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GEOMETRY OF SUBMANIFOLDS
I. THE FIRST CASORATI CURVATURE INDICATRICES

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Dedicated to the Distinguished MSU Professor Bang–Yen Chen, in gratitude for his guidance and friendship for almost 40 *years now, at the occasion of his* 70*th anniversary*

1. From *the Introduction to Vincensini's history of differential geometry* [1] comes the following quote: "... Quant à la notion même de l'intérêt que peut offrir un thème scientifique, lorsqu'on n'en perçoit pas les rapports, immédiats ou lointains, avec les réalités inhérents au monde concret dans lequel nous vivons, elle est elle aussi, dans une large mesure, affaire de foi, et souvent de convention. Il en est ainsi particulièrement en géométrie, òu, plus peut–être que dans les autres branches de la Science, l'intérêt d'un sujet déterminé est susceptible d'affecter des formes trés diverses. Indépendamment des qualités intrinsiques du sujet, considéré dans ses relations avec l'ensemble de l'édifice mathématique, des facteurs tenant plus specialement à la nature de l'esprit humain ou à la sensibilité même de l'âme humaine, peuvent intervenir dans son appréciation. Et il n'est pas jusqú à la simple mode, qui ne puisse influencer la marche de la pensée géométrique, et le jugement que l'on peut être amené à porter sur la valeur de son évolution. Cette évolution est d'ailleurs intimement liée au courant de conscience qui, depuis le fond des âges, anime l'humanité.
Depuis les temps les plus reculés en effet, par instinct de conservation d'abord, puis pour améliorer progressivement leurs conditions de vie, les hommes se sont trouvés dans la nécessité d'inventer et de construire des instruments de plus en plus perfectionnés, de procéder à des comparaisons quantitatives ou qualitatives mettant en

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jeu les notions de nombre ou d'étendue, de s'organiser en groupes ayant certaines préoccupations communes, bref, de se livrer aux différentes opérations qui constituent l'essense même de la science mathématique. La part prise par les différents aspects de la pensée mathématique dans l'évolution du courant de conscience auquel il vient d'être fait allusion a naturellement varié, non seulement avec le temps, mais aussi avec les influences que chaque époque, avec ses besoins propres et les exigeances particulières, spirituelles et matérielles, naturelles ou fortuites, a exercées sur cette même pensée. Mais il semble bien qu' aucune doctrine mathématique n'ait été plus intimement liée que la géométrie à cette évolution.
La géométrie représente incontestablement l'une des expressions les plus parfaites de ce besoin de spéculation qui est le propre et la raison d'être de l'esprit humain, et c'est à cette forme particulière de la pensée mathématique, et plus spécialement à la géométrie différentielle qui en est à la fois la quintescence et le remarquable aboutissement, qu'est consacrée l'étude qui va suivre.".

The above may well be juxtaposed to some parts of the contents of *the Foreword to Bang–Yen Chen's newest book on the geometry of submanifolds* [2], in which the author a.o. states that: "..., at least in my opinion, for specimens of the human kind, 'nature' essentially stands for their organised thoughts about their sensations and perceptions of 'their worlds outside and inside' and 'doing mathematics' basically stands for their thoughtful living in 'the universe' of their idealisations and abstractions of these sensations and perceptions. "The history of mathematics is the kernel of the history of the human culture, the skeleton of which supports and keeps together all the rest of the sciences", (Georges Sarton).".

And by his concrete and subtle psychology as experimentator, Claude Bernard came to the conclusion which was so compactly formulated by Bergson as follows: "(Real -added by L.V.-) scientific research is a dialogue between nature and our mind", cfr. [3].

In this series of papers, we will present some geometrical properties and formula's concerning what we propose to be called *the Trenčevski frame on submanifolds* with arbitrary dimensions and of arbitrary co–dimensions in (semi–) Riemannian ambient spaces, cfr. [4, 5, 6, 7]. In the present part, attention will be confined to *nD submanifolds M^n* of co–dimension *m in Euclidean spaces E^{n+m}*, (only for reasons of simplicity of the statements at this stage concerning some of the notions involved). The Trečevski frames may well be the *adapted frames* on general submanifolds of which *the geometrical characterisations of the tangent and normal orthonormal vectors are as natural as possible*, being essentially involved with our kind's *most intuitive notions of curvature*. Hereafter, as such, only *the tangent and the first principal normal vector fields* will be discussed, the hereby involved curvatures being *the tangential and the first normal Casorati curvatures*, which, in case of surfaces M^2 in E^3 are nothing but *the squares of the Euler normal or principal curvatures and Casorati's "most common sense" curvature*, respectively. Thus the present paper could be seen as a kind of explicitation of the last statement of Section 10 of ARIGATEN [8].

Here is one more citation of Bergson, taken from the Introduction to Tome 1 of the Encyclopédie française [3]: *" L'humanité ne comprend bien le nouveau que s'il prend la place de l'ancien."*. And, in this sense, the interested readers may find much further interest when reading further a. o. in some of the references which are listed, in a sometimes more or less liberal way, in Section 10, like e.g. in [ω].

2. For *curves* $\Gamma = M^1$ *in a Euclidean plane* E^2, basing on the geometrical idea's of Kepler and Descartes concerning *osculating circles* and of Huygens on *focal points*, in full generality, Newton analytically determined their *curvature* κ. Later, Euler gave the interpretation of κ as *the rate of change of the tangent directions* of Euclidean planar curves Γ *with respect to an arclength parameter* s, $\kappa = d\theta/ds = \lim_{\Delta s \to 0}(\Delta\theta/\Delta s)$; (cfr. Figure 1). Essentially this corresponds to *the Frenet formula's* $T' = \kappa\, N$ and $N' = -\kappa\, T$ of $\Gamma = M^1$ in E^2, whereby $\{T, N\}$ is *the Frenet frame* along Γ in E^2, i. e. T is *the unit tangent vector field* which heads in the direction of increasing arclength and N is *the unit normal vector field* such that the orientation of $\{T, N\}$ is *the standard* "counter–clockwise" *positive orientation in the plane*; (cfr. Figure 2). The sign *"plus" or "minus"* of the curvature κ indicates *the direction of the curving* of the curve Γ, or, equivalently, of the turning of its tangent T, *towards* (= "plus") *or away from* (="minus") *its normal* N. Hence, $\kappa^2 = (d\theta/ds)^2$ may be considered as the quantity which, *in accordance with our intuition*, most readily measures *the amount of curvature as such* of a Euclidean planar curve Γ, rather than the more sophisticated measure κ which, besides giving an equally accurate measure of the amount of curvature itself of Γ in E^2, by *its absolute value*, in addition, gives the supplementary information of the direction of turning in which the curve actually realises this curving, by *its sign*.

3. For *curves* $\Gamma = M^1$ *in a 3D Euclidean space* E^3, *the Frenet frame* $\{T, N, B\}$ along Γ in E^3 consists of *the unit tangent vector field* T and *the (first) principal normal vector field* N and *the binormal vector field or the second principal normal vector field* B, and the corresponding *Frenet formula's* are given by $T' = \kappa\, N$, $N' = -\kappa\, T + \tau\, B$ and $B' = -\tau\, N$; these notions and formula's first came up in the works of Pagani, Bartels–Senff, Frenet and Serret (cfr. [9][10]). Space curves originally were called *"curves with two curvatures"* since, formulated more or less as follows by Clairaut, "in the way that such curves are considered, they in some sense always result from the curvatures of two curves". In this context, following the works of Dürer which were at the origin of Monge's *descriptive geometry*, in particular, one might think of a pair of such curves that Clairaut alluded to as being *the projections of a space curve* Γ, say, when working "in" a rectangular Cartesian (x, y, z) co–ordinate system, *onto the perpendicular xy– and xz–planes*, yielding there two Euclidean planar curves, say, Γ_{xy} and Γ_{xz}; (cfr. Figure 3). And, along this line of thought, when focussing on the behaviour of a space curve Γ around one of its points p, one might more specifically consider such two curves as the planar curves Γ_1 and Γ_2 which are *the projections* of Γ *onto two arbitrary mutually orthogonal planes* $\pi_1 = [T \wedge \xi_1](p)$ *and* $\pi_2 = [T \wedge \xi_2](p)$ *through* *its tangent line at* p, say, determined by an arbitrary orthonormal normal frame field $\{\xi_1, \xi_2\}$ on Γ in E^3, or, in particular, as *the projections* Γ_{TN} and Γ_{TB} of Γ *onto its osculating and rectifying planes* $[T \wedge N](p)$ and $[T \wedge B](p)$ *at* p, respectively; (cfr. Figures 4 and 5).

In Lancret's mémoires on curves of double curvature, the meaning of *"double curvature"* was turned into the current one at present, that is, he considered *the curvature* κ (*the first curvature*) and *the torsion* τ (*the second curvature*) of space curves in terms of *the infinitesimal angles between "nearby" normal planes* (or, *tangent lines*, for that matter) and *between "nearby" osculating planes*, in accordance with which, afterwards, Cauchy found the scalar valued expressions for κ and τ.

The *formula of Gauss* of the general theory of submanifolds when written out for the special situation of curves $\Gamma = M^1$ in E^3 becomes $\tilde{\nabla}_T T = T' = \nabla_T T + h(T,T) = \kappa N$, whereby $\tilde{\nabla}$ is *the standard connection* on E^3, (that is the directional differentiation on R^3), $\nabla_T T$ is the tangential component of T', (which vanishes since T has constant lenght 1 and Γ is $1D$), and h is *the second fundamental form* of Γ in E^3, i.e. $h(T,T)$ is the normal component of T'. So, in particular, *the squared norm of the second fundamental form* is nothing but *the squared curvature*, $\kappa^2 = \|h\|^2$; (this fact, of course, also applies to the planar curves Γ in E^2, i.e. for the curves Γ in E^3 with identically vanishing torsion). With respect to an *arbitrary adapted orthonormal moving frame* $\{T, \xi_1, \xi_2\}$ along Γ in E^3, the generalised Frenet formula's are given by $T' = \tilde{\nabla}_T T = a_1\xi_1 + a_2\xi_2$, $\xi_1' = \tilde{\nabla}_T \xi_1 = -a_1 T + b\xi_2$ and $\xi_2' = \tilde{\nabla}_T \xi_2 = -a_2 T - b\xi_1$, for some real valued functions a_1, a_2 and b. Consequently: $\kappa^2 = \|h\|^2 = \|\tilde{\nabla}_T T\|^2 = a_1^2 + a_2^2$, which geometrically means that, *at every point* p *of a curve* Γ *in* E^3, *the square of the curvature* κ *is given by the sum of the squares of the curvatures* a_1 *and* a_2 *at this same point of the two Euclidean planar curves which are the projections of* Γ *onto any pair of mutually orthogonal planes passing through the tangent line of* Γ *at* p; (in the particular case that $\xi_1 = N$ and $\xi_2 = B$, of course, $\kappa(p) = |a_1(p)|$ and $a_2(p) = 0$).

4. *The Frenet theory of curves* $\Gamma = M^1$ *in arbitrary dimensional Euclidean spaces* E^{1+m}, i.e. of Euclidean curves with arbitrary co–dimensions $m \geq 1$, was established by C. Jordan [11]. For *the Frenet frame* $\{T, \eta_1, \eta_2, \ldots, \eta_m\}$ whereby T is *the (unit) tangent vector field* $T = \Gamma' = d\Gamma/ds$, η_1 is *the (unit) first principal normal vector field*, η_2 is *the (unit) second principal normal vector field*, or still *the first binormal vector field*, η_3 is *the (unit) third principal normal vector field*, etc., *the Frenet formula's* are given by $T' = \kappa_1\eta_1$, $\eta_1' = -\kappa_1 T + \kappa_2\eta_2$, $\eta_2' = -\kappa_2\eta_2 + \kappa_3\eta_3, \ldots, \eta_{m-1}' = -\kappa_{m-1}\eta_{m-2} + \kappa_m\eta_m$, $\eta_m' = -\kappa_m\eta_{m-1}$, whereby $\kappa_1, \kappa_2, \ldots, \kappa_m$ are *the* m *curvatures* of the curve Γ in E^{1+m}.

5. According to *the Euler theory of surfaces* M^2 *in* E^3, amongst all adapted orthonormal frame fields $\{E_1, E_2, \eta\}$ defined along M^2 in E^3, i.e. frames for which E_1 and E_2 are *tangent* to M^2 and, or, equivalently, for which η is *normal* to M^2 in E^3, likely, those which are *most adapted* to such a surface in view of their *geometrical specificity* are the orthonormal frames $\{F_1,\, F_2,\, \eta\}$ for which F_1 and F_2 at each point of M^2 determine *the principal tangential directions* of M^2 in E^3, i.e. *the directions in which the curvatures of the normal sections* σ (that is, *the curvatures of the Euclidean planar curves* σ *in which locally* M^2 *is cut by the various planes through its normal line at the points under consideration) attain their extremal values*, being *the principal curvatures* $k_1 \geq k_2$, or, still, *the eigenvalues of their shape operator* A, $A(F_1) = k_1\, F_1$ and $A(F_2) = k_2\, F_2$; (*the principal directions*, determined by F_1 and F_2, *remain unaltered* when *changing the orientation of* η, but under such a change *the principal curvatures switch their signs*). Let p be any point on a surface M^2 in E^3, and put $f_1 = F_1(p)$ and $f_2 = F_2(p)$, and let $k_1(p) \geq k_2(p)$ be the principal curvatures with respect to a unit normal vector field η. Then, *the formula of Euler* expresses *the normal curvature* $k(u)$ of M^2 in E^3 at p *in the tangent direction* $u = f_1.\cos\theta + f_2.\sin\theta$, i.e. the curvature of the Euclidean planar curve $\sigma(u)$ in which M^2 locally around p is cut by the plane $u \wedge \eta(p)$, that is, of *the normal section* of M^2 in E^3 at p in the direction u, as $k(u) = k(\theta) = k_1(p).\cos^2\theta + k_2(p).\sin^2\theta$, whereby $\theta = \measuredangle(f_1, u)$. *The quadratic Taylor–Maclaurin approximation* of the surface M^2 in E^3 in the neighbourhood of a point p is given by $z(x, y) = (1/2).\{k_1(p).x^2 + k_2(p).y^2\}$, whereby reference is made to *a Cartesian* (x, y, z)*–co–ordinate system* which has its origin O at p, of which the xy*–plane* is the tangent plane of M^2 at p, the x*–axis* and the y*–axis* are choosen in the principal directions f_1 and f_2, respectively, and the z*–axis* is pointing as determined by $\eta(p)$. *The conic sections* with equation $k_1(p).x^2 + k_2(p).y^2 = \pm 1$ constitute *the indicatix of Dupin*, on which in particular may be read off the normal curvatures $k(u)$ in all tangent directions u at p, (cfr. [12]).

Alternatively, as *Euler curvature indicatrix of a surface* M^2 *in* E^3 *at any one of its points*, one could consider *the intersection of the "vertical unit cylinder"* $x^2 + y^2 = 1$ *with the quadratic approximation* of M^2 in E^3 at p, since in any tangential direction u, or, still, in any tangent direction determined by an angle θ, *the* z*–co–ordinate or "positive, zero or negative height"* of this curve, according to Euler's formula, equals $(1/2).k(u)$; (cfr. Figure 6, which illustrates the case of an elliptic point p: $k_1(p) > k_2(p) > 0$).

6. Almost in complete analogy with this study of the extremal values of the curvatures of the Euclidean planar normal sections σ of surfaces M^2 in E^3 yielding *the principal curvatures* $k_1 \geq k_2$, and the directions in which these values are reached yielding *the principal tangential directions* F_1 and F_2 of such surfaces, *the critical values* of the curvatures of the Euclidean planar normal sections σ of *hypersurfaces* M^n *of any dimensions* $n \geq 2$ *in a Euclidean space* E^{n+1} yield *the principal curvatures* $k_1 \geq \cdots \geq k_n$ and the mutually orthogonal directions in which these values are reached yield *the principal tangent directions* $F_1,\, \ldots,\, F_n$ of such hypersurfaces M^n in E^{n+1}, and the above recalled classical results of Euler and Dupin for surfaces M^2 in E^3 essentially remain the same for hypersurfaces M^n in E^{n+1} with arbitrary dimensions n. Now, the *Euler curvature indicatrix at some point* p of M^n is *the intersection* of *the "vertical* *unit hypercylinder"* $x_1^2 + \cdots + x_n^2 = 1$ with the hypersurface $z = (1/2).\{k_1(p).x_1^2 + \cdots + k_n(p).x_n^2\}$ which is *the quadratic Taylor–Maclaurin approximation of* M^n *in* E^{n+1} *at* p, whereby reference is made to *a rectangular Cartesion co–ordinate system* $(x_1, \ldots, x_n, z)$ which has its *origin* O at p, of which the nD hyperplane perpendicular to the z*–axis* is the tangent hyperplane T_pM^n in which the x_1*–axis*, $\ldots$, x_n*–axis* are choosen in the principal directions $f_1 = F_1(p)$, $\ldots$, $f_n = F_n(p)$ of M^n in E^{n+1} at p.

7. In the modern approach to submanifold theory, say for *submanifolds* M^n *in ambient Euclidean spaces* E^{n+m}, (and, basically, likewise for submanifolds M^n in arbitrary Riemannian manifolds $\tilde{M}^{n+m}$ as ambient spaces), one proceeds as follows [2, 13]. *Tangent vector fields* being denoted by $X, Y, \ldots$ and *normal vector fields* being denoted by $\xi, \eta, \ldots$, *the formula's of Gauss and of Weingarten*, $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi$, give *the canonical decomposition* of the Euclidean vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$ along M^n in E^{n+m}, whereby $\tilde{\nabla}$ is *the directional derivative in the ambient space, into their tangential and normal components*, hereby ∇ being *the Riemannian connection of the submanifold* M^n, h being *the normal vector valued second fundamental form*, A_ξ being *the shape operator or the Weingarten map of the submanifold with respect to the normal vector field* ξ and $\nabla^\perp$ being *the normal connection* of M^n in E^{n+m}.

In case of hypersurfaces M^n in E^{n+1}, *the principal curvatures* $k_1, \ldots, k_n$ and *the tangential principal directions* F_1, $\ldots$, F_n are *the eigenvalues* and *eigendirections of the* only *shape operator* $A = A_\xi$ (working with unit normal vector fields ξ) that one basically (up to sign) has to deal with in this case, such that $A(F_1) = k_1F_1, \ldots, A(F_n) = k_nF_n$. However, in 1869, when Kronecker generalised the Euler–Dupin–Meusnier theory from surfaces M^2 in E^3 to hypersurfaces M^n in E^{n+1} of arbitrary dimensions $n \geq 2$ [14], this "step" from $n = 2$ to any dimension > 2 was pretty much more involved. And, even much more so, was the extension of this theory to submanifolds M^n in E^{n+m} with arbitrary dimensions $n \geq 2$ and of arbitrary co–dimensions $m \geq 1$. Yet, no later than in 1874, C. Jordan did this "job" [15]. To begin with, in order to develop his general Frenet theory for curves, Jordan already had defined a suitable notion of *the angle between two arbitrary dimensional affine subspaces of arbitrary dimensional Euclidean spaces.* Then, he considered for any submanifold M^n in E^{n+m} *the intuitively most natural curvature* of such submanifold M^n at any of its points p in any tangent direction $u \in T_pM^n$, $\|u\| = 1$, that is $(d\varphi_u/ds)^2(0)$, whereby $\varphi_u \in [0, \pi/2]$ is *the angle between the tangential* nD *spaces at the point* p *and at a nearby point of the submanifold*, say $p+dp$, in the tangent direction u of M^n at p, s being *an arclength parameter* of a curve γ on M^n from $p = \gamma(0)$ going in the direction $u = \gamma'(0)$ to $p+dp = \gamma(s+ds)$, being determined by $\cos^2 \varphi_u(ds) = (det\, M)^2$ whereby M is the $n \times n$ matrix with general element $M_{ij} = v^i_1.w^j_1 + v^i_2.w^j_2 + \cdots + v^i_n.w^j_n$, the vectors $(v^i_1, v^i_2, \ldots, v^i_n)$ and $(w^j_1, w^j_2, \ldots, w^j_n)$ forming *arbitrary orthonormal bases* of T_pM^n and of $T_{p+dp}M^n$, respectively, $(i, j \in \{1, 2, \ldots, n\})$, (by which definition, in particular, all eventually possible confusions related to various choices of orientations are avoided; for further references in this respect, see e.g. [16]); cfr. Figure 7. And, he defined *the critical values of the function* $S_p^{n-1}(1) = \{u \in T_pM | \, \|u\| = 1\} \to \mathbb{R} : u \mapsto c(u) = (d\varphi_u/ds)^2(0)$ as *the principal curvatures* $c_1(p) \geq \cdots \geq c_n(p) \geq 0$ *of* M^n *in* E^{n+m} *at* p, and the tangential directions of M^n in E^{n+m} at p in which these values are reached as *the principal tangential directions of* M^n *in* E^{n+m} *at* p, say determined by *unit vectors* $f_1, \ldots, f_n \in T_pM$.

As the first step in his original general study of *the theory of submanifolds of arbitrary dimensions and co–dimensions*, Kostadin Trenčevski re–obtained this same result in the 19nineties, (cfr. [4][5][6][7]), and, in [16], basically yet again this same result was re–obtained, now following the 1890 Casorati views on the most intuitive scalar valued curvature of surfaces M^2 in E^3, i.e. now rather measuring $(d\psi_u/ds)^2(0)$ whereby ψ_u is *the angle between the normal mD spaces of the submanifold* M^n *in* E^{n+m} *at* p *and at a nearby point* $p + dp$ *on* M^n *in a direction* u; cfr. Figure 8, (and by the way, Jordan had also already proved that -in the above notations- $\psi_u = \varphi_u$).

Actually, as first shown by Trenčevski, these *extrinsic principal tangential directions* $f_1, \ldots, f_n$ at a point p of a submanifold M^n in E^{n+m} and their *corresponding principal curvatures* $c_1(p) \geq \cdots \geq c_n(p)$ essentially turn out to be *the mutually orthogonal eigendirections* and *the corresponding eigenvalues of the symmetric* linear Casorati operator $A^C = \sum_\alpha A_\alpha^2$ at p, (where we have put $A_\alpha = A_{\xi_\alpha}$, $\{\xi_1, \ldots, \xi_m\}$ being any *orthonormal normal local frame field* on M^n in E^{n+m}, $\alpha \in \{1, \ldots, m\}$): $A^C(F_i) = c_i F_i$, $f_i = F_i(p)$. In the particular case of hypersurfaces M^n in E^{n+1}, then having $A^C = A^2$, it follows that *the tangential Casorati principal curvatures* c_i and the Euler principal curvatures k_i are related by $c_1 = k_1^2, \ldots, c_n = k_n^2$ (possibly up to the ordering of these curvatures, which could eventually change depending on the signs of the Euler curvatures) and that *the tangential Casorati principal directions* and the Euler principal directions are the same.

For submanifolds M^n of arbitrary dimensions n and co–dimensions m in E^{n+m}, *the tangential Casorati curvature* $c(u) = (d\varphi_u/ds)^2(0) = (d\psi_u/ds)^2(0)$ *at any point* p *of* M^n *and in any arbitrary direction* $u = f_1.\cos\theta_1 + \cdots + f_n.\cos\theta_n, \theta_i = \measuredangle(f_i, u)$, is given by a formula similar to Euler's one in the case of the normal curvatures of hypersurfaces, namely, $c(u) = c_1(p).\cos^2\theta_1 + \cdots + c_n(p).\cos^2\theta_n$. And, in analogy with the above Euler indicatrix of (hyper)surfaces, *the tangential Casorati indicatrix* $\mathcal{C}^T$ of a submanifold M^n in E^{n+m} at a point p is defined as *the intersection of the quadratic hypersurface* with equation $z = c_1(p).x_1^2 + \cdots + c_n(p).x_n^2$, (*an elliptical paraboloid* -when $c_n(p) > 0$- or *a hypercylinder based on an elliptical paraboloid in some* E^{k+1} *in* E^{n+1} -1 $\leq k < n$ being the number of strictly positive Casorati curvatures $c_i(p)$- or *the hyperplane* $z = 0$ -when all $c_i(p)$'s are zero, i. e. when p is a totally geodesic point on M^n in E^{n+m}-) with the *"vertical unit hypercylinder"* with equation $x_1^2 + \cdots + x_n^2 = 1$, whereby reference is made to a rectangular co–ordinate system $(x_1, \ldots, x_n, z)$ which has its *origin* O at p, of which the nD hyperplane perpendicular to the z*–axis* is the nD tangent plane T_pM^n in which the x_1*–axis*, $\ldots, x_n$*–axis* are choosen in the tangential Casorati principal directions of M^n in E^{n+m}

at p; in any tangential direction $u = \sum_i f_i . \cos\theta_i$, the z–co–ordinate or height of this intersection, according to the above formula, equals $c(u)$, thus indicating "how much or how little" the submanifold M^n in E^{n+m} is curved at p in the tangential direction u in accordance with our common sense, (cfr. Figure 9).

The tangential Casorati principal directions of a submanifold M^n in E^{n+m}, or in any ambient Riemannian space $\tilde{M}^{n+m}$ for that matter, *from the extrinsic point of view*, likely are its most distinguished tangential directions, whereas, *from the intrinsic point of view*, i.e. when focusing on the Riemannian geometry of such a submanifold, *the Ricci principal directions*, i.e. the eigendirections of its Ricci curvature operator, likely are its most distinguished tangential directions. So, it certainly is of natural interest to consider which submanifolds M^n do assume the kind of special shapes in their ambient spaces $\tilde{M}^{n+m}$ such that their corresponding extrinsic Casorati principal directions actually *do coincide* with their intrinsically fixed Ricci principal directions. Since by contraction of the Gauss equation for submanifolds M^n in real space forms $\tilde{M}^{n+m}(c)$ of constant sectional curvature c it follows that $Ric(X,Y) = (n-1)c.g(X,Y) + g(A_{n\vec{H}}(X),Y) - g(A^C(X),Y)$, obviously this in particular is the case for all submanifolds M^n in $\tilde{M}^{n+m}(c)$ which are *minimal* or *pseudo–umbilical* or have *flat normal connection*, and, for the non–minimal submanifolds M^n in $\tilde{M}^{n+m}(c)$ for which this is the case, these common Casorati and Ricci principal directions automatically also are *the principal directions of the mean curvature normal vector field* $\vec{H} = (1/n).tr\, h = (1/n)\sum_\alpha (tr\, A_\alpha)\xi_\alpha$. Further, referring to [17][18], and, in some sense not so surprisingly, *Wintgen ideal submanifolds* and *the first Chen ideal submanifolds*, i. e. *the* $\delta(2)$*–ideal submanifolds*, turn out to enjoy this basic property of submanifolds to have coinciding Casorati and Ricci principal directions.

8. Now follows a reminder of the contribution to the geometry of surfaces M^2 in E^3 that was made by Casorati around 1890, of which one basic ingredient was already discussed in the previous section, but of which the main point, i. e. the geometrical meaning itself of *"the Casorati curvature as such* of surfaces M^2 in E^3", has uptill now remained pretty much ignored, although this curvature already for quite some time has been known to be of great importance in the geometry of submanifolds (cfr. e.g. and a.o. [13][19]) and although this curvature more recently has been shown to be of importance as well in the applications of geometry in the natural sciences and in technology (cfr. e.g. [8][20][21]). Actually, the historical article of Vincensini is, as far as I know, one of the rare texts which at least does mention this curvature, let it be as follows: (after having motivated that *meaningful single scalar valued curvature quantities on a surface* M^2 *in* E^3 *basically should be expressions of the two principal curvatures* k_1 *and* k_2, Vincensini continues like this), "C'est ainsi qu'ont été successivement proposées les expressions: $k_1.k_2$, $k_1 + k_2$, $k_1^2 + k_2^2$, la première desquelles, due à Gauss et désignée sous le nom de courbure totale (...), s'est révélée comme l'une des notions les plus importantes (...) de la géométrie différentielle. La deuxième expression a été introduite par Sophie Germain sous le nom de courbure moyenne. (...), la courbure moyenne n'en est pas moins présidé au développements de théories importantes. Telle par exemple la théorie des surfaces à courbure moyenne partout nulle (ou surfaces minima). La troisième expression, $k_1^2 + k_2^2$, proposée par Casorati, bien que susceptible d'intervenir ultimement dans l'étude de certains problèmes particuliers, n'a pas réussi à prendre place dans la littérature mathématique.". The cause for the serious significance of this curvature in the geometry of submanifolds and in the applications of this geometry, of course, at least in my opinion, completely lies in *the geometrical meaning of*, in the words of Casorati, *"la mesure de la courbure des surfaces suivant l'idée commune"*, which will next be recalled basing on [16] and [22].

To determine *the Casorati curvature (as such)* $C(p)$ *of a surface* M^2 *in* E^3 *at one of its points* p, first of all, consider a small geodesic circle $\gamma_{\Delta\rho}$ on M^2 centered at p with radius $\Delta\rho$; (cfr. Figure 10). Let q be any point on $\gamma_{\Delta\rho}$ and consider the geodesic δ, parametrised by arclength, such that $p = \delta(0)$ and $q = \delta(\Delta\rho)$ and which at p points in the tangent direction $u = \delta'(0)$ to M^2 at p. Let $\eta(p)$ and $\eta(q)$ be *the unit normals* on the surface M^2 in E^3 *at* p *and* q, respectively, corresponding to a choice of local unit normal vector field η on M^2 in E^3 around p. Then, according to our intuition, *the angle* $\Delta\psi_u$ *between* $\eta(p)$ *and* $\eta(q)$ *measures how much the surface* M^2 *at* p *curves in the direction* u: the more the surface thus curves, the larger this angle. Let r be the point on the geodesic δ *at a distance* $\Delta\psi_u$ from p in the direction u, i. e. let r be the point $r = \delta(\Delta\psi_u)$. Joining all such points r for all points q on $\gamma_{\Delta\rho}$, "around p" (actually passing through p whenever in some direction u the surface M^2 is not curved at all in E^3 at p) there results a curve $\Gamma_{\Delta\rho}$. And, clearly, *the larger the area enclosed by this curve* $\Gamma_{\Delta\rho}$ *the more the surface* M^2 *is curved in* E^3 *around* p, and, accordingly and following the idea's within the definitions of Gauss and Germain of their curvatures via the ratio of the area's of a region around p, in Casorati's case the geodesic disc on M^2 centered at p and of radius $\Delta\rho$, and a corresponding region on some surface, (in the case of the Gauss curvature $K(p)$: the spherical image, and in the case of the mean curvature $H(p)$ of Germain: the region on a cylinder perpendicular to a small circle around p in T_pM^2 between T_pM^2 and the surface itself -and, in this case, the ratio is taken with the area of the disc around p in T_pM^2 bounded by this small circle-), *Casorati defined his curvature as* $C(p) = \lim_{\Delta\rho\to 0}\{A(\Gamma_{\Delta\rho})/A(\gamma_{\Delta\rho})\}$ whereby $A(\Gamma_{\Delta\rho})$ and $A(\gamma_{\Delta\rho})$ stand for *the area of the regions on* M^2 *which are enclosed by the curves* $\Gamma_{\Delta\rho}$ and $\gamma_{\Delta\rho}$, respectively, *and he proved that* $C = (1/2).(k_1^2 + k_2^2) = (1/2).tr\,A^2 = (1/2).\,\|h\|^2$; cfr. Figure 10. The following are two quotes from Casorati's paper [22]: (i) "Cette espèce de prééminence que je donne par là à C, comme mesure de courbure, me parait justifiée par plusieurs motifs dont les suivants se présentent immédiatement à l'esprit. C est, (...), une traduction de l'idée commune de courbure d'une surface plus fidéle que K et H. C caractérise par sa valeur zéro le manque total de courbure, de même que la première des deux courbures des lignes (-i.e. of curves in E^3-; L. V.), que l'on a

déjà l'habitude de nommer tout simplement courbure. Avec cette signifaction du mot courbure on peut dire:
Si la courbure est nulle en tout point, la surface est plane (la ligne est droite).
Il n'y a que la surface plane (resp. la ligne droite) dont la courbure soit nulle en tout point.", and, (ii) "Je ne crois pas inutile, (...), de recommander aux jeunes mathématiciens les recherches que suscite tout naturellement la considération de la nouvelle mesure C, et d'exhorter les auteurs de traités, particulièrement d'Analyse et de Géométrie infinitésimale, à lui accorder une place dans leurs livres.", which points could hardly be phrased better today.

The above considerations made by Casorati for surfaces M^2 in E^3 straightforwardly can be taken over to general submanifolds M^n in E^{n+m}, (and also to general submanifolds M^n in arbitrary ambient Riemannian spaces $\tilde{M}^{n+m}$, in the latter situation making use of the Riemannian connection $\tilde{\nabla}$ of $\tilde{M}^{n+m}$ to compare the positions of $T_q^{\perp}M^n$ and $T_p^{\perp}M^n$, i.e. by then measuring the angles at q between the mD normal space $T_q^{\perp}M^n$ at q and the mD subspace $(T_p^{\perp}M^n)^*$ of $T_q\tilde{M}^{n+m}$ which is obtained by $\tilde{\nabla}$–parallely transporting $T_p^{\perp}M^n$ from p to q along δ), as was done in [16]. In particular, from [16], it may be well to recall that *the Casorati curvature (as such) C equals the arithmetic mean of the tangential Casorati curvatures* $c_1, \ldots, c_n$; $C = (1/n).\|h\|^2 = (1/n).tr\, A^C = (1/n). \sum_\alpha tr\, A_\alpha^2 = (1/n). \sum_i c_i$. At this stage it could further be observed that $C_\alpha(p) = (1/n).\, tr\, A_\alpha^2(p)$ is *the Casorati curvature (as such) at p of the projection M_α^n of the submanifold M^n of E^{n+m} onto the $(n+1)$–dimensional subspace E^{n+1} of the ambient space which is spanned by $T_pM^n = \mathbb{R}^n$ together with the normal line $[\xi_\alpha(p)]$ determined by the unit normal vector $\xi_\alpha(p)$*, and, hence, that $C_\alpha(p) = (1/n). \sum_i c_{\alpha i}(p)$, i.e. that $C_\alpha(p)$ is the arithmetic mean of the tangential Casorati curvatures of this hypersurface M_α^n in E^{n+1} at p. In this context, these positive real functions C_α on a submanifold M^n in E^{n+m} will be called normal Casorati curvatures of M^n in E^{n+m}; more precisely: *the normal Casorati curvature of M^n in E^{n+m} in the normal direction determined by a unit normal vector field η* is defined as $C_\eta = (1/n).\, tr\, A_\eta^2$.

9. Let $N1$ be *the first normal space* of M^n in E^{n+m}, i.e. $N1 = im\, h = \{h(X,Y)\,|X,Y \in TM^n\}$, or, still, $N1$ is the orthogonal complement in the normal space $T^{\perp}M$ of M^n in E^{n+m} of the subspace of all normals with vanishing shape operators, or, still, with vanishing corresponding normal Casorati curvatures: $N1 = \{\xi \in T^{\perp}M\,|A_\xi = 0\}^{\perp} = \{\xi \in T^{\perp}M\,|C_\xi = 0\}^{\perp}$, such that $TM^n \oplus N1$ is *the first* osculating space of M^n in E^{n+m}; (cfr. Figure 11). Recently, Trenčevski made an original general study of *the osculating spaces of all orders* for general submanifolds M^n in E^{n+m}, and, in particular obtained the values of their maximal dimensions; moreover, *in each of the successive normal spaces of all possible orders he determined appropriate orthonormal frames of principal normal vector fields and corresponding principal normal curvatures* [4][5][6][7]. In subsequent papers we will return to what next follows for $N1$ then also thus carrying on for the second, third, etc. normal spaces $N2, N3$, etc., hereby specifying the relationships of the successive principal normal vector fields and normal principal curvatures with *the classical curvature varieties* that have been studied in the geometry of submanifolds, going back in particular to Kommerell's 1897 study of surfaces M^2 in E^4 [23], (herefore, especially, cfr. B. Rouxel's mémoire of the Académie royale belge [24] for its mathematical contents as well as for its "indications historiques").

The *factual dimensions of the successive normal spaces* $N1, N2, N3, \dots$ *of a submanifold* M^n *in* E^{n+m} will be denoted by $m1, m2, m3, \dots$ and besides the general normal indices $\alpha \in \{1, \dots, m\}$ we will use normal indices $\alpha 1 \in \{1, \dots, m1\}$ for vector fields $\xi_{\alpha 1} \dots$ in the first normal space $N1$, and similarly for the other normal spaces. In the "full" normal space $T^{\perp}M^n$ of an *n–dimensional submanifold* M^n *with co–dimension* m in a Euclidean space E^{n+m}, consider the following *operator* $a1 : T^{\perp}M \to T^{\perp}M : \xi \mapsto a1(\xi) = (1/n). \sum_{\alpha} tr\,(A_{\xi}A_{\alpha}).\xi_{\alpha}$ which is *a symmetric linear map*; in [14], B.-Y. Chen defined the *allied normal vector field of* ξ by $a(\xi) = (1/n). \sum_{\gamma} tr\,(A_{\xi}A_{\gamma}).\xi_{\gamma}$, whereby $\{\xi_{\gamma}\}$, $(\gamma = 2, 3, \dots, m)$, together with ξ forms an orthogonal frame of $T^{\perp}M$, and in particular initiated the study of the submanifolds for which the allied mean curvature vector field $a(\vec{H})$ vanishes identically, which he called $\mathcal{A}$–submanifolds and which later also were called *Chen submanifolds*, (cfr. a.o. [24][25][26][27][28]). By *the Principal Axes Theorem*, there exists *an orthonormal normal frame field* $\eta_1, \dots, \eta_{m1}, \xi_{m1+1}, \dots, \xi_m$ *of eigenvector fields* for $a1$ with corresponding eigenvalues $C_1 = (1/n).\, trA_1^2 \geq \dots \geq C_{m1} = (1/n).\, trA_{m1}^2 > C_{m1+1} = (1/n).\, trA_{m1+1}^2 = ... = C_m = (1/n).\, trA_m^2 = 0$. Clearly $\eta_1, \dots, \eta_{m1}$ *span the first normal space* $N1 = im\, h$ of M^n in E^{n+m} and following Trenčevski $\eta_1, \dots, \eta_{m1}$ are called *the first principal normal vector fields* and $C_1, \dots, C_{m1}$ are called *the first principal normal curvatures of* M^n *in* E^{n+m}; $a1(\eta_{\alpha 1}) = C_{\alpha 1}.\eta_{\alpha 1}$, whereby $C_{\alpha 1} = (1/n).\, tr\, A_{\alpha 1}^2$ is *the normal Casorati curvature of* M^n *in* E^{n+m} *in the normal direction* $\eta_{\alpha 1}$, as defined in the previous Section. Thus we have the following.

Theorem 1. *The first principal normal vector fields of a submanifold* M^n *in* E^{n+m} *determine the normal directions on* M^n *in* E^{n+m} *in which the normal Casorati curvatures attain their* $m1 = dim\, N1$ *non–zero critical values.* □

And, in this setting, a result of J. Weiner and P. Verheyen (cfr. [26(a)]) could now be reformulated as follows.

Theorem 2. *A non–minimal submanifold* M^n *in* E^{n+m} *is a Chen submanifold if and only if its mean curvature vector field determines a first principal normal direction.* □

The intersection of *the "vertical" unit hypercylinder* $z_1^2 + \dots + z_{m1}^2 = 1$ with *the elliptical paraboloid* $z = C_1(p).z_1^2 + \dots + C_{m1}(p).z_{m1}^2$ in the space $E^{m1+1} = N1(p) \oplus \mathbb{R}$, whereby reference is made to a rectangular Cartesian co–ordinate system $(z_1, \dots, z_{m1}, z)$ which is choosen such that a point p of M^n is the *origin* O, the z_1–*axis*$, \dots, z_{m1}$–*axis* are determined by the first principal normal directions in the first normal space $N1(p)$ of M^n in E^{n+m} at p and the z–*axis* is perpendicular to $N1(p)$ at p, will be called *the first normal Casorati curvature indicatrix* $\mathcal{C}^{\perp}$ *of the submanifold* M^n *in* E^{n+m} *at the point* p. *For any unit normal vector field* η *lying in* $N1$, $\eta =$

$\sum_{\alpha 1} \eta_{\alpha 1} . \cos\theta_{\alpha 1}$, $\theta_{\alpha 1} = \measuredangle(\eta_{\alpha 1}, \eta)$, *the corresponding normal Casorati curvature* $C_\eta(p)$ *of* M^n *in* E^{n+m} *at* p is given by *the height of this indicatrix above the point* η *on the unit hypersphere* $z_1^2 + \cdots + z_{m1}^2 = 1$ *in* $E^{m1} = N1$ *which is centered at* $O = p$: $C_\eta(p) = \sum_{\alpha 1} C_{\alpha 1}(p). \cos^2\theta_{\alpha 1}$; (cfr. Figure 12). And, moreover, and similar as in the situation of the squared curvature of curves $\Gamma = M^1$ in E^{1+m} as mentioned explicitely in Section 3 for the case $m = 2$, but clearly goes through for all m, now *the (total) Casorati curvature (as such)*, $C(p) = (1/n). \|h\|^2(p)$, of a submanifold M^n in E^{n+m} at some point p of M^n is given by *the sum of the normal Casorati curvatures at* p *of the* m nD *hypersurfaces* M^n_α *in* E^{n+1} *which are the projections of the original submanifold* M^n *in* E^{n+m} *onto the* m *Euclidean subspaces* E^{n+1} *of* E^{n+m} *which are spanned by* $T_pM^n = E^n$ *together with, respectively, each of the normal lines* $\mathbb{R} = [\xi_\alpha(p)]$ *through* p *which are generated by each of the normals* $\xi_\alpha(p)$, whereby $\{\xi_\alpha\}$ is any local orthonormal normal frame field on M^n in E^{n+m} around p, since $C = (1/n). \|h\|^2 = (1/n). \sum_\alpha tr\, A_\alpha^2$. In particular, $C(p)$ is the sum of the normal Casorati curvatures at p of the $m1$ projections $M^n_{\alpha 1}$ of M^n onto $E^{n+1} = T_pM^n \oplus [\eta_{\alpha 1}(p)]$, whereby $[\eta_{\alpha 1}(p)]$ is the line in E^{n+m} through p which is generated by the first principal normal vector $\eta_{\alpha 1}(p)$ of M^n in E^{n+m} at p; $C = \sum_{\alpha 1} C_{\alpha 1}$.

10.

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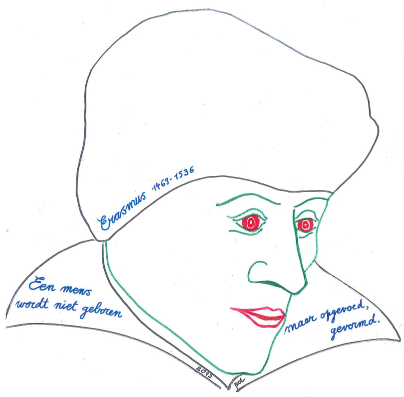
Illustrations

1.



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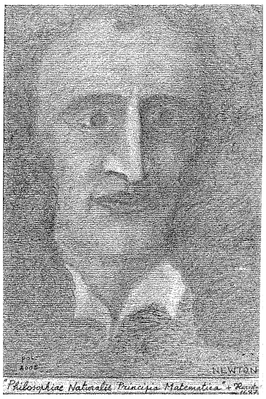

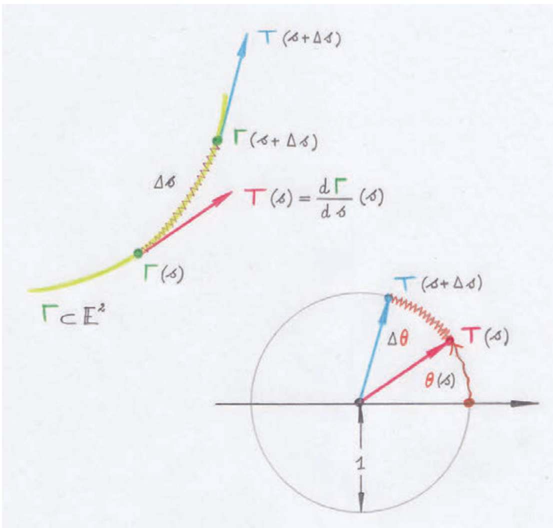

Figure 1

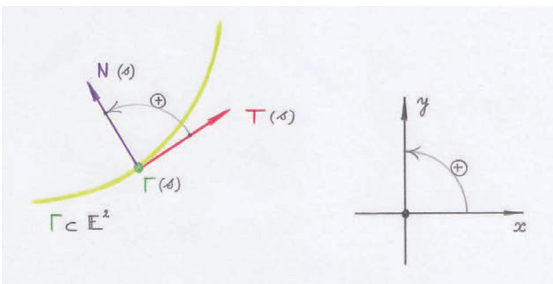

Figure 2

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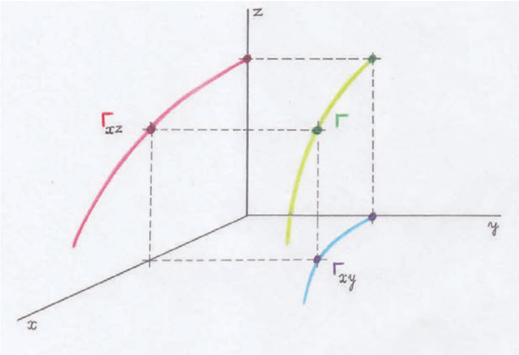


Figure 3

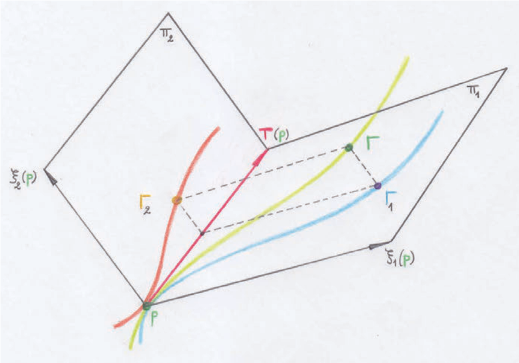


Figure 4

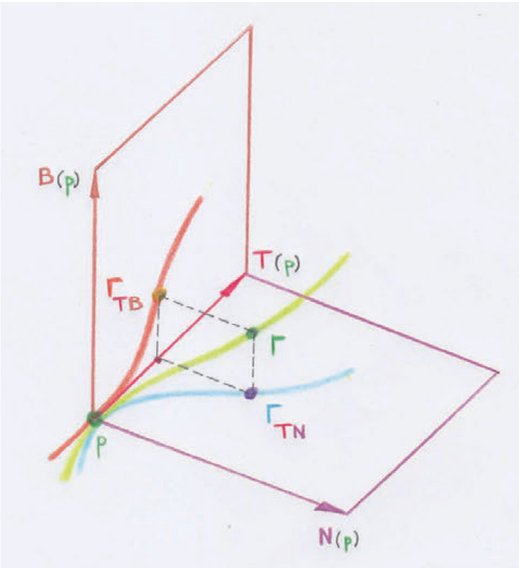
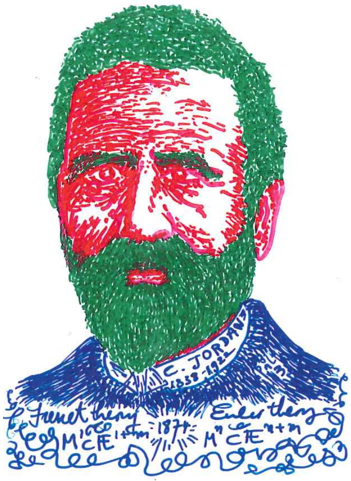


Figure 5

4.


C. JORDAN

5.

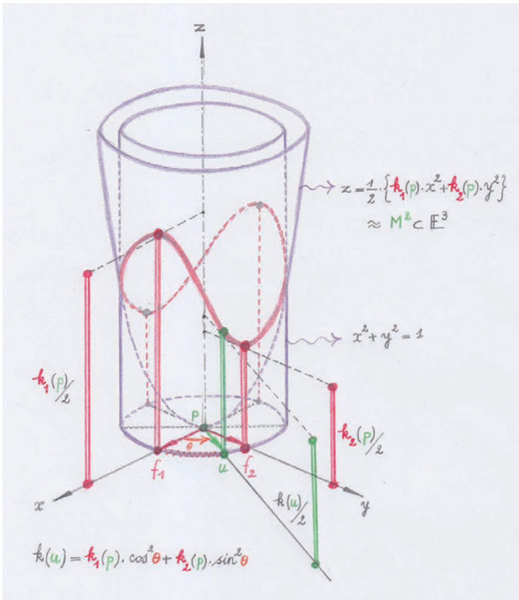

z
z = 1/2 · {k1(p)·x^2 + k2(p)·y^2}
≈ M^2 ⊂ E^3
x^2 + y^2 = 1
k1(p)/2
k2(p)/2
p
f1
u
f2
x
y
k(u)/2
k(u) = k1(p)·cos^2θ + k2(p)·sin^2θ

Figure 6

7.

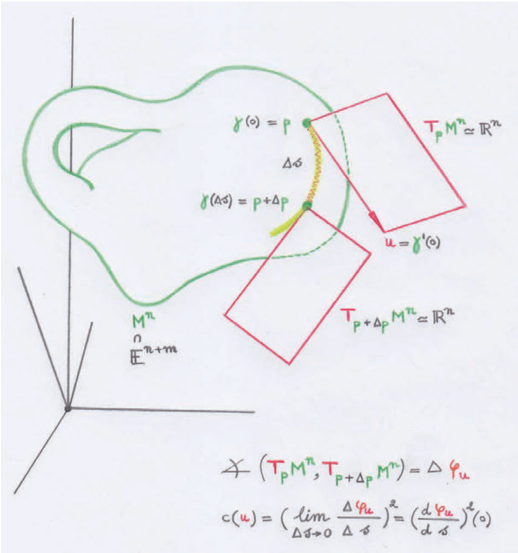


Figure 7

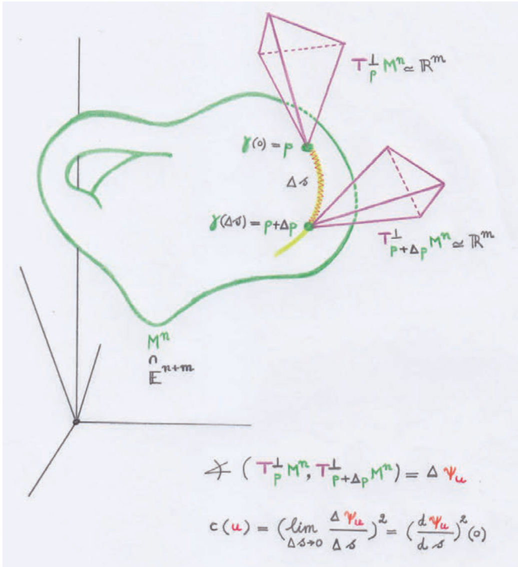


Figure 8

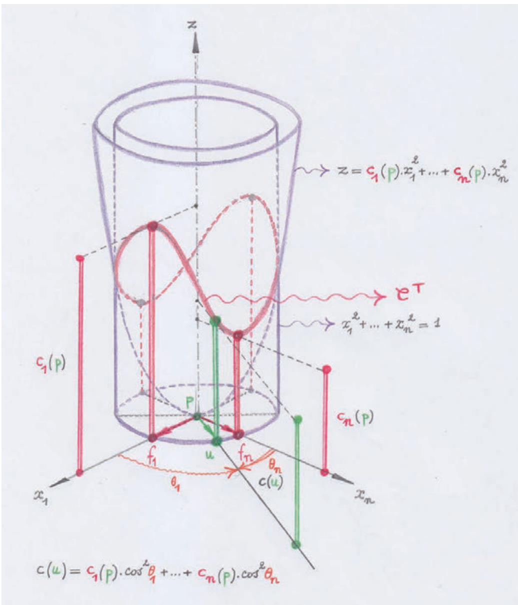
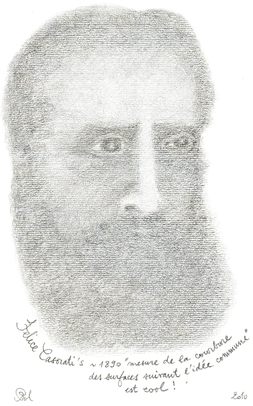


Figure 9

8.



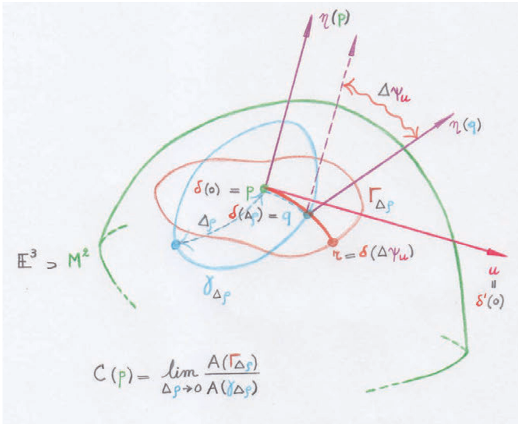


Figure 10

9.

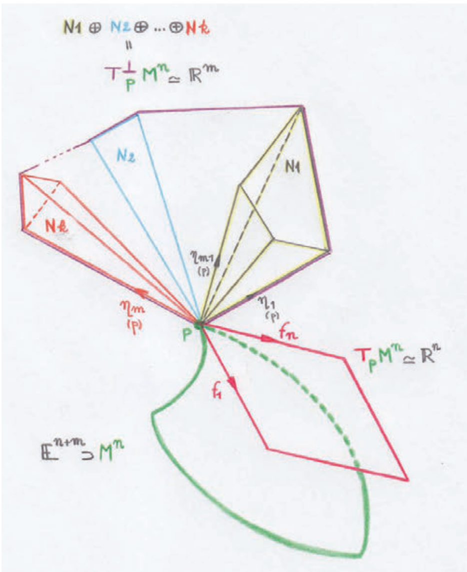


Figure 11

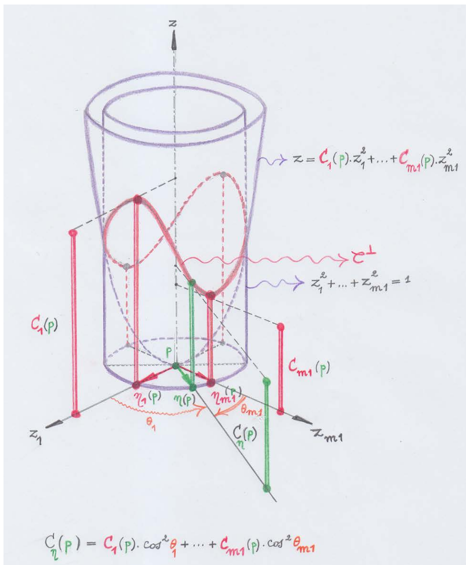


Figure 12

KU Leuven, Section of Geometry, Belgium.