** (2011) 2201-2209.

[15] J. J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order **22** (2005) 223-239.

[16] J. J. Nieto, R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica, Engl. Ser. **23** (12) (2007) 2205-2212.

[17] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004) 1435-1443.

[18] D. O. Regan, A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008) 1241-1252.

[19] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal **72** (2010) 4508-4517.

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