Kragujevac Journal of Mathematics
Volume 36 Number 2 (2012), Pages 261–268.

THE DOUBLE SEQUENCE SPACE $\ell_2(p, f, q, s)$

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Abstract. In this paper we introduce the double sequence space $\ell_2(p, f, q, s)$ on a seminormed complex linear space by using a modulus function. We further examine some properties of this space.

1. Introduction

The idea of modulus was structured in 1953 by Nakano [4]. Following Ruckle [6] and Maddox [3], we recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$,
(iii) f is increasing, and
(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$.

A double sequence $x = (x_{jk})$ of real numbers, $j, k \in \mathbb{N}$, the set of all positive integers, is said to be convergent in the Pringsheim's sense or P-convergent if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ whenever $j, k \geq N$ and L is called Pringsheim limit (denoted by P-$\lim x = L$) [5]. A double sequence x is bounded, i.e. $x \in \ell^2_\infty$, if there exists a positive number M such that $|x_{jk}| \leq M$ for all j and k.

Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $w^2(X)$ we denote the linear space of all double sequences $x = (x_{kl})$ with $x_{kl} \in X$ under the usual coordinate-wise operations:

Key words and phrases. Modulus function, Double sequence space, Seminormed space.
2010 *Mathematics Subject Classification.* 40A05, 40B05.
Received: November 15, 2011.

$\alpha x = (\alpha x_{kl})$ and $x + y = (x_{kl} + y_{kl})$ for each $\alpha \in \mathbb{C}$, the set of all complex numbers. Let $p = (p_{kl})$ be a double sequence of strictly positive real numbers and f be a modulus. We now define the following double sequence space; such type of space for single sequences was defined by Bilgin in [1]:

$$\ell_2(p, f, q, s) = \left\{ x \in w^2(X) : \sum_{k,l=1}^{\infty} (kl)^{-s} \left[f(q(x_{kl}))\right]^{p_{kl}} < \infty, \ s \geq 0 \right\}.$$

If we take $s = 0$, $f(x) = x$ and $q(x) = |x|$ (or $X = \mathbb{C}$), then we obviously get the sequence space $\ell_2(p)$ given in [2].

Let q_1, q_2 be seminorms on X. Then q_1 is said to be stronger than q_2 if there exists a constant L such that $q_2(x) \leq L q_1(x)$.

2. Linear topological structure of $\ell_2(p, f, q, s)$

Theorem 2.1. *$\ell_2(p, f, q, s)$ is a linear space if and only if $\sup_{k,l \geq 1} p_{kl} < \infty$.*

Proof. Let $H = \sup_{k,l \geq 1} p_{kl}$. Then for any $x, y \in \ell_2(p, f, q, s)$, we have

$$|x_{kl} + y_{kl}|^{p_{kl}} \leq C \left\{ |x_{kl}|^{p_{kl}} + |y_{kl}|^{p_{kl}} \right\} \tag{2.1}$$

where $C = \max\left(1, 2^{H-1}\right)$. Hence using this inequality and from the definition of f, we have

$$\begin{aligned}
(kl)^{-s} \{f(q(\lambda x_{kl} + \mu y_{kl}))\}^{p_{kl}} &\leq (kl)^{-s} \{f(q(\lambda x_{kl})) + f(q(\mu y_{kl}))\}^{p_{kl}} \\
&\leq (kl)^{-s} \{f(|\lambda| q(x_{kl})) + f(|\mu| q(y_{kl}))\}^{p_{kl}} \\
&\leq (kl)^{-s} C \{f(|\lambda| q(x_{kl})) + f(|\mu| q(y_{kl}))\}^{p_{kl}} \\
&\leq C \{1 + [|\lambda|]\}^H (kl)^{-s} [f(q(x_{kl}))]^{p_{kl}} \\
&+ C \{1 + [|\mu|]\}^H (kl)^{-s} [f(q(y_{kl}))]^{p_{kl}}
\end{aligned}$$

for any $\lambda, \mu \in \mathbb{C}$, where $[t]$ denotes the integer part of t. When adding the above inequality from $k, l = 1$ to ∞, we get $\lambda x + \mu y \in \ell_2(p, f, q, s)$. Hence $\ell_2(p, f, q, s)$ is a linear space.

Conversely, suppose that $\ell_2(p, f, q, s)$ is a linear space but $\sup_{k,l \geq 1} p_{kl} = \infty$. Then for all positive integers i, j, we have

$$p_{k(i),l(j)} > i + j$$

where both of the sequences $k(i)$ and $l(j)$ are strictly increasing sequences of positive integers or one of them is strictly increasing and the other is nondecreasing. In this case, choose $f(x) = x$, $q(x) = |x|$ and consider the sequence (x_{kl}) where

$$x_{kl} = \begin{cases} (kl)^{s/p_{kl}} \, 2^{-(i+j)/p_{kl}}, & \text{if } k = k(i) \text{ and } l = l(j); \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{k,l=1}^{\infty}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}=\sum_{i,j=1}^{\infty}\frac{1}{2^{i+j}}<\infty$$

whence $x\in\ell_2\left(p,f,q,s\right)$. On the other hand,

$$\sum_{k,l=1}^{\infty}(kl)^{-s}\left[f\left(q\left(2x_{kl}\right)\right)\right]^{p_{kl}}=\sum_{i,j=1}^{\infty}2^{p_{k(i),l(j)}}\frac{1}{2^{i+j}}>\sum_{i,j=1}^{\infty}2^{i+j}\frac{1}{2^{i+j}}=\infty$$

whence $2x\notin\ell_2\left(p,f,q,s\right)$ and this contradicts with the fact that $\ell_2\left(p,f,q,s\right)$ is a linear space. Therefore, $\sup_{k,l\geq1}p_{kl}$ must be finite. This completes the proof. □

Theorem 2.2. *Let* $H=\sup\limits_{k,l\geq1}p_{kl}<\infty$ *and* $M=\max(1,H)$. *Then*

(i) $\ell_2\left(p,f,q,s\right)$ *is a paranormed space with* $g(x)=\left\{\sum\limits_{k,l=1}^{\infty}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}\right\}^{1/M}$.

(ii) *If* (X,q) *is complete, then* $\ell_2\left(p,f,q,s\right)$ *is a complete linear metric space with the paranorm* g *defined in* (i).

Proof. (i) Clearly $g(\overline{\theta})=0$ and $g(-x)=g(x)$ for all $x\in\ell_2\left(p,f,q,s\right)$, where $\overline{\theta}=(\theta,\theta,\dots)$. It also follows from (2.1), Minkowski's inequality and the definition of f that g is subadditive and

$$g\left(\lambda x\right)\leq K_{\lambda}^{H/M}g\left(x\right)$$

where K_λ is an integer such that $|\lambda|\leq K_\lambda$. Therefore the function $(\lambda,x)\to\lambda x$ is continuous at $\lambda=0$, $x=\overline{\theta}$ and when λ is fixed, the function $x\to\lambda x$ is continuous at $x=\overline{\theta}$. If x is fixed and $\varepsilon>0$, we can choose k_0 and l_0 such that

$$\begin{aligned}R(x)=\sum_{k=1}^{k_0}\sum_{l=l_0+1}^{\infty}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}&+\sum_{k=k_0+1}^{\infty}\sum_{l=1}^{l_0}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}\\&+\sum_{k=k_0+1}^{\infty}\sum_{l=l_0+1}^{\infty}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}<\frac{\varepsilon}{2}.\end{aligned}$$

Thus $R(\lambda x)<\varepsilon/2$ since $|\lambda|<1$ and $\delta>0$, so that $|\lambda|<\delta$ gives

$$\sum_{k=1}^{k_0}\sum_{l=1}^{l_0}(kl)^{-s}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}<\frac{\varepsilon}{2}.$$

Therefore $|\lambda|<\min\left(1,\delta\right)$ implies that $g(\lambda x)<\varepsilon/2$. Thus the function $\lambda\to\lambda x$ is continuous at $\lambda=0$ and so $\ell_2\left(p,f,q,s\right)$ is a paranormed space.

(ii) Let $x^{ij}=(x_{kl}^{ij})$ be a Cauchy sequence in $\ell_2\left(p,f,q,s\right)$, i.e, for every $\varepsilon>0$ $(0<\varepsilon<1)$ there exists a positive integer $N=N(\varepsilon)$ such that

$$g(x^{ij}-x^{rt})=\left\{\sum_{k,l=1}^{\infty}(kl)^{-s}\left[f\left(q\left(x_{kl}^{ij}-x_{kl}^{rt}\right)\right)\right]^{p_{kl}}\right\}^{1/M}<\varepsilon \tag{2.2}$$

for all $i, j, r, t > N$. Since f is modulus and

$$\left\{(kl)^{-s}\left[f\left(q\left(x_{kl}^{ij}-x_{kl}^{rt}\right)\right)\right]^{p_{kl}}\right\}^{1/M} \leq g(x^i - x^j)$$

for each fixed k and l, it follows that

$$q\left(x^{ij}-x^{rt}\right) < \varepsilon$$

for all $i, j, r, t > N$. Hence (x_{kl}^{ij}) is a Cauchy sequence in X and therefore $(x_{kl}^{ij}) \to y_{kl}$ as $i, j \to \infty$ for each fixed k and l. Put $y = (y_{kl})$. From (2.2), we may write $\sum_{k,l=1}^{P,Q} (kl)^{-s}\left[f\left(q\left(x_{kl}^{i}-x_{kl}^{j}\right)\right)\right]^{p_{kl}} < \varepsilon^M$ $(P, Q = 1, 2, \ldots)$, for all $i, j, r, t > N$. Since f is continuous, letting $r, t \to \infty$ and then $P, Q \to \infty$ we obtain

$$\sum_{k,l=1}^{P,Q} (kl)^{-s}\left[f\left(q\left(x_{kl}^{ij}-y_{kl}\right)\right)\right]^{p_{kl}} < \varepsilon^M$$

for all $i, j > N$. Therefore we have $g\left(x^{ij}-y\right) < \varepsilon$ for all $i, j > N$ and this proves that $g\left(x^{ij}-y\right) \to 0$ and $y \in \ell_2(p, f, q, s)$. Hence $\ell_2(p, f, q, s)$ is complete. □

3. Inclusion relations

Theorem 3.1. *Let f, f_1, f_2 be modulus functions, q, q_1, q_2 be seminorm functions and let $s, s_1, s_2 \geq 0$.*

(i) *If $s > 1$, then $\ell_2(p, f, q, s) \subset \ell_2(p, f \circ f_1, q, s)$, where $f \circ f_1$ denotes the composition of f_1 and f.*
(ii) *$\ell_2(p, f_1, q, s) \cap \ell_2(p, f_2, q, s) \subset \ell_2(p, f_1 + f_2, q, s)$.*
(iii) *$\ell_2(p, f, q_1, s) \cap \ell_2(p, f, q_2, s) \subset \ell_2(p, f, q_1 + q_2, s)$.*
(iv) *If q_1 is stronger than q_2, then $\ell_2(p, f, q_1, s) \subset \ell_2(p, f, q_2, s)$.*
(v) *If $\limsup_{t\to\infty} \frac{f_1(t)}{f_2(t)} < \infty$, then $\ell_2(p, f_2, q, s) \subset \ell_2(p, f_1, q, s)$.*
(vi) *If $s_1 \leq s_2$, then $\ell_2(p, f, q, s_1) \subset \ell_2(p, f, q, s_2)$.*

Proof. (i) Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $t_{kl} = f_1(q(x_{kl}))$ and consider

$$\sum_{k,l=1}^{\infty} (kl)^{-s}\left[f(t_{kl})\right]^{p_{kl}} = \sum_{\substack{k,l=1 \\ t_{kl}\leq\delta}}^{\infty} (kl)^{-s}\left[f(t_{kl})\right]^{p_{kl}} + \sum_{\substack{k,l=1 \\ t_{kl}>\delta}}^{\infty} (kl)^{-s}\left[f(t_{kl})\right]^{p_{kl}}.$$

For $t_{kl} > \delta$, if we use the fact that

$$t_{kl} < \frac{t_{kl}}{\delta} < 1 + \frac{t_{kl}}{\delta},$$

it follows that

$$f(t_{kl}) < 2f(1)\frac{t_{kl}}{\delta},$$

since f is increasing. Now let $x \in \ell_2(p,f,q,s)$ and $s>1$. Then we have

$$\begin{aligned}\sum_{k,l=1}^{\infty}(kl)^{-s}\left[f\left(t_{kl}\right)\right]^{p_{kl}} &\leq \max\left(1,\varepsilon^{H}\right)\sum_{k,l=1}^{\infty}(kl)^{-s}\\ &+\max\left(1,\left[\frac{2f(1)}{\delta}\right]^{H}\right)\sum_{k,l=1}^{\infty}(kl)^{-s}\left[t_{kl}\right]^{p_{kl}}<\infty.\end{aligned}$$

This completes the proof of (i).

The proofs of (ii) and (iii) follow from inequality (2.1). Also, (iv), (v) and (vi) follow easily. □

We get the following sequence spaces from $\ell_2(p,f,q,s)$ by choosing some of the special p,f,q and s :

$$\ell_2(p,q,s)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}(kl)^{-s}\left[q\left(x_{kl}\right)\right]^{p_{kl}}<\infty,\ s\geq 0\right\},$$
$$\ell_2(f,q,s)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}(kl)^{-s}\left[q\left(x_{kl}\right)\right]<\infty,\ s\geq 0\right\},$$
$$\ell_2(p,f,q)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}\left[f\left(q\left(x_{kl}\right)\right)\right]^{p_{kl}}<\infty,\ s\geq 0\right\},$$
$$\ell_2(p,q)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}\left[q\left(x_{kl}\right)\right]^{p_{kl}}<\infty,\ s\geq 0\right\},$$
$$\ell_2(f,q)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}\left[f\left(q\left(x_{kl}\right)\right)\right]<\infty,\ s\geq 0\right\},$$
$$\ell_2(q)=\left\{x\in w^2(X):\sum_{k,l=1}^{\infty}q\left(x_{kl}\right)<\infty,\ s\geq 0\right\}.$$

From Theorem 3.1, we get the following inclusion relations for these newly defined sequence spaces.

Corollary 3.1.

(i) *If $s>1$ and f is any modulus, then $\ell_2(p,q,s)\subset\ell_2(p,f,q,s)$.*
(ii) $\ell_2(p,f,q)\subset\ell_2(p,f,q,s)$.
(iii) $\ell_2(p,q)\subset\ell_2(p,q,s)$.
(iv) $\ell_2(f,q)\subset\ell_2(f,q,s)$.

Now, let $c_0^2(q)$ and $\ell_\infty^2(q)$ denote the space of all bounded and null sequences in Pringsheim's sense, respectively. That is,

$$\begin{aligned}c_0^2(q) &= \left\{x\in w^2(X): P\text{-}\lim_{k,l\to\infty}q\left(x_{kl}\right)=0\right\},\\ \ell_\infty^2(q) &= \left\{x\in w^2(X):\sup_{kl}q\left(x_{kl}\right)<\infty\right\}.\end{aligned}$$

These spaces are given by Tripathy and Sarma in [7].

Theorem 3.2.

(i) $\ell_2(f,q) \subset \ell_2(q)$.

(ii) $\ell_2(p,f,q) \subset c_0^2(q)$.

Proof. (i) Assume that $x \in \ell_2(f,q)$ but $x \notin \ell_2(q)$. Then there are two cases:
(a) There exist strictly increasing sequences $(m(i))$ of positive integers and $n(1) < n(2) < \cdots < n(j_0)$ for some fixed $j_0 \in \mathbb{N}$ such that

$$\sum_{k=m(i-1)+1}^{m(i)} \sum_{k=n(j-1)+1}^{n(j)} q(x_{kl}) \geq 1 \tag{3.1}$$

for all positive integers i and for $1 \leq j \leq j_0$, where $m(0) = 0$ and $n(0) = 0$ (or there exist strictly increasing sequences $(n(j))$ of positive integers and $m(1) < m(2) < \cdots < m(i_0)$ for some fixed $i_0 \in \mathbb{N}$ such that (3.1) holds for all positive integers j and for $1 \leq i \leq i_0$). Then we have

$$\begin{aligned} f(1) &\leq f\left(\sum_{k=m(i-1)+1}^{m(i)} \sum_{k=n(j-1)+1}^{n(j)} q(x_{kl})\right) \\ &\leq \sum_{k=m(i-1)+1}^{m(i)} \sum_{k=n(j-1)+1}^{n(j)} f(q(x_{kl})) \end{aligned}$$

for each i, j. But since $x \in \ell_2(f,q)$, we have

$$P\text{-}\lim_{i,j\to\infty} \sum_{k=m(i-1)+1}^{m(i)} \sum_{k=n(j-1)+1}^{n(j)} f(q(x_{kl})) = 0,$$

which implies that $f(1) = 0$. This contradicts with the definition of f.
(b) There exist strictly increasing sequences $(m(i))$ and $(n(j))$ such that (3.1) holds for all positive integers i, j. The proof of this case is similar to that of (a).

(ii) Let $x \in \ell_2(p,f,q)$. Then there are integers k_0 and l_0 such that

$$[f(q(x_{kl}))]^{p_{kl}} \leq 1$$

for all $k > k_0$ and $l > l_0$. So

$$[f(q(x_{kl}))]^{H} < [f(q(x_{kl}))]^{p_{kl}}, \quad \text{for } k > k_0 \text{ and } l > l_0.$$

Hence

$$P\text{-}\lim_{k,l\to\infty} [f(q(x_{kl}))]^{H} = 0$$

and since f is modulus, we have

$$P\text{-}\lim_{k,l\to\infty} q(x_{kl}) = 0$$

which completes the proof. □

Theorem 3.3. *Let* $0 < p_{kl} < t_{kl} < \infty$. *Then* $\ell_2(p,f,q) \subset \ell_2(t,f,q)$.

Proof. The proof is easy. So we omit it. □

Corollary 3.2.

(i) *If* $0 < p_{kl} \le 1$ *for all* k *and* l*, then* $\ell_2(p,f,q) \subset \ell_2(f,q)$.
(ii) *If* $p_{kl} \ge 1$ *for all* k *and* l*, then* $\ell_2(f,q) \subset \ell_2(p,f,q)$.
(iii) *If* $0 < p_{kl} \le 1$ *for all* k *and* l*, then* $\ell_2(p,q) \subset \ell_2(q)$.
(iv) *If* $p_{kl} \ge 1$ *for all* k *and* l*, then* $\ell_2(q) \subset \ell_2(p,q)$.

Theorem 3.4. *Let* $s > 1$*. Then,*

(i) $\ell^2_\infty(q) \subset \ell_2(p,f,q,s)$.
(ii) *If* f *is bounded, then* $\ell_2(p,f,q,s) \equiv w^2(X)$.
(iii) *If* q *is bounded, then* $\ell_2(p,q,s) \equiv \ell_2(p,f,q,s) \equiv w^2(X)$.

Proof. (i) If $x = (x_{kl}) \in \ell^2_\infty(q)$, then there exists $K > 0$ such that $q(x_{kl}) \le 1 + [K]$ for k and l. Hence $s > 1$ implies

$$\sum_{k,l=1}^{\infty} (kl)^{-s} [f(q(x_{kl}))]^{p_{kl}} \le (1+[K])^H \sup_{k,l} [f(1)]^{p_{kl}} \sum_{k,l=1}^{\infty} (kl)^{-s} < \infty,$$

which proves (i).

(ii) Let $s > 1$ and f be bounded. Then for any $x \in w^2(X)$, we have

$$\sum_{k,l=1}^{\infty} (kl)^{-s} [f(q(x_{kl}))]^{p_{kl}} \le \max\left(1, [\sup\{f(t) : t \ge 0\}]^H\right) \sum_{k,l=1}^{\infty} (kl)^{-s} < \infty,$$

which proves that $\ell_2(p,f,q,s) \equiv w^2(X)$.

(iii) Let $s > 1$ and q be bounded. Then for every $u \in X$ there exists a positive constant K such that $q(u) \le K$. Therefore for any $x \in w^2(X)$,

$$\sum_{k,l=1}^{\infty} (kl)^{-s} [q(x_{kl})]^{p_{kl}} \le \max\left(1, K^H\right) \sum_{k,l=1}^{\infty} (kl)^{-s} < \infty,$$

so that $\ell_2(p,q,s) \equiv w^2(X)$. Also, since f is continuous and increasing, for any $x \in w^2(X)$,

$$\sum_{k,l=1}^{\infty} (kl)^{-s} [f(q(x_{kl}))]^{p_{kl}} \le \max\left(1, [f(K)]^H\right) \sum_{k,l=1}^{\infty} (kl)^{-s} < \infty.$$

Hence $\ell_2(p,q,s) \equiv \ell_2(p,f,q,s) \equiv w^2(X)$. □

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