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ON THE d_2-SPLITTING GRAPH OF A GRAPH

N. R. SANTHI MAHESWARI [1] AND C. SEKAR [2]

Abstract. For a positive integer d and a vertex v of a graph G, the d^{th} degree of v in G, denoted by $d_d(v)$, is defined as the number of vertices at a distance d away from v. Hence $d_1(v) = d(v)$ and $d_2(v)$ means number of vertices at a distance 2 away from v. A graph G is said to be $(2, k)$-regular if $d_2(v) = k$, for all v in G. In this paper we define d_2-splitting graph of a graph and we study some properties of d_2-splitting graph.

1. Introduction

Throughout this paper we consider only finite, simple and connected graphs. Notations and terminology that we do not define here can be found in [3, 2]. We denote the graph G by $(V(G), E(G))$. The addition of two graphs G_1 and G_2 is a graph $G_1 + G_2$ with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv/u \in V(G_1), v \in V(G_2)\}$. The degree of a vertex v is the number of edges incident at v and we denote it by $d(v)$. A graph G is regular if all its vertices have the same degree. The set of all vertices at a distance one from v is denoted by $N(v)$. Two vertices u and v of G are said to be connected if there is a (u, v)-path in G. In a connected graph G, the distance between two vertices u and v is the length of the shortest (u, v)-path in G and is denoted by $d(u, v)$. Consequently, we define the degree of a vertex v as the number of vertices at a distance 1 from v. This observation suggests a generalization of degree. That is, $d_d(v)$ is defined as the number of vertices at a distance d from v. Hence $d_1(v) = d(v)$ and $N_d(v)$ denote the set of all vertices that are at a distance d away from v in a graph G. Hence $N_1(v) = N(v)$.

A graph G is said to be distance d-regular [1] if every vertex of G has the same number of vertices at a distance d from it. If $k \geq 0$ and if every vertex of G has

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exactly k number of vertices at a distance d from it, then we call this graph by (d,k)-regular graph. That is, a graph G is said to be (d,k)-regular graph if $d_d(v) = k$, for all v in G. The $(1,k)$-regular graphs and regular graphs are the same. (d,k)-regular graphs are natural extension of regular graphs. A graph G is said to be $(2,k)$-regular if $d_2(v) = k$, for all v in G, where $d_2(v)$ denotes number of vertices at a distance 2 from v.

Splitting graph $S(G)$ was introduced by Sampath Kumar and Walikar [5]. For each vertex v of a graph G, take a new vertex v' and join v' to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G.

In the similar way, degree splitting graph $DS(G)$ was introduced by R. Ponraj and S. Somasundaram [4]. Let $G = (V,E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$, where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i $(1 \le i \le t)$.

We define *d_2-splitting graph* of G denoted by $D_2S(G)$ and we investigate some properties of $D_2S(G)$.

2. d_2-SPLITTING GRAPH

Definition 2.1. Let G be a graph with $V(G) = V_1 \cup V_2 \cup V_3 \cup \cdots \cup V_w \cup W$ where each V_i is a set having at least two vertices, all having the same d_2 and $W = V - \cup V_i$. The d_2-splitting graph of G denoted by $D_2S(G)$ is obtained from G by introducing new vertices $u_1, u_2, \ldots, u_w$ and joining u_i to each vertex of V_i $(1 \le i \le w)$.

Example 2.1.

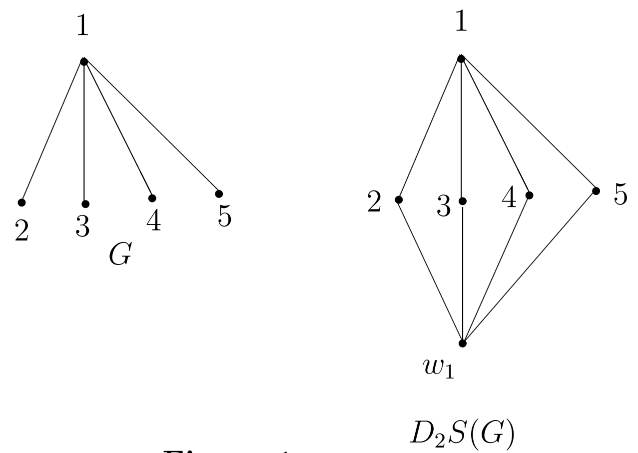


Figure 1.

Here, $V_1 = \{2,3,4,5\}$, $W = \{1\}$.

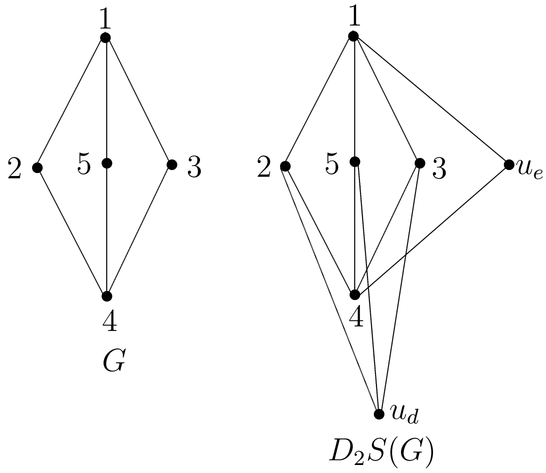


Figure 2.

Here, $V_1 = \{1, 4\}, V_2 = \{2, 3, 5\}, W = \phi$.

Observation 2.1. In a graph with more than one vertex, at least two vertices have the same degree d_2.

Observation 2.2. Trivial graph K_1 is the only graph such that $K_1 = D_2S(K_1)$.

Observation 2.3. For any graph $G \neq K_1$, G is a sub graph of $D_2S(G)$.

Observation 2.4. If $G = K_n^C$, then $D_2S(G) = K_{1,n}$.

Definition 2.2. Consider P_n $(n \geq 6)$ and two new vertices u and v on either side of P_n. Join the vertex v to first two vertices from the left and last two vertices of P_n from the right. Join the vertex u to the remaining vertices of P_n in the middle. The resulting graph is called Shipping graph and is denoted by SP_n.

Example 2.2. For a path on 6 vertices, the shipping graph SP_6 is shown in Figure 3.

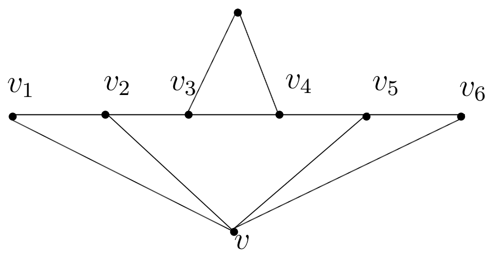


Figure 3.

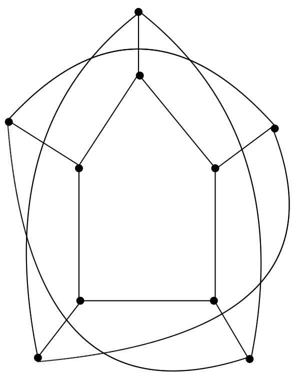
Observation 2.5. If $G = P_n$ $(n \geq 6)$, then $D_2S(G) = SP_n$.

Observation 2.6. If $G = C_n$, then $D_2S(G) = W_n$.

Observation 2.7. If $G = W_4$, then $D_2S(G) = K_5$.

Observation 2.8. If $G = K_n$ $(n > 1)$, then $D_2S(G) = K_{n+1}$.

Observation 2.9. If G is a $(2,k)$-regular, then $D_2S(G) = G + K_1$.



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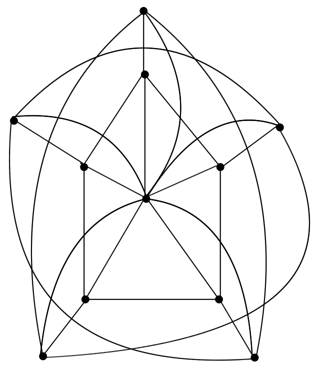

$D_2S(G)$

Figure 4.

Theorem 2.1. *Let G be a graph with p vertices and q edges. If $G(\neq K_p)$ is $(2,k)$-regular, then $D_2S(G)$ is not $(2,k)$-regular.*

Proof. Let G be $(2,k)$-regular with $k > 0$, that is, $d_2(v) = k$ for all $v \in V(G)$. Let $V(D_2S(G) - V(G)) = \{u\}$. Since u is adjacent to all the vertices of G, in $D_2S(G)$, $d_2(u) = 0$. That is, $d_2(u) \neq d_2(v)$, for all $v \in V(G)$. Hence $D_2S(G)$ is not $(2,k)$-regular. □

Theorem 2.2. *If G is a connected graph with at least one edge, then $D_2S(G)$ contains a cycle.*

Proof. Let G be a connected graph with $|E(G)| \geq 1$.
Case 1. If G contains a cycle, then $D_2S(G)$ also contains a cycle.
Case 2. Suppose G contains no cycle. Since G is a connected graph with $|E(G)| \geq 1$, G contains more than one vertex. By Observation 2.1 G contains at least two vertices having the same d_2.

Without loss of generality, let x and y be two vertices in G such that $d_2(x) = d_2(y)$. By definition of $D_2S(G)$, it contains a vertex u such that u is adjacent with both x and y.
Subcase 1. If x and y are adjacent, then u, x, y, u form a cycle in $D_2S(G)$.
Subcase 2. If x and y are not adjacent, then they are connected by a path $x = v_1, v_2, \ldots, v_n = y$. Since G is connected, $u, v_1, \ldots, v_n, u$ is a cycle in $D_2S(G)$. □

Remark 2.1. If G is a disconnected graph with $|E(G)| \geq 1$, then at least one component of G has edge set, so by Theorem 2.2, $D_2S(G)$ contains a cycle.

Result 2.1. Let G be a bipartite graph with bipartition (V_1, V_2), where $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_1^1, v_2^1, \ldots, v_n^1\}$. If there is a pair of vertices v_i and v_j^1 such that the length of the v_i - v_j^1 path is odd and $d_2(v_i) = d_2(v_j^1)$, then $D_2S(G)$ is not bipartite. Also, if there is no pair of vertices v_i and v_j^1 such that $d_2(v_i) = d_2(v_j^1)$, then $D_2S(G)$ is bipartite.

Theorem 2.3. *$D_2S(K_{m,n})$ is a bipartite graph if and only if $m \neq n$.*

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_1^1, v_2^1, \ldots, v_n^1\}$ be the partition of $V(K_{m,n})$. Therefore $d_2(v_i) = m - 1$ for $i = 1, 2, \ldots, m$ and $d_2(v_j^1) = n - 1$ for $j = 1, 2, \ldots, n$.

Suppose $m \neq n$. Then $m - 1 \neq n - 1$. Therefore, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, there is no pair v_i and v_j^1 such that $d_2(v_i) = d_2(v_j^1)$. Let $V(D_2S(K_{m,n}))\setminus V(K_{m,n}) = \{u_1, u_2\}$. Let u_1 be adjacent to every vertex in V_2 and u_2 be adjacent to every vertex in V_1. Clearly, $(V_1 \cup \{u_1\}, V_2 \cup \{u_2\})$ is a bipartition of $D_2S(K_{m,n})$. Therefore, $(D_2S(K_{m,n}))$ is a bipartite graph when $m \neq n$.

Conversely, let $D_2S(K_{m,n})$ be a bipartite graph. Suppose $m = n$. Then $m - 1 = n - 1$, that is, $d_2(v) = m - 1$ for all $v \in K_{m,n}$. Therefore, there exists a pair of adjacent vertices v_i and v_j^1 such that $d_2(v_i) = d_2(v_j^1)$. By definition of $D_2S(K_{m,n})$, there exists a vertex u which is adjacent to both v_i and v_j^1. Therefore, $D_2S(K_{m,n})$ will contain the odd cycle $u_1 v_i v_j^1 u_1$. This implies that $D_2S(K_{m,n})$ is not a bipartite graph, which is a contradiction. Hence $m \neq n$. □

Theorem 2.4. *$D_2S(K_{n,n})$ is a tripartite graph.*

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_n\}$ are the partition of $V(K_{n,n})$. Then $d_2(v_i) = n-1$ for $i = 1, 2, \ldots, n$ and $d_2(u_j) = n-1$ for $j = 1, 2, \ldots, n$. By definition of $D_2S(G)$, $D_2S(K_{n,n})$ contains a vertex u such that u is adjacent to all u_i $(i = 1, 2 \ldots, n)$ and v_j $(j = 1, 2, \ldots, n)$, that is, u is adjacent with all vertices of $K_{n,n}$. $D_2S(K_{n,n})$ is $K_{1,n,n}$ and hence tripartite. □

Result 2.2. For any graph G, $\omega(D_2S(G)) \leq \omega(G)$, where $\omega(G)$ denotes the number of components of G.

Proof. Case 1. If G is a connected graph, then $D_2S(G)$ is connected. Therefore, $\omega(G) = 1 = \omega(D_2S(G))$.
Case 2. If G is a disconnected graph then G has more than one component. Let us assume that G has two components G_1 and G_2. Let $x \in V(G_1)$ and $y \in V(G_2)$ such that $d_2(x) = d_2(y)$. By definition of $D_2S(G)$, there exists a vertex u such that u is adjacent to both x and y. Hence $\omega(D_2S(G)) = 1 < \omega(G)$. Suppose that either x and y are in $V(G_1)$ or x and y are in $V(G_2)$. Then $\omega(D_2S(G)) = 2 = \omega(G)$. Hence $\omega(D_2S(G)) \leq \omega(G)$. □

Theorem 2.5. *If G is an Eulerian graph, then $D_2S(G)$ is not an Eulerian graph.*

Proof. Let G be an Eulerian graph. Since G is an Eulerian graph, each vertex in G is of even degree. By Observation 2.1, G contains at least two vertices having the same d_2. Let x and y be two vertices in G such that $d_2(x) = d_2(y)$. By definition of $D_2S(G)$, there exits a vertex u which is adjacent to both x and y. Therefore, degree of x in $D_2S(G)$ = (degree of x in G) + 1 = even + 1 = odd. Degree of x and y in $D_2S(G)$ are odd. Therefore, $D_2S(G)$ is not an Eulerian graph. □

Theorem 2.6. *Let G be a graph with p vertices and q edges and let s be the number of vertices in W. Then $|E(D_2S(G))| = p + q - s$ where W is as in Definition* 2.1.

Proof. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ and $V(D_2S(G)) - V(G) = \{u_1, u_2, \ldots, u_s\}$. Let $d'(v)$ denote the degree of a vertex v in $D_2S(G)$ (clearly $d'(v) \leq d(v)$, for all v in G). Then $|E(D_2S(G))| = \frac{1}{2}\sum d'(v) = \frac{1}{2}\left[\sum_{i=1}^{p}(d(v_i)+1) - s + p - s\right] = p + q - s.$ □

Remark 2.2. If G is a $(2, k)$-regular graph, then $|E(D_2S(G))| = p + q$.

Theorem 2.7. *$D_2S(K_{n,n})$ is a Hamiltonian graph.*

Proof. For $n \geq 1$, the number of vertices in $D_2S(K_{n,n}) = 2n + 1 = p \geq 3$. Minimum degree of the graph $D_2S(K_{n,n})$ is $n+1$, that is, $p = 2n+1$ and $\delta = n+1$. Therefore, $\delta \geq \frac{p}{2}$. By Dirac's theorem, $D_2S(K_{n,n})$ is a Hamiltonian graph. □

Theorem 2.8. *$D_2S(K_{m,n})$ is a Non-Hamiltonian graph with $m \neq n$.*

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_n\}$ be the partition of $V(K_{m,n})$. Assume $m < n$. Here $V(D_2S(K_{m,n})) = \{V_1 \cup \{u_1\}\} \cup \{V_2 \cup \{u_2\}\}$, u_1 is adjacent with all the vertices of V_2 and u_2 is adjacent to all the vertices of V_1. Therefore, $|V_1 \cup \{u_1\}| = m + 1$ and $|V_2 \cup \{u_2\}| = n + 1$. Then

$$(\omega(D_2S(K_{m,n})) - \{V_1 \cup \{u_1\}\}) = n + 1 > m + 1 = |V_1 \cup \{u_1\}|.$$

Hence $D_2S(K_{m,n})$ is a Non-Hamiltonian. □

Note 2.1. $D_2S(G)$ of a disconnected graph G may be connected. For instance, let G be a graph with two components G_1 and G_2 such that G_1 and G_2 are $(2, k)$-regular and each vertex of G_1 and G_2 have the same d_2. Then, by definition of $D_2S(G)$, there exists a vertex which is adjacent to all the vertices of G_1 and G_2. Therefore $D_2S(G)$ is connected.

Theorem 2.9. *Let G be a connected graph. Then $K(D_2S(G)) \geq K(G)$.*

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $V(D_2S(G)) - V(G)) = \{u_1, u_2, u_3, \ldots, u_w\}$. Since G is a connected graph, by Observation 2.1, G contains at least two vertices having same d_2 and they are connected by a path. Let v_i and v_j be the vertices of G such that $d_2(v_i) = d_2(v_j)$ and v_i and v_j are connected by a path. Suppose G is k-connected. Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ be the minimum vertex cut of G. Since $G - S$ is disconnected, $G - S$ has at least two components. Take two components G_1 and G_2.
Case 1. Suppose v_i and v_j are in the same component. Then $K(D_2S(G)) = K(G)$.
Case 2. Suppose v_i and v_j belong to different components. Without loss of generality, let $v_i \in G_1$ and $v_j \in G_2$. Then there is no v_i-v_j path in $G - S$. But v_i and v_j are connected by a path $v_iu_iv_j$ in $(D_2S(G) - S)$. That is, $D_2S(G) - S$ is connected. Hence $K(D_2S(G)) \geq K(G)$. □

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[1] Department of Mathematics,
G. Venkataswamy Naidu College,
Kovilpatti-628 502, Tamil Nadu, India.
E-mail address: `nrsmaths@yahoo.com`

[2] Department of Mathematics,
Aditanar College of Arts and Science,
Tiruchendur, Tamil Nadu, India.
E-mail address: `sekar.acas@gmail.com`