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UPPER SIGNED TOTAL DOMINATION NUMBER OF DIRECTED GRAPHS

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Abstract. Let $D = (V, A)$ be a finite simple digraph in which $d_D^-(v) \geq 1$ for all $v \in V$. A function $f : V \longrightarrow \{-1, 1\}$ is called a signed total dominating function (STDF) if $\sum_{u \in N^-(v)} f(u) \geq 1$ for each vertex $v \in V$. A STDF f of a digraph D is minimal if there is no STDF $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The maximum value of $\sum_{v \in V} f(v)$, taken over all minimal signed total dominating functions f, is called the *upper signed total domination number* $\Gamma_t^s(D)$. In this paper, we present a sharp upper bound for $\Gamma_t^s(D)$.

1. Introduction

In this paper, D is a finite simple digraph with vertex set $V(D) = V$ and arc set $A(G) = A$. The *order* $n(D) = n$ of a digraph D is the number of its vertices. We write $d_D^+(v) = d^+(v)$ for the outdegree of a vertex v and $d_D^-(v) = d^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex $v \in V$, let $N_D^-(v) = N^-(v)$ be the set consisting of all vertices of D from which arcs go into v. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v and $d_X^-(v) = |E(X, v)|$. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Consult [5] for the notation and terminology which are not defined here.

Let D be a digraph such that $\delta^-(D) \geq 1$. A *signed total dominating function* (abbreviated STDF) of D is a function $f : V \to \{-1, 1\}$ such that $f[v] = f(N^-(v)) \geq 1$

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for every $v \in V$. A STDF f of a digraph D is minimal if there is no STDF $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The maximum value of $\sum_{v \in V} f(v)$, taken over all minimal signed total dominating functions f, is called the *upper signed total domination number* $\Gamma_t^s(D)$. The concept of the signed total dominating function of digraphs was introduced by Sheikholeslami [4].

The concept of the upper signed total domination number $\Gamma_t^s(G)$ of undirected graphs G was introduced by Henning [2] and has been studied by several authors (see for example [1, 3]).

In this paper, we present an upper bound on the upper signed total domination number of digraphs. We make use of the following observations.

Observation 1.1. *For any digraph D of order $n \geq 2$ with minimum indegree $\delta^-(D) \geq 1$, $\Gamma_t^s(D) \leq n$ and $\Gamma_t^s(D) \equiv n \pmod 2$.*

Proof. Let f be a minimal signed total dominating function of D with $f(V(D)) = \Gamma_t^s(D)$ and let $P = \{v \in V \mid f(v) = 1\}$ and $M = \{v \in V \mid f(v) = -1\}$. Then clearly $\Gamma_t^s(D) = |P| - |M|$ and $n = |P| + |M|$. Thus $\Gamma_t^s(D) = n - 2|M|$ implying that $\Gamma_t^s(D) \leq n$ and $\Gamma_t^s(D) \equiv n \pmod 2$ as desired. □

Observation 1.2. *A STDF f of a digraph D is minimal if and only if for every $v \in V$ with $f(v) = 1$, there exists at least one vertex $u \in N^+(v)$ such that $f[u] = 1$ or 2.*

Proof. Let f be a minimal signed total dominating function of D. Suppose to the contrary that there exists a vertex $v \in V(D)$ such that $f(v) = 1$ and $f[u] \geq 3$ for each $u \in N^+(v)$. Then the mapping $g : V(D) \to \{-1, 1\}$ defined by $g(v) = -1$ and $g(x) = f(x)$ for $x \in V(D) - \{v\}$, is clearly a STDF of D such that $g \neq f$ and $g(u) \leq f(u)$ for each $u \in V(D)$, a contradiction.

Conversely, let f be a signed total dominating function of D such that for every $v \in V$ with $f(v) = 1$, there exists at least one vertex $u \in N^+(v)$ such that $f[u] = 1$ or 2. Suppose to the contrary that f is not minimal. Then there is a STDF g of D such that $g \neq f$ and $g(u) \leq f(u)$ for each $u \in V(D)$. Since $g \neq f$, there is a vertex $v \in V$ such that $g(v) < f(v)$. Then $g(v) = -1$ and $f(v) = 1$ because $f(v), g(v) \in \{-1, 1\}$. Since g is a STDF of D, $g[u] \geq 1$ for each $u \in N^+(v)$. It follows that $f[u] \geq g[u] + 2 \geq 3$ for each $u \in N^+(v)$ which is a contradiction. This completes the proof. □

2. An upper bound for $\Gamma_t^s(D)$

Theorem 2.1. *Let D be a digraph of order n with minimum indegree $\delta^- \geq 1$ and maximum indegree Δ^-. Then*

$$\Gamma_t^s(D) \leq \begin{cases} \dfrac{\Delta^-(\delta^- + 5) - \delta^- + 1}{\Delta^-(\delta^- + 5) + \delta^- - 1} n, & \text{if} \quad \delta^- \text{ is odd,} \\ \dfrac{\Delta^-(\delta^- + 4) - \delta^- + 2}{\Delta^-(\delta^- + 4) + \delta^- - 2} n, & \text{if} \quad \delta^- \text{ is even.} \end{cases}$$

Proof. If $\delta^- = 1$ or 2, then the result is true by Observation 1.1. Let $\delta^- \geq 3$ and let f be a minimal STDF such that $\Gamma_t^s(D) = f(V(D))$. Suppose that $P = \{v \in V(D) \mid f(v) = 1\}$, $M = \{v \in V(D) \mid f(v) = -1\}$, $p = |P|$ and $q = |M|$. Then $\Gamma_t^s(D) = f(V) = |P| - |M| = p - q = n - 2q$.

Since f is a STDF,

$$(d^-(v) - d_M^-(v)) - d_M^-(v) \geq 1$$

for each $v \in V(D)$. It follows that $d_M^-(v) \leq \frac{\Delta^- - 1}{2}$ when $v \in V(D)$. Define $A_i = \{v \in P \mid d_M^-(v) = i\}$, $a_i = |A_i|$, $B_i = \{v \in M \mid d_M^-(v) = i\}$ and $b_i = |B_i|$ for each $0 \leq i \leq \lfloor \frac{\Delta^- - 1}{2} \rfloor$. Then the sets $A_0, A_1, \ldots, A_{\lfloor (\Delta^- - 1)/2 \rfloor}$ form a partition of P and $B_0, B_1, \ldots, B_{\lfloor (\Delta^- - 1)/2 \rfloor}$ form a partition of M.

Since f is a minimal STDF, it follows from Observation 1.2 that for every $v \in P$, there is at least one vertex $u_v \in N^+(v)$ such that $f[u_v] \in \{1, 2\}$. For each $v \in A_0$, since v has no in-neighbor in M,

$$f[v] = d^-(v) \geq \delta^- \geq 3.$$

Therefore $u_v \notin A_0$ for each $v \in P$.

Let $T = \{u \mid u \in N^+(A_0) \text{ and } f[u] = 1 \text{ or } 2\}$. If $0 \leq i \leq \lfloor \frac{\delta^- - 3}{2} \rfloor$ and $v \in A_i \cup B_i$, then we have $f[v] = d^-(v) - 2i \geq 3$. This implies that

$$T \subseteq \bigcup_{\lfloor (\delta^- - 1)/2 \rfloor}^{\lfloor (\Delta^- - 1)/2 \rfloor} (A_i \cup B_i).$$

If $\lfloor \frac{\delta^- - 1}{2} \rfloor \leq i \leq \lfloor \frac{\Delta^- - 1}{2} \rfloor$ and $v \in T \cap (A_i \cup B_i)$, then $d^-(v) - 2i = f[v] = 1$ or 2 which implies that $d^-(v) = 2i + 1$ or $2i + 2$. Hence each $v \in T \cap (A_i \cup B_i)$ has at most $i + 2$ in-neighbor in A_0 and so $T \cap (A_i \cup B_i)$, has at most $(i + 2)(|T \cap A_i| + |T \cap B_i|)$ in-neighbors in A_0.

Since f is a minimal STDF of D, it follows from Observation 1.2 that $N^+(v) \cap T \neq \emptyset$ for every $v \in A_0$. Note that

$$A_0 \subseteq \bigcup_{\lfloor (\delta^- - 1)/2 \rfloor}^{\lfloor (\Delta^- - 1)/2 \rfloor} N^-(T \cap (A_i \cup B_i)).$$

Thus

$$\begin{aligned} a_0 &\leq |\bigcup_{\lfloor (\delta^- - 1)/2 \rfloor}^{\lfloor (\Delta^- - 1)/2 \rfloor} N^-(T \cap (A_i \cup B_i))| \\ &\leq \sum_{\lfloor (\delta^- - 1)/2 \rfloor}^{\lfloor (\Delta^- - 1)/2 \rfloor} (i + 2)(|N^-(T \cap A_i)| + |N^-(T \cap B_i)|) \\ &\leq \sum_{\lfloor (\delta^- - 1)/2 \rfloor}^{\lfloor (\Delta^- - 1)/2 \rfloor} (i + 2)(a_i + b_i). \end{aligned} \tag{2.1}$$

Obviously,

$$n = \sum_{i=0}^{\lfloor(\Delta^- -1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\Delta^- -1)/2\rfloor} b_i. \tag{2.2}$$

Since the number $e(M, V(D))$ of arcs cannot be more than $q\Delta^-$, we have

$$\sum_{i=1}^{\lfloor(\Delta^- -1)/2\rfloor} i(a_i + b_i) \leq q\Delta^-. \tag{2.3}$$

Case 1. δ^- is odd.
Then $\lfloor(\delta^- - 1)/2\rfloor = (\delta^- - 1)/2$. Note that

$$i + 3 \leq \frac{i(\delta^- + 5)}{\delta^- - 1} \quad \text{when} \quad i \geq \frac{\delta^- - 1}{2}.$$

By (2.1), (2.2) and (2.3),

$$\begin{aligned}
n &= \sum_{i=0}^{\lfloor(\Delta^- -1)/2\rfloor} (a_i + b_i) \\
&= \sum_{i=0}^{(\delta^- -3)/2} (a_i + b_i) + \sum_{i=(\delta^- -1)/2}^{\lfloor(\Delta^- -1)/2\rfloor} (a_i + b_i) \\
&\leq \sum_{i=1}^{(\delta^- -3)/2} a_i + \sum_{i=(\delta^- -1)/2}^{\lfloor(\Delta^- -1)/2\rfloor} (i+3)(a_i + b_i) + \sum_{i=0}^{(\delta^- -3)/2} b_i \\
&\leq b_0 + \frac{\delta^- + 5}{\delta^- - 1}\Big(\sum_{i=1}^{\lfloor(\Delta^- -1)/2\rfloor} i(a_i + b_i)\Big) \\
&\leq q + \frac{\delta^- + 5}{\delta^- - 1} q\Delta^-.
\end{aligned}$$

By solving the above inequality for q, we obtain that

$$q \geq \frac{n(\delta^- - 1)}{\Delta^-(\delta^- + 5) + \delta^- - 1}.$$

Hence,

$$\Gamma_t^s(D) = n - 2q \leq \frac{\Delta^-(\delta^- + 5) - \delta^- + 1}{\Delta^-(\delta^- + 5) + \delta^- - 1} n.$$

Case 2. δ^- is even.
Then $\lfloor(\delta^- - 1)/2\rfloor = (\delta^- - 2)/2$. Note that

$$i + 3 \leq \frac{i(\delta^- + 4)}{\delta^- - 2} \quad \text{when} \quad i \geq \frac{\delta^- - 2}{2}.$$

By (2.1), (2.2) and (2.3),

$$
\begin{aligned}
n &= \sum_{i=0}^{\lfloor(\Delta^- -1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\Delta^- -1)/2\rfloor} b_i \\
&= \sum_{i=0}^{(\delta^- -4)/2} a_i + \sum_{i=(\delta^- -2)/2}^{\lfloor(\Delta^- -1)/2\rfloor} (a_i+b_i) + \sum_{i=0}^{\lfloor(\delta^- -4)/2\rfloor} b_i \\
&\leq \sum_{i=1}^{(\delta^- -4)/2} a_i + \sum_{i=(\delta^- -2)/2}^{\lfloor(\Delta^- -1)/2\rfloor} (i+3)(a_i+b_i) + \sum_{i=0}^{\lfloor(\delta^- -4)/2\rfloor} b_i \\
&\leq b_0 + \frac{\delta^- +4}{\delta^- -2} \sum_{i=1}^{\lfloor(\Delta^- -1)/2\rfloor} i(a_i+b_i) \\
&\leq q + \frac{\delta^- +4}{\delta^- -2} q\Delta^-. \qquad (2.4)
\end{aligned}
$$

Solving the inequality (2.4) for q,

$$q \geq \frac{n(\delta^- -2)}{\Delta^-(\delta^- +4)+\delta^- -2}.$$

Thus

$$\Gamma_t^s(D) = n - 2q \leq \frac{\Delta^-(\delta^- +4)-\delta^- +2}{\Delta^-(\delta^- +4)+\delta^- -2} n.$$

This completes the proof. □

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. We denote the associated digraph $D(K_n)$ of the complete graph K_n of order n by K_n^*, the associated digraph $D(\overline{K_n})$ of the complement of complete graph K_n by $\overline{K_n^*}$ and the associated digraph $D(C_n)$ of the cycle C_n of order n by C_n^*.

For two digraphs G and H, the *join* $G \leftrightarrow H$ is defined as the digraph consisting of G and H with extra arcs from each vertex of G to every vertex of H and arcs from each vertex of H to every vertex of G.

Let $V(K_4^*) = \{v_1, v_2, v_3, v_4\}$, $V(\overline{K_2^*}) = \{w_1, w_2\}$ and $V(C_{36}^*) = \{u_1, \ldots, u_{36}\}$. Assume that D is obtained from $(K_4^* \leftrightarrow \overline{K_2^*}) + C_{36}^*$ by adding arcs which go from w_1, w_2, v_1 to u_j for $1 \leq j \leq 36$. Then $\delta^-(D) = 4$ and $\Delta^-(D) = 5$. Define $f : V(D) \to \{-1, 1\}$ by $f(w_1) = f(w_2) = -1$ and $f(x) = 1$ for $x \in V(D) - \{w_1, w_2\}$. Obviously f is a minimal signed total dominating function of D with $\omega(f) = 38$. This example shows that the bound in Theorem 2.1 is sharp.

Corollary 2.1. *Let D be an r-inregular digraph of order n. Then*

$$\Gamma_t^s(D) \leq \begin{cases} \dfrac{r^2+4r+1}{r^2+6r-1} n, & \text{if } \delta^- \text{ is odd,} \\ \dfrac{r^2+3r+2}{r^2+5r-2} n, & \text{if } \delta^- \text{ is even.} \end{cases}$$

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