Kragujevac Journal of Mathematics
Volume 37(1) (2013), Pages 25–31.

RICCI AND CASORATI PRINCIPAL DIRECTIONS OF $\delta(2)$ CHEN IDEAL SUBMANIFOLDS

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Dedicated to Professor Bang–Yen Chen at the occasion of his 70th anniversary

Abstract. We show that *for $\delta(2)$ Chen ideal submanifolds in Euclidean spaces* the (intrinsic) *Ricci principal directions* and the (extrinsic) *Casorati principal directions* coincide.

1. $\delta(2)$ Chen ideal submanifolds of Euclidean spaces

Let M^n be an n–dimensional Riemannian submanifold of an $(n+m)$–dimensional *Euclidean space* E^{n+m}, $(n \geq 2, m \geq 1)$ and let g, ∇ and $\tilde{g}, \tilde{\nabla}$ be the *Riemannian metric* and the corresponding *Levi–Civita connection* on M^n and on E^{n+m}, respectively. *Tangent vector fields* on M^n will be written as $X, Y, \ldots$ and *normal vector fields* on M^n in E^{n+m} will be written as $\xi, \eta, \ldots$. The *formulae of Gauss and Weingarten*, concerning the decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$, respectively, into their tangential and normal components along M^n in E^{n+m}, are given by $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi$, respectively, whereby h is the *second fundamental form* and A_ξ is the *shape operator* or *Weingarten map* of M^n with respect to the normal vector field ξ, such that $\tilde{g}(h(X,Y),\xi) = g(A_\xi(X), Y)$, and $\nabla^\perp$ is the *connection in the normal bundle*. The *mean curvature vector field* $\vec{H}$ is defined by $\vec{H} = \frac{1}{n} \operatorname{tr} h$ and its length $\|\vec{H}\| = H$ is the (extrinsic) *mean curvature* of M^n in E^{n+m}. A submanifold M^n in E^{n+m} is *totally geodesic* when $h = 0$, *totally umbilical* when $h = g\vec{H}$, *minimal* when $H = 0$ and *pseudo–umbilical* when $\vec{H}$ is an umbilical

Key words and phrases. $\delta(2)$ Chen ideal submanifolds, Casorati curvature, Ricci principal directions, Casorati principal directions.
2010 *Mathematics Subject Classification.* Primary: 53B20, 53B25, 53A07; Secondary: 53C42 .
Received: March 10, 2013.

normal direction. Let $\{E_1, \ldots, E_n, \xi_1, \ldots, \xi_m\}$ be any *adapted orthonormal* local frame field on the submanifold M^n in E^{n+m}, denoted for short also as $\{E_i, \xi_\alpha\}$, whereby $i \in \{1, 2, \ldots, n\}$ and $\alpha \in \{1, 2, \ldots, m\}$. By the *equation of Gauss*, the $(0,4)$ *Riemann–Christoffel curvature tensor* of a submanifold M^n in E^{n+m} is given by $R(X,Y,Z,W) = \tilde{g}(h(Y,Z), h(X,W)) - \tilde{g}(h(X,Z), h(Y,W))$. The $(0,2)$ *Ricci curvature tensor* of M^n is defined by $S(X,Y) = \sum_i R(X, E_i, E_i, Y)$ and the metrically corresponding $(1,1)$ tensor or *Ricci operator* will also be denoted by S: $g(S(X), Y) = S(X,Y)$. Since S is *symmetric* there exists on M^n an *orthonormal set of eigenvector fields* $R_1, \ldots, R_n$ which determine the (intrinsic) *Ricci principal directions* of the Riemannian manifold M^n, and the corresponding *eigenfunctions* $Ric_1, \ldots, Ric_n$ are the *Ricci curvatures* of M^n: $S(R_i) = Ric_i\, R_i$. A Riemannian manifold M^n is an *Einstein space* when $S = Ric\, g$, or still when *all Ricci curvatures are equal* $Ric_1 = \cdots = Ric_n = Ric$, M^n is a *quasi–Einstein space* when *it has a Ricci curvature of multiplicity* $\geq n-1$ and M^n is a 2*–quasi–Einstein space* when *it has a Ricci curvature of multiplicity* $\geq n-2$. The *scalar curvature* of a Riemannian manifold M^n is defined by $\tau = \sum_{i<j} K(E_i \wedge E_j)$ whereby $K(E_i \wedge E_j) = R(E_i, E_j, E_j, E_i)$ is the *sectional curvature* for the plane section $\pi = E_i \wedge E_j$, $(i \neq j)$. By the *equation of Ricci*, the *normal curvature tensor* of a submanifold M^n in E^{n+m} is given by $R^\perp(X,Y,\xi,\eta) = g([A_\xi, A_\eta](X), Y)$, whereby $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$, which, as already observed by Cartan [1], implies that *the normal connection is flat or trivial* if and only if *all shape operators* A_ξ *are simultaneously diagonalisable*.

The function $\inf K : M^n \to \mathbb{R}$ is defined by $(\inf K)(p) = \inf\{K(p,\pi)\,|\pi$ *is a plane section of* $T_p(M^n)\}$. In [2], B.-Y. Chen introduced the $\delta(2)$*–curvature* as $\delta(2) = \tau - \inf K$, which clearly is a *Riemannian scalar invariant* of the manifold (M^n, g). Later B.-Y. Chen introduced many further new scalar Riemannian invariants, together with $\delta(2)$ called his *delta–curvatures* $\delta(n_1, n_2, \ldots, n_k)$; (cfr. [3][4][5][6]). And, for all submanifolds M^n of Euclidean spaces E^{n+m}, or of arbitrary Riemannian ambient spaces $\tilde{M}^{n+m}$ for that matter, B.-Y. Chen established *optimal pointwise inequalities between these intrinsic delta–curvatures* of M^n *and the squared mean curvature* H^2, and some number determined by the curvature of the ambient space $\tilde{M}^{n+m}$, which is zero for Euclidean spaces. Such inequalities can be considered as imposing definite lower bounds, basically dictated by these delta–curvatures, to the extrinsic squared mean curvature or surface tension H^2 which results from the kind of shape of the submanifold M^n in the ambient space $\tilde{M}^{n+m}$. From this point of view, the submanifolds M^n which actually do realise such lower bound for their surface tension are called *Chen ideal submanifolds*.

For surfaces M^2 in E^3, the *Euler* inequality $K \leq H^2$, whereby K is the *Gauss curvature* of M^2 at once follows from the fact that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$, whereby k_1 and k_2 are the *principal curvatures* of M^2 in E^3, and, moreover, $K = H^2$ if and only if M^2 is *totally umbilical* i.e. if $k_1 = k_2$, or still, by a Theorem of Meusnier, if M^2 is (part of) a *plane* E^2 or of a *round sphere* S^2 in E^3. The inequalities of Chen do generalise this Euler inequality for the submanifolds M^n in general ambient

Riemannian manifolds $\tilde{M}^{n+m}$, and, in particular, for ambient Euclidean spaces they take the following form, (for more details, cfr. [6]).

Theorem A. *For any submanifold M^n in E^{n+m}, $\delta(n_1,\dots,n_k) \le c(n_1,\dots,n_k).H^2$, $\forall(n_1,\dots,n_k) \in S(n)$, and equality holds at a point p if and only if, with respect to some suitable adapted orthonormal frame $\{E_i,\xi_\alpha\}$ around p along M^n in E^{n+m}, the shape operators of M^n in E^{n+m} are given*

$$A_\alpha = \begin{pmatrix} \begin{matrix} A_1^\alpha & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k^\alpha \end{matrix} & 0 \\ 0 & \mu_\alpha I \end{pmatrix},$$

whereby I is an identity matrix and A_1^α, $\dots$, A_k^α are symmetric $n_1 \times n_1$, $\dots$, $n_k \times n_k$ matrices, respectively, for which $\operatorname{tr} A_1^\alpha = \cdots = \operatorname{tr} A_k^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

The next result is the special case of Theorem A for $k = 1$ and $n_1 = 2$ [2].

Theorem B. *For any submanifold M^n in E^{n+m},*

$$\delta(2) \le \{[n^2(n-2)]/[2(n-1)]\}.H^2, \qquad (*)$$

and equality holds at a point p of M^n if and only if, with respect to some suitable adapted orthonormal frame $\{E_i,\xi_\alpha\}$ around p along M^n in E^{n+m}, the shape operators of M^n in E^{n+m} are given by

$$A_\alpha = \begin{pmatrix} A_1^\alpha & 0 \\ 0 & \mu_\alpha I \end{pmatrix},$$

whereby I is an identity matrix and A_1^α is a symmetric 2×2 matrix for which $\operatorname{tr} A_1^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

And, one may further specialise to an orthonormal frame $\{F_1, \dots, F_n, \eta_1, \dots, \eta_m\}$ such that η_1 lies in the direction of $\vec{H}$ and such that $F_1, \dots, F_n$ diagonalise A_1, so that Theorem B can also be formulated as follows.

Theorem C. *For any submanifold M^n in E^{n+m},*

$$\delta(2) \le \{[n^2(n-2)]/[2(n-1)]\}.H^2, \qquad (*)$$

and equality holds at a point p if and only if, with respect to a suitable adapted orthonormal frame $\{F_i,\eta_\alpha\}$ around p along M^n in E^{n+m}, the shape operators of M^n in E^{n+m} are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \quad A_\gamma = \begin{pmatrix} c_\gamma & d_\gamma & 0 & \dots & 0 \\ d_\gamma & -c_\gamma & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$(\gamma \in \{2, \dots, m\})$, *whereby* $\mu = a + b$, $($*and* $\inf K = ab - \sum_\gamma (c_\gamma^2 + d_\gamma^2))$.

Such frame $\{F_i, \eta_\alpha\}$ will be called *an adapted Chen frame on* $\delta(2)$ *Chen ideal submanifolds*.

According to Theorems B and C it is clear that such ideal submanifolds M^n in E^{n+m} in general admit two *mutually orthogonal and complementary distinguished (tangent) distributions*: (i) the $2D$ distribution $E_1 \wedge E_2 = F_1 \wedge F_2$, which will be called *the distribution of the Chen planes* and which is the $2D$ distribution on which the Riemannian manifolds M^n *point by point* do attain their *minimal sectional curvatures*, and (ii) the orthogonally complementary distribution $E_3 \wedge \dots \wedge E_n = F_3 \wedge \dots \wedge F_n$. For a detailed description of the *minimal* $\delta(2)$ Chen ideal submanifolds referring to [2], we recall that for the *non–minimal* such submanifolds η_1 determines their mean curvature vector field $\vec{H}$.

2. The Casorati and the Ricci principal directions on $\delta(2)$ Chen ideal submanifolds

For any submanifold M^n in some ambient Riemannian manifold $\tilde{M}^{n+m}$, the $(1,1)$ tensor field $A^C = \sum_\alpha A_\alpha^2$ is called its *Casorati operator* and the *Casorati curvature (as such)* of M^n in $\tilde{M}^{n+m}$ is defined by $C = \frac{1}{n} \operatorname{tr} A^C = \frac{1}{n}\|h\|^2$. The Casorati operator being *symmetric* there exists on M^n an *orthonormal set of eigenvector fields* $F_1, \dots, F_n$ which determine *the extrinsic* or *Casorati principal directions* of the submanifold M^n in $\tilde{M}^{n+m}$, and the corresponding eigenfunctions $c_1, \dots, c_n$ (all ≥ 0), are the *extrinsic (tangential) principal curvatures* or the *(tangential) Casorati principal curvatures* of M^n in $\tilde{M}^{n+m}$; $A^C(F_i) = c_i F_i$. For the *geometrical meanings* of these notions, which essentially go back to Jordan and Casorati, see [7][8][9][10].

A *hypersurface* M^n in a Riemannian space $\tilde{M}^{n+1}$ is called *umbilical* when *its shape operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity* n, or, still, when *all its principal curvatures are equal.* A hypersurface M^n in $\tilde{M}^{n+1}$ is called *quasi–umbilical* when *its shape operator has an eigenvalue of multiplicity* $\geq n-1$, (see e.g. [11]), and it is called 2–*quasi–umbilical* when *its shape operator has an eigenvalue of multiplicity* $\geq n-2$ ([12][13]). Similarly, a general submanifold M^n in some ambient Riemannian space $\tilde{M}^{n+m}$ is called *Casorati umbilical* when *its Casorati operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity* n, or, still, when *all its (tangential) Casorati principal curvatures are equal.* A submanifold M^n in $\tilde{M}^{n+m}$ is called *Casorati quasi–umbilical* when *its Casorati operator has an eigenvalue of* *multiplicity* $\geq n-1$, and it is called *Casorati* 2*–quasi–umbilical* when *its Casorati operator has an eigenvalue of multiplicity* $\geq n-2$.

From Theorem C it follows that

$$A_1^2 = \begin{pmatrix} a^2 & 0 & 0 & \dots & 0 \\ 0 & b^2 & 0 & \dots & 0 \\ 0 & 0 & \mu^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu^2 \end{pmatrix},$$

$$A_\gamma^2 = \begin{pmatrix} c_\gamma^2 + d_\gamma^2 & 0 & 0 & \dots & 0 \\ 0 & c_\gamma^2 + d_\gamma^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$A^C = \begin{pmatrix} a^2 + \sum_\gamma (c_\gamma^2 + d_\gamma^2) & 0 & 0 & \dots & 0 \\ 0 & b^2 + \sum_\gamma (c_\gamma^2 + d_\gamma^2) & 0 & \dots & 0 \\ 0 & 0 & \mu^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu^2 \end{pmatrix}.$$

This shows that for $\delta(2)$ Chen ideal submanifolds *the eigendirections* $F_1, F_2, F_3, \dots,$ F_n *of the mean curvature vector field* $\vec{H}$ also are the eigendirections of the *Casorati operator*. In particular, it follows from these expressions of the matrices A_1 and A^C that for both of them the $(n-2)D$ subspaces $F_3 \wedge \dots \wedge F_n$ are eigenspaces corresponding to their *common eigenvalue* μ^2 of multiplicity $n-2$. And since $S = nHA_1 - A^C$, as follows by contraction of the Gauss equation for any submanifold M^n in E^{n+m}, we have the following.

Theorem 2.1.

(i) *On all* $\delta(2)$ *Chen ideal submanifolds* M^n *in* E^{n+m} *the (extrinsic) principal (tangential) Casorati directions and the (intrinsic) principal Ricci directions do coincide.*

(ii) *Every generic such submanifold is Ricci and Casorati* 2*–quasi–umbilical, whereby the corresponding common* $(n-2)D$ *eigenspaces are the orthogonal complements of the Chen planes on which the Riemannian manifolds* M^n *realise their minimal sectional curvatures.*

Remark. Part (i) of this result had been stated already in [14]. Part (ii), as also various other kinds of results, from e.g. [2] [15] and [16], could be reflected upon in the light of some questions raised by Professor Berger (see e.g. [17]) concerning *the distribution of the* 2*D–planes on which Riemannian manifolds take extremal values.*

3. $\delta(2,2,\ldots,2)$ Chen ideal submanifolds

The special case of B.-Y. Chen's Theorem A for $n_1 = \cdots = n_k = 2$ and for $k \geq 2$ is the following.

Theorem D. *For any submanifold M^n in E^{n+m},*

$$\delta(2,2,\ldots,2) \leq \{n^2[(n-k)-1]/[2(n-k)]\}.H^2, \qquad (**)$$

and equality holds at a point p, if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ around p along M^n in E^{n+m}, the shape operators of M^n in E^{n+m} are given by

$$A_\alpha = \begin{pmatrix} b_1^\alpha & c_1^\alpha & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ c_1^\alpha & d_1^\alpha & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & . & \ldots & . & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & . & \ldots & . & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & b_k^\alpha & c_k^\alpha & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & c_k^\alpha & d_k^\alpha & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \mu_\alpha & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \mu_\alpha \end{pmatrix},$$

whereby $b_1^\alpha + d_1^\alpha = \cdots = b_k^\alpha + d_k^\alpha = \mu_\alpha : M^n \to \mathbb{R}$.

The submanifolds M^n of E^{n+m} for which $(**)$ at all of their points actually is an equality are called $\delta(2,2,\ldots,2)$ *Chen ideal submanifolds.* And since the algebraic considerations of the eigenvectors and eigenvalues of the matrices $A_{\vec{H}}$ and A^C of such submanifolds, just like in Section 2, essentially concern the 2×2 blocks involved, we also have the following.

Theorem 3.1. *On all $\delta(2,2,\ldots,2)$ Chen ideal submanifolds M^n in E^{n+m} the principal Casorati directions and the principal Ricci directions do coincide.*

Acknowledgment: Anica Pantić, Miroslava Petrović–Torgašev and Leopold Verstraelen thank the Center for Scientific Research of the Serbian Academy of Sciences and Arts and the University of Kragujevac for their partial support of the research done for this paper.

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