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SUBMANIFOLD THEORY AND PARALLEL TRANSPORT

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Dedicated to Professor Bang–Yen Chen at the occasion of his 70th anniversary

Abstract. A geometrical interpretation of semi-parallel submanifolds is presented in terms of parallel transport of the second fundamental form around an infinitesimal coordinate parallelogram. Further, a new scalar curvature invariant of the immersion is introduced. Isotropy of this invariant at every point means that the submanifold is pseudo-parallel.

1. Introduction

The most symmetric of all Riemannian manifolds (M, g) are the *real space forms*, i.e. the manifolds with *constant sectional curvature* $K = c$. Their (0,4)-*Riemann-Christoffel curvature tensor* R is given by $R = \frac{c}{2} g \wedge g$, ($\wedge$ denoting the Nomizu-Kulkarni product of (0,2)-tensors), and they were characterized by Riemann, Helmholtz and Lie as the Riemannian spaces which satisfy the *axiom of free mobility*. The class of the real space forms can be obtained by applying *projective transformations* to the locally flat spaces, i.e. to the manifolds (M, g) for which $K = 0$, or equivalently, for which $R = 0$. The Riemann-Christoffel curvature tensor according to Schouten [15] essentially measures the *change of direction* when a vector $v \in T_pM$ is *parallelly transported* all around *infinitesimal coordinate parallelograms* to a vector $v^\star \in T_pM$. The locally flat spaces are characterized by the fact that such parallel transport leaves v invariant, i.e. such that $v^\star = v$ for all such coordinate parallelograms cornered at p.

In the 1920ties, Cartan introduced the *locally symmetric spaces*, i.e. the Riemannian manifolds (M, g) for which R is parallel, $\nabla R = 0$, where ∇ denotes the *Levi-Civita connection* of the metric [2]. As shown by Cartan, the locally symmetric spaces are

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the Riemannian manifolds for which locally all *geodesic reflections or symmetries* σ_p in all points p of M actually are *isometries*, and, as shown by Levy [13], they can also be characterized as the Riemannian manifolds for which the *sectional curvature* $K(p,\pi)$ remains *invariant under parallel transport along any curve in* M, i.e. for which $K(p^\star,\pi^\star) = K(p,\pi)$, where $\pi^\star \subset T_{p^\star}M$ is the plane obtained by moving π parallelly from p to $p^\star$ along any curve γ joining p and $p^\star$. The study of the locally symmetric spaces was independently started by Shirokov [16][17].

Every locally symmetric space satisfies $R \cdot R = 0$, whereby the first R stands for the *curvature operator* of (M,g), i.e. for tangent vector fields X and Y one has $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, which acts as a derivation on the second R which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general. The Riemannian manifolds for which $R \cdot R = 0$ are called *semi-symmetric spaces* and were classified by Szabó [18][19]. They can be characterized by the geometric property that, up to second order, $K(p,\pi^\star) = K(p,\pi)$, whereby π is any tangent 2-plane to M at p and $\pi^\star$ is the tangent 2-plane to M at p obtained by parallelly transporting π all around any infinitesimal coordinate parallelogram cornered at p. For short, their *sectional curvatures are invariant under parallel transport around infinitesimal coordinate parallelograms* [10].

In the 1980ties, Deszcz [6][7][11] introduced the *pseudo-symmetric spaces* as follows. Let $Q(g,R) \equiv \wedge_g \cdot R$ be the Tachibana tensor of a Riemannian manifold (M,g), i.e. the $(0,6)$-tensor $\wedge_g \cdot R$ where the metrical endomorphism $(X \wedge_g Y) : TM \to TM$ given by $(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$ acts as a derivation on the $(0,4)$-curvature tensor R, then M is said to be pseudo-symmetric if the $(0,6)$-tensors $R \cdot R$ and $Q(g,R)$ are proportional, i.e. if $R \cdot R = L\,Q(g,R)$ for some scalar valued function $L : M \to \mathbb{R}$. This function is called the *double sectional curvature* or the *sectional curvature of Deszcz* [10]. The class of pseudo-symmetric manifolds can be obtained by applying *projective transformations* to the semi-symmetric manifolds, i.e. to the manifolds for which $L = 0$ [4][20].

In analogy with the above *intrinsic symmetries* of Riemannian manifolds concerning their Riemann-Christoffel curvature tensor R, table 1 lists the corresponding *extrinsic symmetries* of *submanifolds* concerning their *second fundamental form.*

TABLE 1. Comparison between intrinsic and extrinsic symmetries

Intrinsic		**Extrinsic**	
Flat space	$R = 0$	Totally geodesic	$h = 0$
Space form	$R = \frac{c}{2}\, g \wedge g$	Totally umbilical	$h = g\,\mathbf{H}$
Locally symmetric	$\nabla R = 0$	Parallel	$\overline{\nabla} h = 0$
Semi-symmetric	$R \cdot R = 0$	Semi-parallel	$\overline{R} \cdot h = 0$
Pseudo-symmetric	$R \cdot R = L\,Q(g,R)$	Pseudo-parallel	$\overline{R} \cdot h = L\,Q(g,h)$

In section 2 we will recall some basic definitions and notations. In section 3 we will give some geometrical interpretations of totally geodesic and parallel submanifolds in terms of parallel transport. Further, we will present a new geometrical interpretation of semi-parallel submanifolds. Every semi-parallel submanifold satisfies the condition $R^{\perp}(X,Y)\mathbf{H}=0$. However, the converse is not true in general. We call spaces which satisfy this condition H-semi-parallel because for these spaces the mean curvature vector $\mathbf{H}$ is invariant under parallel transport around infinitesimal coordinate parallelograms.

In section 4 we introduce a new scalar curvature invariant of submanifolds which depends on the point, a tangent direction and a plane, as well as on a normal direction to the submanifold. If this invariant is *isotropic*, i.e. only depends on the point, the submanifold turns out to be pseudo-parallel at this point.

2. Notation

Let (M^n,g) and $(\widetilde{M}^{n+m},\widetilde{g})$ be two Riemannian manifolds with dimension n and $n+m$, and with respective Levi-Civita connections ∇ and $\widetilde{\nabla}$. Assume that (M,g) is isometrically immersed in $(\widetilde{M},\widetilde{g})$. We can decompose the covariant derivative $\widetilde{\nabla}$ of two tangent vector fields X and Y to M, i.e. $X,Y\in\mathfrak{X}(M)$, into its tangential and normal part as follows,

$$\widetilde{\nabla}_X Y=\nabla_X Y+h(X,Y), \tag{2.1}$$

where $h(X,Y)$ is normal to M and h is called the second fundamental form. Equation (2.1) is known as the Gauss formula. A submanifold is called *totally geodesic* if $h=0$.

We can further define the normal connection $\nabla^{\perp}$ through the decomposition of the covariant derivative in $\widetilde{M}$ of a normal vector field ξ of M in $\widetilde{M}$ with respect to a tangent vector field $X\in\mathfrak{X}(M)$ into its tangential and normal parts,

$$\widetilde{\nabla}_X\xi=-A_\xi X+\nabla^{\perp}_X\xi, \tag{2.2}$$

where A_ξ is called the shape operator with respect to ξ. The shape operator is related to the second fundamental form by $g(A_\xi X,Y)=\widetilde{g}(h(X,Y),\xi)$. Equation (2.2) is known as the Weingarten formula. A point p of M is called umbilic if $h(u,v)=g(u,v)\mathbf{H}$, for all $u,v\in T_pM$. $\mathbf{H}$ is called the mean curvature vector field, or the Bompiani vector field, of the submanifold M in $\widetilde{M}$.

The Bortolotti - Van der Waerden connection $\overline{\nabla}$ of M in $\widetilde{M}$ acting on h gives,

$$(\overline{\nabla}h)(X,Y,Z):=\nabla^{\perp}_X\Big(h(Y,Z)\Big)-h(\nabla_X Y,Z)-h(Y,\nabla_X Z),$$

with $X,Y,Z\in\mathfrak{X}(M)$.

3. A geometrical interpretation of semi-parallel submanifolds

Let $\gamma:I\subseteq\mathbb{R}\to M$ be a curve in a Riemannian manifold M. Choose points $p=\gamma(t_0)$ and $p^\star=\gamma(t_0^\star)$ on the curve and a vector $v\in T_pM$. Let V be the unique vector field along γ such that

$$V(t_0)=v,\qquad \nabla_{\gamma'}V=0,$$

where ∇ is the Levi-Civita connection of (M, g). Then we call $v^\star = V(t_0^\star)$ the parallel transport of v from p to $p^\star$ along the curve γ with respect to the connection ∇.

If M is immersed in another Riemannian manifold $\widetilde{M}$, we can also transport the vector v parallel along γ in $\widetilde{M}$ with respect to the Levi-Civita connection $\widetilde{\nabla}$ of $(\widetilde{M}, \widetilde{g})$. The following result can be proven straightforwardly.

Proposition 3.1. *A submanifold M in $\widetilde{M}$ is totally geodesic if and only if the parallel transports of tangent vectors to M with respect to the connections ∇ on M and $\widetilde{\nabla}$ on $\widetilde{M}$ are the same.*

Given a curve γ in M and two vectors $u, v \in T_pM$, with $\gamma(t_0) = p$, we have the vector $h(u, v)$ in the normal space of M at the point p, $T_p^\perp M$. At the point $\gamma(t_0^\star) = p^\star$, we can consider two normal vectors. First, the parallel translate of $h(u, v)$ by $\nabla^\perp$, which we denote by $h(u, v)^{\star\perp}$, and secondly, the vector $h(u^\star, v^\star)$ obtained after first parallelly translating u and v by ∇, and then applying h.

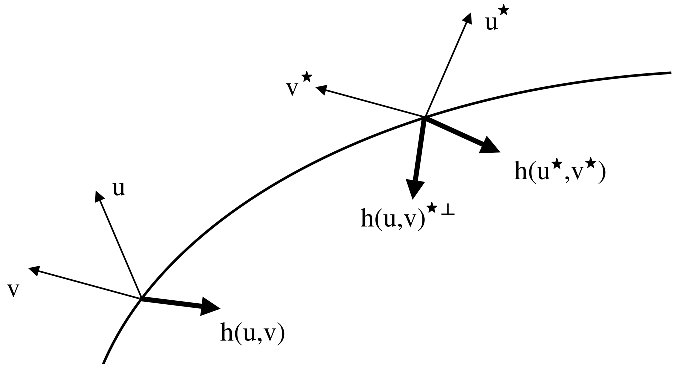

FIGURE 1. A geometrical interpretation of parallel submanifolds.

A submanifold is called parallel or extrinsically symmetric when $\overline{\nabla} h = 0$.

Proposition 3.2. *A submanifold M in $\widetilde{M}$ is parallel if and only if the parallel transport of the second fundamental form with respect to $\nabla^\perp$ along any curve in M is equal to the second fundamental form acting on the parallel transport of two tangent vectors to M along the same curve.*

Proof. Let $p \in M$ and $\gamma : I \subset \mathbb{R} \to M$ a curve in M with $\gamma(t_0) = p$. Consider two vector fields $U, V \in \mathfrak{X}(\gamma)$ so that $U_p = u$ and $V_p = v$, and $\nabla_{\gamma'} U = \nabla_{\gamma'} V = 0$.

Assume that M is parallel, i.e. $\overline{\nabla} h = 0$. Because the parallel transport defines a unique vector field it is sufficient to prove that $\nabla^\perp_{\gamma'} h(U, V) = 0$. In fact,

$$\nabla^\perp_{\gamma'} h(U, V) = h(\nabla_{\gamma'} U, V) + h(U, \nabla_{\gamma'} V) = 0.$$

Conversely, let us assume that $h(u,v)^{\star\perp} = h(u^\star, v^\star)$. Then,

$$\begin{aligned}(\overline{\nabla} h)(\gamma'(t_0), u, v) \mid_p &= \Big(\nabla^\perp_{\gamma'} h(U,V) - h(\nabla_{\gamma'} U, V) - h(U, \nabla_{\gamma'} V)\Big) \mid_p \\ &= \nabla^\perp_{\gamma'} h(U,V) \mid_p = 0.\end{aligned}$$

Because p, u, v and γ can be chosen arbitrary this implies $\overline{\nabla} h = 0$. $\square$

The parallel submanifolds were introduced in [8][9] as an extrinsic analogue of the symmetric Riemannian manifolds introduced by Cartan. In a space of constant curvature, they are the submanifolds which are invariant with respect to the reflections in the normal space. For example, the only surfaces in $\mathbb{E}^3$ which are parallel are the open parts of planes, spheres and round cylinders [9][14]. Remark that every extrinsic symmetric submanifold in a space form is also an intrinsically symmetric manifold in the sense of Cartan.

If we are given an infinitesimal coordinate parallelogram cornered in p, with tangents x and y in M, the parallel translate of a tangent vector $v \in T_pM$ around the whole parallelogram with respect to ∇, is given by

$$v^\star = v - \big[R(x,y)v\big]\Delta x \Delta y + O^{>2}(\Delta x, \Delta y),$$

while the parallel translate of a normal vector ξ with $\nabla^\perp$ is given by

$$\xi^{\star\perp} = \xi - \big[R^\perp(x,y)\xi\big]\Delta x \Delta y + O^{>2}(\Delta x, \Delta y).$$

The *normal curvature tensor* $R^\perp$ is defined by $R^\perp(u,v)\xi := \Big(\nabla^\perp_u \nabla^\perp_v - \nabla^\perp_v \nabla^\perp_u - \nabla^\perp_{[u,v]}\Big)\xi$, for all $u, v \in T_pM$. Just as the Riemann-Christoffel curvature tensor R can be geometrically interpreted as measuring the second order difference in the direction of a vector after parallel transport around an infinitesimal coordinate parallelogram (see [15]), the normal curvature tensor $R^\perp$ can be analogously interpreted as measuring the second order difference in the direction of a normal vector after parallel transport with the normal connection $\nabla^\perp$ around an infinitesimal coordinate parallelogram in the submanifold.

We can then consider the second fundamental form after parallel transport of u and v around the parallelogram,

$$h(u^\star, v^\star) = h(u,v) - \Big[h\Big(R(x,y)u, v\Big) + h\Big(u, R(x,y)v\Big)\Big]\Delta x \Delta y + O^{>2}(\Delta x, \Delta y),$$

and the parallel translate of the second fundamental form itself around the parallelogram with $\nabla^\perp$,

$$h(u,v)^{\star\perp} = h(u,v) - \Big[R^\perp(x,y)h(u,v)\Big]\Delta x \Delta y + O^{>2}(\Delta x, \Delta y).$$

Subtracting both expressions gives

$$h(u^\star, v^\star) - h(u,v)^{\star\perp} = (\overline{R} \cdot h)(u, v; x, y)\Delta x \Delta y + O^{>2}(\Delta x, \Delta y),$$

where

$$(\overline{R}\cdot h)(U,V;X,Y) = (\overline{R}(X,Y)\cdot h)(U,V)$$
$$:= R^{\perp}(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V),$$

with $X,Y,U,V \in \mathfrak{X}(M)$.

A submanifold is called semi-parallel when $\overline{R}\cdot h = 0$ [5].

Proposition 3.3. *A submanifold is semi-parallel if and only if, for all $p \in M$, the normal vectors $h(u,v)^{\star\perp}$ and $h(u^{\star},v^{\star})$ coincide for all $u,v \in T_pM$ and for every coordinate parallelogram in M, up to second order.*

Remark that the results in Proposition 3.1 and 3.2 are exact results, while the interpretation in Proposition 3.3 only holds up to second order.

In [5] it was shown that a semi-parallel submanifold of the Euclidean space $\mathbb{E}^n$ is intrinsically semi-symmetric.

4. A new scalar invariant and pseudo-parallel submanifolds

Probably the most simple (0,4)-tensor acting on tangent vectors which has the same symmetry properties as $\overline{R}\cdot h$, is given by

$$Q(g,h)(U,V;X,Y) = -((X\wedge Y)\cdot h)(U,V) = h((X\wedge Y)U,V) + h(U,(X\wedge Y)V),$$

with $X,Y,U,V \in \mathfrak{X}(M)$. The wedge product $\wedge$ between two vector fields $X,Y \in \mathfrak{X}(M)$ is defined as

$$(X\wedge Y)U = g(U,X)Y - g(U,Y)X,$$

for all $U \in \mathfrak{X}(M)$.

Proposition 4.1. *A submanifold M in $\widetilde{M}$ is totally umbilical if and only if $Q(g,h) = 0$.*

Proof. Assume that $Q(g,h) = 0$. In particular, if $X,Y \in \mathfrak{X}(M)$ are orthonormal vectors,

$$Q(g,h)(X,Y;X,Y) = h(Y,Y) - h(X,X) = 0.$$

Moreover, it holds that

$$Q(g,h)(X,X;X,Y) = 2h(X,Y) = 0,$$

and hence M is totally umbilical. The other direction is trivial. □

We can give the following geometrical interpretation of the vector $Q(g,h)(u,v;x,y)$. Let $u,v,x,y \in T_pM$ and assume that $\{x,y\}$ are orthonormal. We extend these vectors to an orthonormal basis $\{x,y,e_3,\ldots,e_n\}$ of T_pM. Both vectors u and v can be decomposed with respect to this basis. Consider the vectors $\widehat{u}$ and $\widehat{v}$ which are obtained after a rotation of the components of u and v in the plane spanned by the basis vectors x and y about an infinitesimal angle ε, while keeping the other components fixed. Up to first order in ε, these vectors read

$$\begin{aligned}\widehat{u} &= u + \varepsilon\,\{g(u,x)y - g(u,y)x\} + O(\varepsilon^2)\\ &= u + \varepsilon\,(x\wedge y)u + O(\varepsilon^2),\end{aligned}$$

and analogously for $\widehat{v}$. The second fundamental form applied to these two vectors gives

$$\begin{aligned}h(\widehat{u},\widehat{v}) &= h(u,v) + \varepsilon\,\{h((x\wedge y)u,v) + h(u,(x\wedge y)v)\} + O(\varepsilon^2)\\ &= h(u,v) + \varepsilon\,Q(g,h)(u,v;x,y) + O(\varepsilon^2).\end{aligned}$$

If we compare our interpretation of $Q(g,h)$ with $\overline{R}\cdot h$ we find the following. The tensor $\overline{R}\cdot h$ measures the first order difference in direction of two second fundamental forms obtained after parallel transport of, first two vectors around an infinitesimal coordinate parallelogram, and secondly, the parallel transport of the second fundamental form itself around the parallelogram. The vector $Q(g,h)(u,v;x,y)$ on the other hand measures the first order difference in direction of the second fundamental form with the second fundamental form obtained after rotating the vectors in a particular plane. Notice that this is a movement *in* p, while $\overline{R}\cdot h$ measures a difference in direction after a movement *away* from p.

It seems therefore natural to consider $Q(g,h)$ as some kind of normalization of $\overline{R}\cdot h$. Using this idea we can define a new invariant of the immersion as follows.

Definition 4.1. Let $f : M \to \widetilde{M}$ be an isometric immersion. At $p \in M$ consider a tangent direction d, spanned by a vector $u \in T_pM$, a tangent plane $\pi = \operatorname{span}\{x,y\}$ in T_pM and a normal direction $\xi \in T_p^{\perp}M$. The direction d, plane π and normal direction ξ are said to be *fundamentally independent* if $\widetilde{g}(Q(g,h)(u,u;x,y),\xi) \neq 0$.

Definition 4.2. Let $p \in M$ and consider a fundamentally independent tangent direction d, spanned by a vector $u \in T_pM$, tangent plane $\pi = \operatorname{span}\{x,y\}$ in T_pM and normal direction $\xi \in T_p^{\perp}M$. We can then define the scalar $L_\xi(p,d,\pi)$ as

$$L_\xi(p,d,\pi) = \frac{\widetilde{g}\left((\overline{R}\cdot h)(u,u;x,y),\xi\right)}{\widetilde{g}\left(Q(g,h)(u,u;x,y),\xi\right)}.$$

Note that this definition is independent of the choice of basis of π, and of the vectors u and ξ which span the tangent and normal directions, respectively.

Theorem 4.1. *Let $f : M \to \widetilde{M}$ be an isometric immersion and $p \in \mathcal{U} \subset M$, whereby $\mathcal{U}$ is the set of points where $Q(g,h)$ is not identically zero. If the function L is isotropic in p, i.e. $L_\xi(p,d,\pi) = L(p)$, for all tangent directions d, tangent planes π and normal directions $\xi \in T_p^{\perp}M$, then*

$$\overline{R}\cdot h = L(p)\,Q(g,h) \qquad \text{at } p\in\mathcal{U}. \tag{4.1}$$

Proof. At the point $p \in \mathcal{U}$, define the tensor

$$S := \overline{R} \cdot h - L(p)\, Q(g, h).$$

It holds that $S(u, u; x, y) = 0$, for all $u, x, y \in T_pM$. Using the symmetries of the tensor S it follows that $S(u, v; x, y) = 0$, for all $u, v, x, y \in T_pM$. $\square$

Definition 4.3. Let $f : M \to \widetilde{M}$ be an isometric immersion. A point $p \in \mathcal{U}$ is called pseudo-parallel if (4.1) holds at p. A submanifold M is called pseudo-parallel if every point of M is pseudo-parallel.

From the Gauss equation it follows that a hypersurface in a space with constant curvature c is pseudo-parallel if and only if the shape operator A has *at most two distinct eigenvalues* λ and μ [1]. If $\lambda \neq \mu$, then $L = \lambda\mu + c$.

The concept of pseudo-parallel submanifolds is the extrinsic analogue of pseudo-symmetric manifolds. Moreover, if a manifold M can be L-pseudo-parallel immersed in a space form, then M itself is intrinsically L-pseudo-symmetric [1][21].

Proposition 4.2. *Let $f : M \to \widetilde{M}$ be an isometric immersion, $p \in \mathcal{U}$, $\pi = \mathrm{span}\{x, y\} \subset T_pM$ and $\xi \in T_p^{\perp}M$. If $L_\xi(p, d, \pi) = L_\xi(p, \pi)$, for all tangent directions d, then*

$$\widetilde{g}\Big(R^{\perp}(x, y)\mathbf{H},\, \xi\Big) = 0.$$

Proof. We have that

$$\widetilde{g}\Big((\overline{R} \cdot h)(u, u; x, y) - L_\xi(p, \pi)\, Q(g, h)(u, u; x, y),\, \xi\Big) = 0,$$

for all $u \in T_pM$. Let $\{e_1, \ldots, e_n\}$ be the orthonormal basis of T_pM which diagonalizes A_ξ. Then

$$\sum_{i=1}^{n} \widetilde{g}\Big((\overline{R} \cdot h)(e_i, e_i; x, y),\, \xi\Big) - L_\xi(p, \pi) \sum_{i=1}^{n} \widetilde{g}\Big(Q(g, h)(e_i, e_i; x, y),\, \xi\Big) = 0. \tag{4.2}$$

The last term becomes

$$\begin{aligned}
\widetilde{g}\Big(Q(g, h)(e_i, e_i; x, y),\, \xi\Big) &= 2, \widetilde{g}\Big(h((x \wedge y)e_i, e_i),\, \xi\Big) \\
&= 2\, \widetilde{g}\Big(A_\xi e_i,\, (x \wedge y)e_i\Big) \\
&= 2\lambda_{\xi i}\, \widetilde{g}\Big(e_i,\, (x \wedge y)e_i\Big) = 0.
\end{aligned}$$

The first term in (4.2) reads

$$\begin{aligned}\sum_{i=1}^{n}\widetilde{g}\Big((\overline{R}\cdot h)(e_i,e_i;x,y),\,\xi\Big) &= \sum_{i=1}^{n}\widetilde{g}\Big(R^{\perp}(x,y)h(e_i,e_i),\,\xi\Big)-2\sum_{i=1}^{n}\widetilde{g}\Big(h(R(x,y)e_i,e_i),\,\xi\Big)\\ &= n\,\widetilde{g}\Big(R^{\perp}(x,y)\mathbf{H},\,\xi\Big)-2\sum_{i=1}^{n}\lambda_{\xi i}\,\widetilde{g}\Big(e_i,\,R(x,y)e_i\Big)\\ &= n\,\widetilde{g}\Big(R^{\perp}(x,y)\mathbf{H},\,\xi\Big).\end{aligned}$$

Hence, $\widetilde{g}\Big(R^{\perp}(x,y)\mathbf{H},\,\xi\Big)=0$. $\square$

Corollary 4.1. *Let $f : M \to \widetilde{M}$ be a pseudo-parallel immersion, then, for every $X,Y\in\mathfrak{X}(M)$, there holds that $R^{\perp}(X,Y)\mathbf{H}=0$.*

In view of the previous result we introduce the following class of submanifolds.

Definition 4.4. Let $f : M \to \widetilde{M}$ be a n-dimensional Riemannian submanifold of $\widetilde{M}$ and denote by $\mathbf{H}$ the mean curvature vector. M is called H-semi-parallel if $R^{\perp}(X,Y)\mathbf{H}=0$, for all $X,Y\in\mathfrak{X}(M)$.

From the previous remarks it is obvious that H-semi-parallel submanifolds can be geometrically understood as those submanifolds for which the mean curvature vector $\mathbf{H}$, up to second order, remains invariant after parallel transport with $\nabla^{\perp}$ around an infinitesimal coordinate parallelogram.

Let us recall that a submanifold is pseudo-umbilical if the shape operator associated with $\mathbf{H}$ is a scalar multiple of the identity [3].

Theorem 4.2. *Let M be a H-semi-parallel surface in a Riemannian space form $\widetilde{M}^{2+m}(c)$. Then, M is either minimal, pseudo-umbilical or has trivial normal connection, i.e. $R^{\perp}=0$.*

Proof. The Ricci equation in this case reads

$$\widetilde{g}(R^{\perp}(X,Y)\xi,\eta)=\widetilde{g}([A_{\xi},A_{\eta}]\,X,Y),$$

with $X,Y\in\mathfrak{X}(M)$ and $\xi,\eta\in\mathfrak{X}^{\perp}(M)$. Let $\{\xi_1,\dots,\xi_m\}$ be an orthonormal frame in the normal bundle $T^{\perp}M$. If $\mathbf{H}\neq 0$, assume $\xi_1=\frac{\mathbf{H}}{\|\mathbf{H}\|}$ and choose a tangent orthonormal frame $\{e_1,e_2\}$ so that $A_{\xi_1}e_i=\alpha_i e_i$, $i=1,2$. Because M is H-semi-parallel we have that $\widetilde{g}(R^{\perp}(X,Y)\xi_1,\xi_a)=0$, for all ξ_a, $a=2,\dots,m$. Hence $[A_{\xi_1},A_{\xi_a}]=0$, for all ξ_a, $a=2,\dots,m$. If we write

$$A_{\xi_a}=\begin{pmatrix}(\beta_a)_{11} & (\beta_a)_{12}\\ (\beta_a)_{12} & (\beta_a)_{22}\end{pmatrix},$$

we obtain the relations $(\beta_a)_{12}(\alpha_1-\alpha_2)=0$, for all $a=2,\dots,m$. Either $\alpha_1=\alpha_2$ and thus M is pseudo-umbilical, or $(\beta_a)_{12}=0$, $\forall a$ and thus $R^{\perp}=0$. $\square$

This result corrects Theorem 5 of [12].

Proposition 4.3. *Let $f : M \to \widetilde{M}(c)$ be an n-dimensional submanifold of a real space form $\widetilde{M}(c)$. If the first normal subspace has maximal dimension at a point $p \in M$, i.e.* $\dim\{h(u,v) \mid u,v \in T_pM\} = \frac{n(n+1)}{2}$, *and M is H-semi-parallel it follows that M is pseudo-umbilical at p.*

Proof. Because M is H-semi-parallel, it follows from the Ricci equation that A_H commutes with every A_ξ, $\xi \in \mathfrak{X}^\perp(M)$. Choose an orthonormal basis for T_pM. Because the first normal subspace has maximal dimension, the shape operators span the space of all symmetric matrices. Hence, since A_H commutes with every shape operator, it commutes with every symmetric matrix. Thus we find that A_H must be a multiple of the identity. □

The following corollaries show that the pseudo-parallel submanifolds are a proper subset of the H-semi-parallel submanifolds (see also [1]).

Corollary 4.2. *Every minimal surface with non-vanishing normal connection in a 4-dimensional space form, for which there exists points where the ellipse of curvature is not a circle, is H-semi-parallel but not pseudo-parallel.*

Corollary 4.3. *Every non-isotropic, pseudo-umbilical surface with non-vanishing normal connection in a space form, is H-semi-parallel but not pseudo-parallel.*

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