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ON WEAKLY SYMMETRIC SPACETIMES

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Abstract. In the present paper we study weakly symmetric spacetimes. The existence of such a spacetime has been proved by a non-trivial example.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a special type of spacetime which is called weakly symmetric spacetime. The notion of weakly symmetric manifold was introduced by Tamassy and Binh [8]. A non-flat semi-Riemannian manifold is called weakly symmetric if the curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + C(Z)R(Y,X)W \\ +D(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,\end{aligned} \tag{1.1}$$

where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, C, D and ρ are 1-forms and a vector field respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. It was proved in [6] that the 1-forms and the vector field must be related as follows:

$$B(X) = C(X) = D(X), \quad g(X,\rho) = D(X),$$

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for all X. That is, the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + D(Y)R(X,Z)W + D(Z)R(Y,X)W + D(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho. \tag{1.2}$$

The 1-forms A and D are called the associated 1-forms and the vector field ρ is called the associated vector field of the manifold. If $A = D = 0$, the manifold reduces to a symmetric manifold in the sense of Cartan. This justifies the name weakly symmetric manifold defined by (1.1). The existence of a $(WS)_n$ was proved by M. Prvanović [7] and a concrete example was given by De and Bandyopadhyay in [1].

A non-flat semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.3}$$

where a, b are reals and A is a non-zero 1-form such that $g(X,U) = A(X)$, for all vector fields X.

The paper is organized as follows:
After preliminaries, in Section 3 we study perfect fluid weakly symmetric spacetime admitting cyclic parallel Ricci tensor and we study the nature of Segre' characteristic of such a spacetime. In Section 4 we show that the spacetimes under consideration do not admit heat flux. Finally, in the last section we construct an example of $(WS)_4$ spacetimes.

2. Preliminaries

Let S and r denote the Ricci tensor of type $(0,2)$ and the scalar curvature respectively, and let L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX,Y) = S(X,Y), \tag{2.1}$$

for any vector fields X, Y. Let $\bar{D}$ be a 1-form defined by

$$\bar{D}(X) = D(LX). \tag{2.2}$$

From (1.2) we get on contraction

$$(\nabla_X S)(Z,W) = A(X)S(Z,W) + D(Z)S(X,W) + D(W)S(Z,X) + D(R(X,Z)W) + D(R(X,W)Z). \tag{2.3}$$

Again contracting (2.3) we get

$$dr(X) = A(X)r + 4S(X,\rho). \tag{2.4}$$

3. $(WS)_4$ Perfect Fluid Spacetimes with Cyclic parallel Ricci Tensor

A semi-Riemannian manifold is said to have cyclic parallel Ricci tensor if the Ricci tensor S is non-zero and satisfies the relation

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0. \tag{3.1}$$

Then from (2.3) and (3.1) it follows that

$$\tilde{T}(X)S(Y,Z) + \tilde{T}(Y)S(X,Z) + \tilde{T}(Z)S(X,Y) = 0, \tag{3.2}$$

where $\tilde{T}(X) = A(X) + 2D(X)$. In local coordinates the equation (3.2) can be written as follows:

$$\tilde{T}_i R_{jk} + \tilde{T}_j R_{ki} + \tilde{T}_k R_{ij} = 0. \tag{3.3}$$

Now we state:

Walker's Lemma: [9] If a_{ij}, b_i are numbers satisfying $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for i, j, $k = 1, 2, 3, \ldots, n$, then either all a_{ij} are zero or, all b_i are zero.

Hence by Walker's Lemma from (3.2), we obtain either $\tilde{T}(X) = 0$ or, $S(Y,Z) = 0$. Since $S \neq 0$, $\tilde{T} = 0$. Thus

$$A(X) = -2D(X). \tag{3.4}$$

Now we consider a weakly symmetric relativistic spacetime $(WS)_4$ satisfying cyclic parallel Ricci tensor as a perfect fluid spacetime with cosmological constant λ in which the associated vector field ρ is the velocity vector field of the fluid. It is known [4] that a general relativistic spacetime (M^4, g) is a 4-dimensional Lorentzian manifold having the matter content as a perfect fluid with unit timelike vector field.

The Einstein's field equation can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{3.5}$$

where k is the gravitational contant, T is the energy-momentum tensor of type (0,2) given by

$$T(X,Y) = (\sigma + p)D(X)D(Y) + pg(X,Y), \tag{3.6}$$

with σ and p denoting the density and pressure of the fluid respectively and D being given by $g(X,\rho) = D(X)$ for all X, ρ is the flow vector field of the fluid such that $g(\rho,\rho) = -1$. We can express (3.5) as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = k[(\sigma + p)D(X)D(Y) + pg(X,Y)]. \tag{3.7}$$

Since $(WS)_4$ spacetime satisfies cyclic parallel Ricci tensor, the scalar curvature r is constant. Hence from (2.4) we get

$$S(X,\rho) = -\frac{r}{4}A(X). \tag{3.8}$$

Using (3.4) and (3.8) we obtain

$$S(X,\rho)=\frac{r}{2}D(X)=\frac{r}{2}g(X,\rho). \tag{3.9}$$

Now putting $Y=\rho$ in (3.7) we get

$$S(X,\rho)-\frac{r}{2}g(X,\rho)+\lambda g(X,\rho)=k[(\sigma+p)D(X)D(\rho)+pg(X,\rho)]. \tag{3.10}$$

In virtue of (3.9) and taking into account the fact that $D(\rho)=-1$ we can write (3.10) as follows:

$$\lambda=-k\sigma. \tag{3.11}$$

Again taking frame field and contracting (3.7) and using (3.11) we obtain

$$p=\frac{3\lambda-r}{3k}. \tag{3.12}$$

Spacetimes are sometimes classified according to the nature of the Segre' characteristic [5] of the Ricci tensor. We now investigate the nature of the Segre' characteristic of the Ricci tensor for perfect fluid $(WS)_4$ spacetime.

From (3.9) it follows that $\frac{r}{2}$ is an eigenvalue of the Ricci tensor and ρ is an eigenvector corresponding to this eigenvalue.

Let ξ be another eigenvector of S different from ρ. Then ξ must be orthogonal to ρ. Hence $g(\rho,\xi)=0$. That is,

$$D(\xi)=0. \tag{3.13}$$

Putting $Y=\xi$ in (3.7) and using (3.13) we obtain

$$S(X,\xi)=(\frac{r}{2}-\lambda+pk)g(X,\xi). \tag{3.14}$$

Using (3.12) in (3.14) we obtain

$$S(X,\xi)=\frac{r}{6}g(X,\xi). \tag{3.15}$$

From (3.15) it follows that $\frac{r}{6}$ is another eigenvalue of S and ξ is an eigenvector corresponding to this eigenvalue. Since for a given eigenvector there is only one eigenvalue and $\frac{r}{2}$ and $\frac{r}{6}$ are different, it follows that the Ricci tensor has only two distinct eigenvalues, namely $\frac{r}{2}$ and $\frac{r}{6}$.

Let the multiplicity of $\frac{r}{2}$ be m. Then the multiplicity of $\frac{r}{6}$ is $(4-m)$, since the dimension of the spacetime is 4.

Hence, $m(\frac{r}{2})+(4-m)\frac{r}{6}=0$ which gives $m=1$. Therefore, the multiplicity of $\frac{r}{2}$ is 1 and the multiplicity of $\frac{r}{6}$ is 3. $m=4$ implies that there is only one eigenvalue $\frac{r}{2}$ of multiplicity 4. But we have proved that there exist two eigenvalues $\frac{r}{2}$ and $\frac{r}{6}$. So we can not take $m=4$. Hence the Segre' characteristic of S is [(111),1]. This leads to the following result:

Theorem 3.1. *A perfect fluid $(WS)_4$ spacetime satisfying cyclic parallel Ricci tensor with the basic vector field of $(WS)_4$ as the velocity vector field of the fluid is of Segre' characteristic* $[(111), 1]$.

In a subsequent paper [2] Gazi and De obtained the following Theorem:

Theorem 3.2. *If a perfect fluid weakly symmetric spacetime satisfies cyclic parallel Ricci tensor, then the spacetime becomes a quasi-Einstein spacetime.*

Proof. We consider a weakly symmetric relativistic spacetime $(WS)_4$ satisfying cyclic parallel Ricci tensor as a perfect fluid spacetime without cosmological constant λ in which the associated vector field ρ is the velocity vector field of the fluid, that is, $g(\rho,\rho) = -1$. Then the Einstein's equation can be expressed as

$$S(X,Y) - \frac{r}{2}g(X,Y) = k[(\sigma+p)D(X)D(Y) + pg(X,Y)]. \tag{3.16}$$

Taking a frame field and contracting (3.16) over X, Y we get

$$r = k(\sigma - 3p). \tag{3.17}$$

Since, here $\lambda = 0$ we get from (3.11)

$$\sigma = 0. \tag{3.18}$$

Hence from (3.16), (3.17) and (3.18) we have

$$S(X,Y) = \frac{r}{6}g(X,Y) - \frac{r}{3}D(X)D(Y), \tag{3.19}$$

which implies that the manifold is a quasi-Einstein manifold. □

Let the $(WS)_4$ spacetime be conformally flat. Then the curvature tensor R of type (1,3) is of the following form:

$$\begin{aligned} R(X,Y)Z = \frac{1}{2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ -\frac{r}{6}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.20}$$

where Q is the Ricci operator. From (3.19) we have

$$QX = \frac{r}{6}X - \frac{r}{3}D(X)\rho. \tag{3.21}$$

Using (3.19) and (3.21), we can express (3.20) as follows:

$$\begin{aligned} R(X,Y)Z = -\frac{r}{6}[D(Y)D(Z)X - D(X)D(Z)Y + D(X)g(Y,Z)\rho \\ -D(Y)g(X,Z)\rho]. \end{aligned} \tag{3.22}$$

Let $\rho^{\perp}$ be the 3-dimensional distribution orthogonal to the generator ρ. Then from (3.22) we have

$$R(X,Y)Z = 0, \tag{3.23}$$

for all X, Y, $Z \in \rho^{\perp}$ and hence

$$R(X, \rho)\rho = 0, \tag{3.24}$$

for all $X \in \rho^{\perp}$.

According to Karchar [3] a Lorentzian manifold is called infinitesimally spatially isotropic relative to a timelike unit vector field ρ if its curvature tensor R satisfies the relations

$$R(X, Y)Z = l[g(Y, Z)X - g(X, Z)Y], \ \ \forall X, Y, Z \in \rho^{\perp}$$

and

$$R(X, \rho)\rho = mX, \ \ \forall X \in \rho^{\perp},$$

where l, m are real valued functions on the manifold. So by virtue of (3.23) and (3.24) we can state the following:

Theorem 3.3. *A conformally flat perfect fluid weakly symmetric spacetime satisfying cyclic parallel Ricci tensor is infinitesimally spatially isotropic to the unit timelike vector field ρ.*

Corollary 3.1. *A conformally flat perfect fluid weakly symmetric spacetime satisfying cyclic parallel Ricci tensor having the basic vector field ρ as the velocity vector field of the fluid has the property that all planes perpendicular to ρ have sectional curvature zero and all planes containing ρ have also sectional curvature zero.*

Again using (3.5) and (3.19) we have

$$(\frac{r}{6} - \frac{r}{2} + \lambda)g(X, Y) - \frac{r}{3}D(X)D(Y) = kT(X, Y). \tag{3.25}$$

Since the spacetime under consideration satisfies cyclic parallel Ricci tensor, the scalar curvature r is constant. Let us suppose that the generator ρ is a Killing vector field. Then

$$(\pounds_{\rho} g)(X, Y) = 0, \tag{3.26}$$

where $\pounds$ denotes the Lie derivative with respect to ρ.

Now from (3.25) we obtain

$$(\frac{r}{6} - \frac{r}{2} + \lambda)(\pounds_{\rho} g)(X, Y) = k(\pounds_{\rho} T)(X, Y). \tag{3.27}$$

Since $k \neq 0$, (3.26) and (3.27) yield

$$(\pounds_{\rho} T)(X, Y) = 0. \tag{3.28}$$

Thus we can state the following:

Theorem 3.4. *If in a perfect fluid weakly symmetric spacetime with cyclic parallel Ricci tensor obeying Einstein's equation, the generator ρ of the spacetime is a Killing vector field, then the Lie derivative of the energy-momentum tensor with respect to ρ is zero and conversely.*

4. Possibility of a fluid $(WS)_4$ spacetime to admit heat flux

In this section we shall give an answer to the following question:
If in a $(WS)_4$ spacetime with cyclic parallel Ricci tensor, the matter distribution is a fluid with the basic vector field of $(WS)_4$ as the velocity vector field of the fluid, can this distribution be described by the following form of the energy-momentum tensor

$$T(X,Y) = (\sigma + p)D(X)D(Y) + pg(X,Y) + D(X)A(Y) + D(Y)A(X), \tag{4.1}$$

where $g(X,\eta) = A(X), \forall X$ and η being the heat flux vector field? Then

$$g(\rho,\eta) = A(\rho) = 0. \tag{4.2}$$

If possible let $T(X,Y)$ be of the form (4.1). Then Einstein's equation can be written as follows:

$$\begin{aligned} S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k[(\sigma + p)D(X)D(Y) + pg(X,Y) \\ +D(X)A(Y) + D(Y)A(X)]. \end{aligned} \tag{4.3}$$

Putting $Y = \rho$ in (4.3) and using (3.9) and (4.2) we obtain

$$kA(X) = -(\lambda + k\sigma)D(X). \tag{4.4}$$

Hence from (4.4), it follows that $A(X) = 0$, because $k \neq 0$ and $\lambda + k\sigma = 0$.

Therefore the answer to the question raised in the beginning of the section is negative. Thus we can state the following:

Theorem 4.1. *If in a $(WS)_4$ spacetime with cyclic parallel Ricci tensor, the matter distribution is a fluid with the basic vector field of $(WS)_4$ as the velocity vector field of the fluid, then such a fluid can not admit heat flux.*

Note: It is to be noted that the absence of heat flux is due to the relation $S(X,\rho) = \frac{r}{2}D(X)$, i.e., due to the fact that ρ is a timelike eigenvector of the Ricci tensor S.

5. Example of a Weakly Symmetric Spacetime

In this section we prove the existence of a weakly symmetric spacetime by constructing a non-trivial concrete example.

Example: Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{5.1}$$

where $i,j = 1,2,3,4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the derivatives of the components of curvature tensors are

$$\Gamma^2_{11} = \Gamma^2_{33} = -\frac{1}{2x^2}, \quad \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2x^2},$$

$$R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0,$$

$$R_{1221,2} = R_{2332,2} = \frac{1}{2(x^2)^2}, \quad R_{1331,2} = -\frac{1}{4(x^2)^2},$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor R_{ij} are

$$R_{11} = R_{33} = -\frac{1}{4(x^2)^2}, \quad R_{22} = -\frac{1}{(x^2)^2}, \quad R_{44} = 0.$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is $-\frac{3}{2(x^2)^3} \neq 0$. We shall now show that $(\mathbb{R}^4, g)$ is a weakly symmetric spacetime.

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\dfrac{1+x^2}{(x^2)^2}, & i = 2 \\ 0, & \text{otherwise}, \end{cases} \tag{5.2}$$

$$D_i(x) = \begin{cases} \dfrac{1}{2(x^2)^2}, & i = 2 \\ 0, & \text{otherwise}, \end{cases} \tag{5.3}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (1.2), it is sufficient to check the following equations:

$$R_{1221,2} = (A_2 + 2D_2)R_{1221}, \tag{5.4}$$

$$R_{2332,2} = (A_2 + 2D_2)R_{2332}, \tag{5.5}$$

$$R_{1331,2} = (A_2 + 2D_2)R_{1331}, \tag{5.6}$$

since for the other cases (1.2) holds trivially. By (5.2) and (5.3) we get

$$\begin{aligned} \text{R.H.S. of (5.4)} &= (A_2 + 2D_2)R_{1221} \\ &= [-\frac{1+x^2}{(x^2)^2} + \frac{1}{(x^2)^2}][-\frac{1}{2x^2}] \\ &= \frac{1}{2(x^2)^2} = R_{1221,2} \\ &= \text{L.H.S. of (5.4)}. \end{aligned}$$

By similar argument it can be shown that (5.5) and (5.6) are also true. So $(\mathbb{R}^4, g)$ is a weakly symmetric spacetime. Thus we can state the following:

Theorem 5.1. *Let a Lorentzian metric g on $\mathbb{R}^4$ be given by*

$$ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{5.7}$$

where $i, j = 1, 2, 3, 4$. Then $(\mathbb{R}^4, g)$ is a weakly symmetric spacetime whose scalar curvature is non-zero and non-constant.

6. Conclusion

In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat. Relativistic fluid models are of considerable interest in several areas of astrophysics, plasma physics and nuclear physics. Theories of relativistic stars (which would be models for supermassive stars) are also based on relativistic fluid models. The problem of accretion onto a neutron stars or a blackhole is usually set in the framework of relativistic fluid models.

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. In the present paper it is shown that perfect fluid $(WS)_4$ spacetime satisfying cyclic parallel Ricci tensor with the basic vector field as the velocity vector field is of Segre' characteristic [(111),1]. We also prove that if in a $(WS)_4$ spacetime with cyclic parallel Ricci tensor, the matter distribution is a fluid with the basic vector field of $(WS)_4$ as the velocity vector field of the fluid, then such a fluid can not admit heat flux. Next we show that a conformally flat perfect fluid $(WS)_4$ spacetime satisfying cyclic parallel Ricci tensor is infinitesimally spatially isotropic to the unit timelike vector field ρ. Finally, we construct an example of such type of spacetimes.

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