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THE RIESZ ASPECTS OF χ^2 SEQUENCE SPACES

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Abstract. Let χ^2 denote the space of all double gai sequences. Let Λ^2 denote the space of all double analytic sequences. In this paper we introduce the concept of sectional analyticity. We also prove a theorem on type $M_{(\chi^2:\chi^2)}$ and also the mean value condition is established. We construct a mildly conservative FK-space, which is not semi-conservative.

1. Introduction

Throughout the paper w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinatewise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \left\{(x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},\\
\mathcal{C}_p(t) &:= \left\{(x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn}-l|^{t_{mn}} = 1 \text{for some } l \in \mathbb{C}\right\},\\
\mathcal{C}_{0p}(t) &:= \left\{(x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},
\end{aligned}$$

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$$\begin{aligned}
\mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}, \\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t), \text{ and} \\
\mathcal{C}_{0bp}(t) &:= \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);
\end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21, 22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently, Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently, Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces of strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]–[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to multiply sequence spaces.

In what follows in this paper, we need the following inequality. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non-zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

If X is a sequence space, we give the following definitions:

(i) X'= the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \text{for each } x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \text{ is convergent, for each } x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{mn}) : \sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn} x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^{\delta} = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\}$;

(vii) $X^{\Lambda} = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty\right\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called α-(or Köthe-Toeplitz) dual of X, β-(or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X, Λ-dual of X, respectively. X^{α} is defined by Gupta and Kamptan [20]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\begin{aligned} \|x\| &= |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and} \\ \|x\|_{bv_p} &= \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty). \end{aligned}$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $g : X \to R$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous; that is $|\alpha_{mn} - \alpha| \to 0$ and $g(x_{mn} - x) \to 0$ imply $g(\alpha_{mn} x_{mn} - \alpha x) \to 0$ for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that $p = (p_{mn})$ is a double analytic sequence of strictly positive real numbers with $\text{supp}_{mn} = H$ and $M = \max(1, H)$.

Let λ and μ be two sequence spaces and $A = \left(a_{k,\ell}^{mn}\right)$ be a four dimensional infinite matrix of real numbers $\left(a_{k,\ell}^{mn}\right)$, where $m, n, k, \ell \in \mathbb{N}$. Then we say A defines a matrix mapping from λ into μ and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{(Ax)_{k\ell}\}$, the A-transform of x, is in μ, where

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn} \, (k, \ell \in \mathbb{N}). \tag{1.2}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each $k, \ell \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

Let (q_{mn}) be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} \, (k, \ell \in \mathbb{N}). \tag{1.3}$$

Then the matrix $R^q = (r_{k\ell}^{mn})^q$ of the Riesz mean is given by

$$(r_{k\ell}^{mn})^q = \begin{cases} \dfrac{q_{mn}}{Q_{k\ell}}, & \text{if } 0 \leq m, n \leq k, \ell; \\ 0, & \text{if } (m, n) > k\ell. \end{cases} \tag{1.4}$$

The double Riesz sequence spaces are defined as follows:

$$\Lambda_r^{2q} = \left\{ x = (x_{mn}) \in w^2 : \sup_{k\ell \in N} \left| \frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} (x_{mn})^{1/m+n} \right| < \infty \right\},$$

$$\chi_r^{2q} = \left\{ x = (x_{mn}) \in w^2 : \lim_{k\ell \to \infty} \left| \frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} ((m+n)! x_{mn})^{1/m+n} \right| = 0 \right\},$$

which are the sequence spaces of the sequences x whose R^q-transforms are in Λ^2 and χ^2, respectively.

The main purpose of this paper is to introduce the Riesz sequence spaces Λ_r^{2q} and χ_r^{2q} of the sequences whose transform are in Λ^2 and χ^2, respectively and to investigate some topological and geometric properties of the following results.

(1) $z \in X^{f\Lambda} \Leftrightarrow z^{-1} X \supset \chi_r^{2q}$.
(2) The Taylor method is of type $M_{\left(\chi_r^{2q} : \chi_r^{2q}\right)}$.
(3) Let A be a triangular matrix which is absolutely regular. If A satisfies the mean value property, then $\left| a_{mm}^{kk} \right| \neq 0$ for all m and k.
(4) Mildly conservative matrices are introduced.

Example 1.1. Let $\{x_{mn}\} \in \Lambda_r^{2q}$. Take $X = 1^*$, where

$$1^* = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ & & \ddots & \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Then $1^* = \Lambda_r^{2q}$.

Example 1.2. $\phi^{\Lambda_r^{2q}} = w^2$.

Definition 1.1. Let X be an FK-space containing ϕ. Then E^+ or $E^+(X)$ is defined as $z \in w^2 : \left\{ z_{mn} f(\delta^{mn}) \in \Lambda_r^{2q} \forall f \in X' \right\}$ and we put $E = E^+ \cap X$.

Definition 1.2. Let X be an FK-space containing ϕ. Then X is said to have sectional analyticity if $X = E$.

Definition 1.3. Let $A \in (\chi_r^{2q} : \chi_r^{2q})$. Suppose that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} a_{k\ell}^{mn} = 0 \; (k, \ell \in \mathbb{N})$ with $\{\alpha_{mn}\} \in \Lambda^2$ implies that $\alpha_{mn} = 0$ for all m, n. Then A is said to be type $M(\chi_r^{2q} : \chi_r^{2q})$.

Definition 1.4. A four dimensional triangular matrix $A = (a_{k\ell}^{mn})$ is said to satisfy the mean value condition $M_{mn}(A)$ if $\left|\sum_{m=1}^{p} \sum_{n=1}^{q} a_{k\ell}^{mn} x_{mn}\right| \leq K \sup_{i \leq p, j \leq q} |y_{ij}|$ where $p \leq k, \ell$ and $q \leq m, n$ is independent of p, q and $\{x_{mn}\}$.

Definition 1.5. A FK-space X is mildly double conservative if $X^f \subset c^2$; where c^2 is double convergent sequence spaces.

Definition 1.6. The set of all sequences $\{x_{mn}\}$ such that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}$ converges is denoted by cs^2.

Definition 1.7. The set of all sequences $\{x_{mn}\}$ such that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|$ converges is denoted by ℓ^2.

Definition 1.8. For any complex number α, let
Take

$$a_{k\ell}^{mn} = \begin{cases} (k-1)(\ell-1)\,\alpha^{mn}(1-\alpha)^{(k-m)(\ell-n)}, & \text{if } 0 \le m,n \le p,q \\ 1, & \text{if } m,n = k,\ell = 0; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $A = (a_{k\ell}^{mn})$ is an upper triangular matrix.

2. Main Results

Theorem 2.1. *Let X be an FK-space containing ϕ. Let $z \in w^2$. Then $z \in E^{+} \Leftrightarrow z^{-1}X \supset \chi_r^{2q}$. Here, $z^{-1}X = \{y \in w^2 : \{y_{mn}z_{mn}\} \in X\}$.*

Proof. Let $f \in (z^{-1}X)^{'}$.
Then

$$\begin{aligned} f(\delta^{mn}) &= \alpha_{mn} + g(z\delta^{mn}) \\ &= \alpha_{mn} + g(z_{mn}\delta^{mn}) \\ &= \alpha_{mn} + z_{mn}g(\delta^{mn}) \qquad \text{where } \alpha \in \phi \text{ and } g \in X^{'}. \end{aligned}$$

Hence, if $z \in E^{+}$, then $\{f(\delta^{mn})\} \in \Lambda^2$, $\forall f \in (z^{-1}X)^{'}$. That is, $(z^{-1}X)^{f} \subset \Lambda^2$. But $\Lambda^2 = (\chi_r^{2q})^{f}$. Hence $(z^{-1}X)^{f} \subset \Lambda^2 = (\chi_r^{2q})^{f}$ and consequently $\chi_r^{2q} \subset z^{-1}X$. Thus, $z \in E^{+} \Rightarrow z^{-1}X \supset \chi_r^{2q}$. The reverse implication follows similarly. ☐

Theorem 2.2. *Let X be an FK-space containing ϕ. Then $z \in X^{f\Lambda} \Leftrightarrow z^{-1}X \supset \chi_r^{2q}$.*

Proof. By Definition 1.3, $z \in E^{+} \Leftrightarrow zu \in \Lambda^2\ \forall u \in X^{f}$. Hence $E^{+} = X^{f\Lambda}$. By Theorem 2.1, $z \in E^{+} \Leftrightarrow z^{-1}X \supset \chi_r^{2q}$. Hence $z \in X^{f\Lambda} \Leftrightarrow z^{-1}X \supset \chi_r^{2q}$. ☐

Theorem 2.3. *Let X be an FK-space containing ϕ. Then X is said to have sectional analyticity if and only if $X^{f} = X^{\Lambda}$.*

Proof. Suppose X has sectional analyticity. Then $X = E = E^{+} \cap X$, so that $X \subset E^{+} = X^{f\Lambda}$. Hence $x^{\Lambda} \supset X^{f\Lambda} \supset X^{f}$ and so $X^{\Lambda} \supset X^{f}$. But always $X^{\Lambda} \subset X^{f}$. Hence $X^{\Lambda} = X^{f}$.

Conversely, suppose that $X^{\Lambda} = X^{f}$. But $E^{+} = X^{f\Lambda} = X^{\Lambda\Lambda} \supset X$. Thus, $E^{+} \supset X$ or equivalently, $E = X$, so that X has sectional analyticity. ☐

Theorem 2.4. *Let $A = (a_{k\ell}^{mn})$ is any four dimensional upper triangular matrix, that is $a_{k\ell}^{mn} = 0$ for $k > \ell$ and $m > n$, then a sufficient condition for $A \in (\chi_r^{2q} : \chi_r^{2q})$ is that the elements should be analytic. Further if, $a_{kk}^{mm} \neq 0$, $\forall k, m$, then A is of type $M(\chi_r^{2q} : \chi_r^{2q})$.*

Proof. Suppose that $|a_{k\ell}^{mn}| \leq K, \forall k, \ell, m, n$. For any given q, we have

$$q^{k\ell} |a_{k\ell}^{mn}| \leq q^{mn} K, \quad (k, m \leq \ell, n) \tag{2.1}$$

and if $k, m > \ell, n$ the expression on the left is 0, and thus (2.1) still holds. Thus we have that $A \in (\chi_r^{2q} : \chi_r^{2q})$, since Brown's condition holds with $p = q$ and $M = K$.

Conversely suppose that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{mn} a_{k\ell}^{mn} = 0, \quad \{\alpha_{mn}\} \in \Lambda^2 \tag{2.2}$$

By induction, $\alpha_{mn} = 0, \forall m, n$. Taking $k, \ell = 0$, since $a_{k\ell}^{mn} = 0$ for $k, m > \ell, n$, the equation (2.1) reduces to $\alpha_{00} (a_{00}^{00}) = 0$. But $(a_{00}^{00}) \neq 0$. Hence $\alpha_{00} = 0$. Now take $m, n \geq 1$ and suppose that

$$\begin{pmatrix} \alpha_{01} & \alpha_{02} & \ldots & \alpha_{0n-1} \\ \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n-1} \\ & & \ddots & \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{m-1n-1} \end{pmatrix} = 0.$$

Since $a_{k\ell}^{mn} = 0$ for $k, \ell > m, n$, the equation (2.2) gives

$$\textstyle\sum_{m=0}^{k} \sum_{n=0}^{\ell} \alpha_{mn} a_{k\ell}^{mn} = 0.$$

By induction hypothesis this reduces to $\alpha_{mm} a_{kk}^{mn} = 0$. But $a_{kk}^{mn} \neq 0$. Hence this gives $\alpha_{mm} = 0$. $\square$

Theorem 2.5. *Taylor method is of type $M\,(\chi_r^{2q} : \chi_r^{2q})$.*

Proof. This follows from Theorem 2.4. $\square$

Theorem 2.6. *Let A be a four dimensional triangular matrix which is absolutely regular. Suppose that $M_{mn}(A)$ holds. Then $\left|a_{pp}^{qq}\right| \neq 0$ for all p, q.*

Proof. Since A is absolutely regular, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| \text{ is convergent } \forall k, \ell \leq m, n. \tag{2.3}$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| = 1, \quad \forall k, \ell \leq m, n. \tag{2.4}$$

Assume that for some positive integer P and Q

$$\sum_{m=1}^{P} \sum_{n=1}^{Q} |a_{k\ell}^{mn}| = 0, \quad \forall k, \ell. \tag{2.5}$$

Then, for any given $\epsilon > 0$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| \leq \sum_{m=1}^{P} \sum_{n=1}^{Q} |a_{k\ell}^{mn}| + \sum_{m=P+1}^{\infty} \sum_{n=Q+1}^{\infty} |a_{k\ell}^{mn}| < \epsilon$$

by (2.3) and (2.5). Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} = 0$, $\forall k,\ell$. This contradicts (2.4). This contradiction shows that

$$\sum_{m=1}^{P}\sum_{n=1}^{Q} a_{k\ell}^{mn} \neq 0 \text{ for some positive integer } P \text{ and } Q \text{ and } \forall k,\ell \leq m,n. \tag{2.6}$$

Since $M_{mn}(A)$ holds, we have

$$\left|\sum_{m=1}^{p}\sum_{n=1}^{q} a_{k\ell}^{mn} x_{mn}\right| \leq K \sup_{i\leq p,\, j\leq q} |y_{ij}| .$$

Take

$$x_{mn} = \begin{cases} 1, & \text{if } m,n = p,q \\ 0, & \text{otherwise.} \end{cases}$$

Then $\left|a_{kp}^{mq}\right| \leq K \left|a_{pp}^{qq}\right|$. This implies that $\sum_{m=1}^{P}\sum_{n=1}^{Q}\left|a_{kp}^{mq}\right| \leq KPQ\left|a_{pp}^{qq}\right|$. This is turn gives that $\left|a_{pp}^{qq}\right| \neq 0$ by using (2.6). □

Problem 2.1. *Give an example of a midly double conservative space, which is not semi conservative.*

Solution. Let T denote the set of all FK-spaces X with $F^f \subset c^2$. This is more restrictive then supposing that X^f is contained in the space of sequences summable $(C,1)$. Now cs^2 even belongs to T. But cs^2 is not semi conservative.

To prove this, let $x = \{x_{mn}\} \in cs^2$ write

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} & 0 \\ x_{21} & x_{22} & \dots & x_{2n} & 0 \\ & & \ddots & & \\ x_{m1} & x_{m2} & \dots & x_{mn} & 0 \end{pmatrix}.$$

The most general linear continuous functional on cs^2 is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} S_{mn} + \lambda \lim_{mn\to\infty} S_{mn}$$

where $\{\lambda_{mn}\} \in \ell^2$. If $x = \delta^{pq}$, then

$$S_{mn} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ & & \ddots & \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad \forall m,n \geq p,q$$

and so that $f\left(\delta^{pq}\right) = \sum_{m=p}^{\infty}\sum_{n=q}^{\infty} \lambda_{mn} + \lambda$. It follows easily that $(cs)^f$ is the set of all sequences $\{y_{mn}\}$ such that $\{\Delta y_{mn}\} \in \ell^2$. This implies that $\{y_{mn}\} \in c^2$; but does not imply that $\sum\sum y_{mn}$ converges. Thus $\{y_{mn}\} \in cs^2$.

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References

[1] T. Apostol, *Mathematical Analysis*, Addison-wesley , London, 1978.
[2] M. Basarir and O. Solancan, *On some double sequence spaces*, J. Indian Acad. Math., **21 (2)** (1999), 193–200.
[3] C. Bektas and Y. Altin, *The sequence space $\ell_M(p,q,s)$ on seminormed spaces*, Indian J. Pure Appl. Math., **34 (4)** (2003), 529–534.
[4] T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co.Ltd. , New York, 1965.
[5] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc., **19** (1917), 86–95.
[6] M. A. Krasnoselskii and Y. B. Rutickii, *Convex functions and Orlicz spaces*, Gorningen, Netherlands, 1961.
[7] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379–390.
[8] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc, **100 (1)** (1986), 161–166.
[9] F. Moricz, *Extentions of the spaces c and c_0 from single to double sequences*, Acta. Math. Hung., **57 (1-2)**, (1991), 129–136.
[10] F. Moricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104**, (1988), 283–294.
[11] M. Mursaleen, M. A. Khan and Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Math., **Vol. XXXII** (1999), 145–150.
[12] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5** (1953), 29–49.
[13] W. Orlicz, *Über Raume* (L^M), Bull. Int. Acad. Polon. Sci. A, (1936), 93–107.
[14] S. D. Parashar and B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math., **25 (4)** (1994), 419-428.
[15] K. Chandrasekhara Rao and N. Subramanian, *The Orlicz space of entire sequences*, Int. J. Math. Math. Sci., **68** (2004), 3755–3764.
[16] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25** (1973), 973–978.
[17] B. C. Tripathy, *On statistically convergent double sequences*, Tamkang J. Math., **34 (3)**, (2003), 231–237.
[18] B. C. Tripathy, M. Et and Y. Altin, *Generalized difference sequence spaces defined by Orlicz function in a locally convex space*, J. Anal. Appl., **1 (3)** (2003), 175–192.
[19] A. Turkmenoglu, *Matrix transformation between some classes of double sequences*, J. Inst. Math. Comp. Sci. Math. Ser., **12 (1)**, (1999), 23–31.
[20] P. K. Kamthan and M. Gupta, *Sequence spaces and series, Lecture notes, Pure and Applied Mathematics*, 65 Marcel Dekker, In c., New York, 1981.
[21] A. Gökhan and R. Çolak, *The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$*, Appl. Math. Comput., **157 (2)**, (2004), 491–501.
[22] A. Gökhan and R. Çolak, *Double sequence spaces ℓ_2^∞*, ibid., **160 (1)**, (2005), 147–153.
[23] M. Zeltser, *Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25*, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, **2001**.
[24] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288(1)**, (2003), 223-231.

[25] M. Mursaleen, *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl., **293(2)**, (2004), 523–531.

[26] M. Mursaleen and O. H. H. Edely, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293 (2)**, (2004), 532–540.

[27] B. Altay and F. Basar, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309(1)**, (2005), 70–90.

[28] F. Basar and Y. Sever, *The space $\mathcal{L}_p$ of double sequences*, Math. J. Okayama Univ, **51**, (2009), 149–157.

[29] N. Subramanian and U. K. Misra, *The semi normed space defined by a double gai sequence of modulus function*, Fasciculi Math., **46**, (2010).

[30] H. Kizmaz, *On certain sequence spaces*, Cand. Math. Bull., **24 (2)**, (1981), 169-176.

[31] B. Kuttner, *Note on strong summability*, J. London Math. Soc., **21** (1946), 118–122.

[32] I. J. Maddox, *On strong almost convergence*, Math. Proc. Cambridge Philos. Soc., **85 (2)**, (1979), 345–350.

[33] J. Cannor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32 (2)**, (1989), 194–198.

[34] A. Pringsheim, *Zurtheorie derzweifach unendlichen zahlenfolgen*, Math. Ann., **53**, (1900), 289–321.

[35] H. J. Hamilton, *Transformations of multiple sequences*, Duke Math. J., **2**, (1936), 29–60.

[36] H. J. Hamilton, *A Generalization of multiple sequences transformation*, Duke Math. J., **4**, (1938), 343–358.

[37] H. J. Hamilton, *Change of Dimension in sequence transformation*, Duke Math. J., **4**, (1938), 341–342.

[38] H. J. Hamilton, *Preservation of partial Limits in Multiple sequence transformations*, Duke Math. J., **4**, (1939), 293–297.

[39] G. M. Robison, *Divergent double sequences and series*, Amer. Math. Soc. Trans., **28**, (1926), 50–73.

[40] L. L. Silverman, *On the definition of the sum of a divergent series*, unpublished thesis, University of Missouri studies, Mathematics series.

[41] O. Toeplitz, *Über allgenmeine linear mittel bridungen*, Prace Matemalyczno Fizyczne (Warsaw), **22**, (1911).

[42] F. Basar and B. Atlay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55(1)**, (2003), 136-147.

[43] B. Altay and F. Basar, *The fine spectrum and the matrix domain of the difference operator Δ on the sequence space $\ell_p, (0 < p < 1)$*, Commun. Math. Anal., **2(2)**, (2007), 1-11.

[44] R. Çolak, M. Et and E. Malkowsky, *Some Topics of Sequence Spaces, Lecture Notes in Mathematics*, Firat Univ. Elazig, Turkey, **2004**, pp. 1–63, Firat Univ. Press, (2004), ISBN: 975-394-0386-6.

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