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A GENERAL SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this present work, we consider a general subclass $C^*[A,B]$ of close-to-convex functions, which denote by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $U = \{z : |z| < 1\}$ and which satisfy the following condition:

$$\left|\frac{(zf'(z))'}{g'(z)} - 1\right| < \left|A - B\frac{(zf'(z))'}{g'(z)}\right|, \qquad (-1 \leqslant B < A \leqslant 1),$$

where $g(z)$ is convex univalent function in U. It gives a sufficient condition for functions to belong to the class investigated. Moreover, we derive some properties including the coefficient bounds as well as distortion theorem. Some radius problems and relationship with other class are also solved. The results presented here would generalize many earlier work.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathcal{C}, |z| < 1\}$.

Let $\mathcal{K}$ and C be the usual classes of convex and close-to-convex functions in U, respectively.

Suppose f, g are analytic in the unit disk U. f is said to be subordinate to g, written $f \prec g$, if there exists an analytic function $\omega(z)$ with $|\omega(z)| \leqslant |z| < 1$, such that $f(z) = g(\omega(z))$.

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A function $f(z) \in \mathcal{A}$ is said to be in the $S^*[A,B]$ if it also satisfies the following relationship:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \qquad -1 \leqslant B < A \leqslant 1,\ z \in U. \tag{1.2}$$

In [11], Silvia defined and studied the class

$$C_\beta[A,B] = \left\{ f \in \mathcal{A} \mid \Re e \frac{zf'(z)}{g(z)} > \beta, 0 \leqslant \beta < 1, g(z) \in S^*[A,B], z \in U \right\}.$$

K.I. Noor [5] extended the results of Silvia by investigating the class

$$C_\beta^*[A,B] = \left\{ f \in \mathcal{A} \mid \Re e \frac{(zf'(z))'}{g'(z)} > \beta, 0 \leqslant \beta < 1, g(z) \in S^*[A,B], z \in U \right\}.$$

In 2004, C. Selvaraj [10] studied various properties of the class C' of functions $f \in \mathcal{A}$ satisfying

$$\Re e \frac{zf'(z)}{g(z)} > 0, \qquad g(z) \in \mathcal{K}, z \in U.$$

Furthermore, Z.G. Peng [9] and B.S. Mehork et al [4] defined and discussed more general class $C'[\alpha,\beta]$ and $J[A,B]$, respectively. Here,

$$C'[\alpha,\beta] = \left\{ f \in \mathcal{A} | \frac{zf'(z)}{g(z)} \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}, g(z) \in \mathcal{K}, 0 \leqslant \alpha < 1, 0 < \beta \leqslant 1, z \in U \right\},$$

$$J[A,B] = \left\{ f \in \mathcal{A} | \frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, g(z) \in \mathcal{K}, -1 \leqslant B < A \leqslant 1, z \in U \right\}.$$

In this paper, we consider the class $C^*[A,B]$ of functions $f(z) = \sum\limits_{n=2}^{\infty} a_n z^n$ analytic in U satisfying for some $g(z) \in \mathcal{K}$ the condition

$$\left| \frac{(zf'(z))'}{g'(z)} - 1 \right| < \left| A - B\frac{(zf'(z))'}{g'(z)} \right|, \qquad (-1 \leqslant B < A \leqslant 1). \tag{1.3}$$

By simple calculations we see that the inequality (1.3) is equivalent to

$$\frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}, \qquad -1 \leqslant B < A \leqslant 1, z \in U, \tag{1.4}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}. \tag{1.5}$$

In particular, $C^*[1,-1] \equiv C^*(0), C^*[1-2\beta,-1] \equiv C^*(\beta)$, $(0 \leqslant \beta < 1)$. $C^*(0)$ and $C^*(\beta)$ have been investigated in [6] and [7].

It is clear that

$$C^*[A,B] \subset J[A,B] \subset C' \subset C.$$

Obviously, the functions in $C^*[A,B]$ are close-to-convex and hence univalent.

2. Preliminary results

In order to discuss the class $C^*[A,B]$, we need the following important lemmas:

Lemma 2.1. [3] *If a function*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \prec \frac{1+Az}{1+Bz}, \qquad -1 \leqslant B < A \leqslant 1,$$

then $|p_n| \leqslant A - B$. *The inequality is sharp.*

Lemma 2.2. [2] *If a function*

$$p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}, \qquad -1 \leqslant B < A \leqslant 1,$$

and $\omega(z)$ *is regular in* U *with* $\omega(0) = 0$ *and* $|\omega(z)| < 1$, *then on* $|z| = r < 1$

$$\mathcal{R}e \frac{zp'(z)}{p(z)} \geqslant \begin{cases} -\frac{(A-B)r}{(1-Ar)(1-Br)}, & R_1 \leqslant R_2, \\ 2\frac{\sqrt{(1-B)(1-A)(1+Ar^2)(1+Br^2)}-(1-ABr^2)}{(A-B)(1-r^2)} + \frac{A+B}{A-B}, & R_1 \geqslant R_2. \end{cases}$$

where

$$R_1 = \frac{\sqrt{(1-A)(1+Ar^2)}}{\sqrt{(1-B)(1+Br^2)}}, \quad R_2 = \frac{1-Ar}{1-Br}.$$

Lemma 2.3. [1, 8] *Let* N *and* D *be analytic in* U, D *maps onto a many-sheeted starlike region,* $N(0) = 0 = D(0)$. *Then*

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz} \Rightarrow \frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Lemma 2.4. *Let* $g(z)$ *be a convex function, and definite* $G(z) = \int_0^z \frac{g(t)}{t} dt$. *Then* $G(z) \in \mathcal{K}$.

Proof. Since

$$G(z) = \int_0^z \frac{g(t)}{t} dt,$$

we have

$$\frac{(zG'(z))'}{G'(z)} = \frac{zg'(z)}{g(z)}. \tag{2.1}$$

Now, $g(z) \in \mathcal{K}$, so

$$\frac{(zg'(z))'}{g'(z)} \prec \frac{1+z}{1-z}. \tag{2.2}$$

Using (2.1), (2.2) and Lemma 2.3, we have

$$\frac{(zG'(z))'}{G'(z)} = \frac{zg'(z)}{g(z)} \prec \frac{1+z}{1-z},$$

which completes the proof of Lemma 2.4. □

3. Main results and their demonstrations

Theorem 3.1. *If a analytic function f in U defined by* (1.1) *satisfies the inequality*

$$\sum_{n=2}^{\infty} n^2(1+|B|)|a_n| + \sum_{n=2}^{\infty} n(1+|A|)|b_n| \leqslant A-B, \quad -1 \leqslant B < A \leqslant 1, \tag{3.1}$$

where for $n = 2, 3, \ldots$, the coefficients b_n are given by (1.5), *then $f \in C^*[A,B]$.*

Proof. We set f to be given by (1.1) and $g \in \mathcal{K}$ defined by (1.5). Then

$$\begin{aligned}
\mathfrak{T} &= |(zf'(z))' - g'(z)| - |Ag'(z) - B(zf'(z))'| \\
&= |\sum_{n=2}^{\infty} n^2 a_n z^{n-1} - \sum_{n=2}^{\infty} n b_n z^{n-1}| - |(A-B) + \sum_{n=2}^{\infty} A n b_n z^{n-1} + \sum_{n=2}^{\infty} (-B) n^2 a_n z^{n-1}| \\
&\leqslant \sum_{n=2}^{\infty} n^2 |a_n||z|^{n-1} + \sum_{n=2}^{\infty} n|b_n||z|^{n-1} \\
&- [(A-B) - \sum_{n=2}^{\infty} |A| n |b_n||z|^{n-1} - \sum_{n=2}^{\infty} |B| n^2 |a_n||z|^{n-1}] \\
&= -(A-B) + \sum_{n=2}^{\infty} (n^2|a_n| + n^2|B||a_n|)|z|^{n-1} + \sum_{n=2}^{\infty} (n|b_n| + |A|n|b_n|)|z|^{n-1}.
\end{aligned}$$

Hence, for $z \in U$, using (3.1) we have

$$\begin{aligned}
\mathfrak{T} &\leqslant -(A-B) + \sum_{n=2}^{\infty} (n^2|a_n| + n^2|B||a_n|) + \sum_{n=2}^{\infty} (n|b_n| + |A|n|b_n|) \\
&= -(A-B) + \sum_{n=2}^{\infty} n^2(1+|B|)|a_n| + \sum_{n=2}^{\infty} n(1+|A|)|b_n| \\
&\leqslant 0.
\end{aligned}$$

From the above calculation we have

$$|(zf'(z))' - g'(z)| < |Ag'(z) - B(zf'(z))'|,$$

which is equivalent to the inequality (1.3). Thus $f \in C^*[A,B]$. $\square$

Theorem 3.2. *If the function $f(z) \in C^*[A,B]$ is defined by* (1.1), *then*

$$|a_n| \leqslant \frac{1}{n} + \frac{n-1}{2n}(A-B), \qquad n \geqslant 2.$$

The result is sharp.

Proof. Suppose the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C^*[A,B]$. Then according to the definition, there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}$, such that

$$\frac{(zf'(z))'}{g'(z)} = p(z) \prec \frac{1+Az}{1+Bz}. \tag{3.2}$$

So $(zf'(z))' = g'(z)p(z)$. Further, by letting $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$ and using (3.2), we have

$$1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} = (1 + \sum_{n=2}^{\infty} n b_n z^{n-1})(1 + \sum_{n=1}^{\infty} p_n z^n), \qquad n \geqslant 2. \tag{3.3}$$

Equating the coefficients of z^{n-1} in (3.3), then

$$n^2 a_n = p_{n-1} + 2b_2 p_{n-2} + 3b_3 p_{n-3} + \cdots + (n-1)b_{n-1}p_1 + nb_n, \qquad n \geqslant 2.$$

Using the Lemma 2.1, we have

$$\begin{aligned} n^2|a_n| &\leqslant n|b_n| + (A-B)[1 + 2|b_2| + 3|b_3| + \cdots + (n-1)|b_{n-1}|] \\ &\leqslant n + (A-B)[1 + 2 + 3 + 4 + \dots. + (n-1)] \\ &= n + (A-B)\frac{n(n-1)}{2}, \qquad n \geqslant 2. \end{aligned}$$

So

$$|a_n| \leqslant \frac{1}{n} + \frac{n-1}{2n}(A-B), \qquad n \geqslant 2.$$

For $n = 2$, equality holds with the functions $f_n(z)$ defined by

$$(zf_n'(z))' = \frac{1}{(1-x_1 z)^2}\frac{1 + Ax_2 z^n}{1 + Bx_2 z^n}, \quad |x_1| = 1, \quad |x_2| = 1.$$

□

Remark 3.1.

(I) As the special case when $A = 1$ and $B = -1$ in Theorem 3.2, the result was proved by K.I. Noor [6, Theorem 2.2(i)].

(II) If we set $A = 1 - 2\beta$ $(0 \leqslant \beta < 1)$, and $B = -1$ in Theorem 3.2, we obtain the result proved by K.I. Noor [6, Theorem 4.4(i)].

Theorem 3.3. *If $f \in C^*[A, B], -1 \leqslant B \leqslant A \leqslant 1$, $|z| = r < 1$, then*

(a) *If $B \neq -1$, then $\mathfrak{L}_1 \leqslant |f'(z)| \leqslant \mathfrak{L}_2, \mathfrak{L}_3 \leqslant |f(z)| \leqslant \mathfrak{L}_4$, where*

$$\mathfrak{L}_1 = \frac{A-B}{r(1+B)^2}\ln\frac{1-Br}{1+r} + \frac{1+A}{(1+B)(1+r)}, \tag{3.4}$$

$$\mathfrak{L}_2 = \frac{A-B}{r(1+B)^2}\ln\frac{1-r}{1+Br} + \frac{1+A}{(1+B)(1-r)},$$

$$\mathfrak{L}_3 = \int_0^r \left[\frac{A-B}{t(1+B)^2}\ln\frac{1-Bt}{1+t} + \frac{1+A}{(1+B)(1+t)}\right]dt, \tag{3.5}$$

$$\mathfrak{L}_4 = \int_0^r \left[\frac{A-B}{t(1+B)^2}\ln\frac{1-t}{1+Bt} + \frac{1+A}{(1+B)(1-t)}\right]dt.$$

(b) *If* $B=-1$, *then* $\mathcal{M}_1 \leqslant |f'(z)| \leqslant \mathcal{M}_2, \mathcal{M}_3 \leqslant |f(z)| \leqslant \mathcal{M}_4$, *where*

$$\mathcal{M}_1 = -\frac{1+A}{2r(1+r)^2} + \frac{A}{r(1+r)} + \frac{1}{2r}(1-A), \tag{3.6}$$

$$\mathcal{M}_2 = \frac{1+A}{2r(1-r)^2} - \frac{A}{r(1-r)} + \frac{1}{2r}(A-1),$$

$$\mathcal{M}_3 = \int_0^r \left[-\frac{1+A}{2t(1+t)^2} + \frac{A}{t(1+t)} + \frac{1}{2t}(1-A) \right] dt, \tag{3.7}$$

$$\mathcal{M}_4 = \int_0^r \left[\frac{1+A}{2t(1-t)^2} - \frac{A}{t(1-t)} + \frac{1}{2t}(A-1) \right] dt.$$

Estimates are sharp.

Proof. As $f \in C^*[A,B]$, there exists a function $g \in \mathcal{K}$, such that $(zf'(z))' = g'(z)h(z)$, where

$$h(z) \prec \frac{1+Az}{1+Bz}.$$

It is easy to see that

$$\frac{1-Ar}{1-Br} \leqslant |h(z)| \leqslant \frac{1+Ar}{1+Br}.$$

Again, $g(z) \in \mathcal{K}$ implies that (see [12])

$$\frac{1}{(1+r)^2} \leqslant |g'(z)| \leqslant \frac{1}{(1-r)^2}.$$

Setting $F(z) = zf'(z)$, we have

$$\frac{1-Ar}{(1-Br)(1+r)^2} \leqslant |F'(z)| \leqslant \frac{1+Ar}{(1+Br)(1-r)^2}. \tag{3.8}$$

For $z = re^{i\theta}$, we have

$$F(z) = \int_0^r F'(\sigma e^{i\theta}) e^{i\theta} d\sigma. \tag{3.9}$$

Following the (3.8) and (3.9), we get

$$\begin{aligned} |F(z)| \leqslant \int_0^r |F'(\sigma e^{i\theta})| d\sigma &\leqslant \int_0^r \frac{1+A\sigma}{(1+B\sigma)(1-\sigma)^2} d\sigma \\ &\leqslant \begin{cases} \frac{A-B}{(1+B)^2} \ln \frac{1-r}{1+Br} + \frac{1+A}{1+B}\frac{r}{1-r}, & B \neq -1, \\ \frac{1+A}{2(1-r)^2} - \frac{A}{1-r} + \frac{1}{2}(A-1) & B=-1. \end{cases} \end{aligned} \tag{3.10}$$

To prove the lower bound of $|F(z)|$ we proceed as follows. Let δ be the radius of the open disc contained in the map of U by $F(z)$. Let z_0 be the point of $|z| = r$ for which $|F(z)|$ attains its minimum value. This minimum increases with r and is less than δ. Hence, the linear segment Γ connecting the origin with the point $F(z_0)$ will be covered entirely by the values of $F(z)$ in U. Let γ be the arc in U which is mapped by $\omega = F(z)$ onto this liner segment. Then

$$\begin{aligned} |F(z)| \geqslant |F(z_0)| = \int_\Gamma |d\omega| &= \int_\gamma |F'(z)dz| \geqslant \int_\gamma |F'(z)|d|z| \\ &\geqslant \int_0^r \frac{1-A\sigma}{(1-B\sigma)(1+\sigma)^2} d\sigma \\ &\geqslant \begin{cases} \frac{A-B}{(1+B)^2} \ln \frac{1-Br}{1+r} + \frac{1+A}{1+B}\frac{r}{1+r} & B \neq -1, \\ -\frac{1+A}{2(1+r)^2} + \frac{A}{1+r} + \frac{1}{2}(1-A) & B = -1. \end{cases} \end{aligned} \tag{3.11}$$

Since $F(z) = zf'(z)$, combining (3.10) and (3.11) we get the required result (3.4) and (3.6). Furthermore, with the same progress, if we integrate (3.4) and (3.6), then (3.5) and (3.7) follows.

For $n = 2$, equality holds with the functions $f_n(z)$ defined by

$$(zf_n'(z))' = \frac{1}{(1-x_1z)^2}\frac{1+Ax_2z^n}{1+Bx_2z^n}, \quad |x_1| = 1, \quad |x_2| = 1.$$

□

Remark 3.2.

(I) As the special case when $A = 1$ and $B = -1$ in Theorem 3.3, the result was proved by K.I. Noor [6, Theorem 2.2(ii,iii)].

(II) If setting $A = 1 - 2\beta$ $(0 \leqslant \beta < 1)$ and $B = -1$ in Theorem 3.3, we obtain the result proved by K.I. Noor [6, Theorem 4.4(ii)].

Theorem 3.4. *Let $F(z) = zf'(z)$, where $f(z) \in C^*[A, B]$ $(-1 \leqslant B < A \leqslant 1)$, then*

$$\mathfrak{Re}\frac{(zF'(z))'}{F'(z)} \geqslant \begin{cases} \frac{1-r}{1+r} - \frac{(A-B)r}{(1-Ar)(1-Br)}, & R_1 \leqslant R_2, \\ \frac{1-r}{1+r} + 2\frac{\sqrt{(1-B)(1-A)(1+Ar^2)(1+Br^2)}-(1-ABr^2)}{(A-B)(1-r^2)} + \frac{A+B}{A-B}, & R_1 \geqslant R_2, \end{cases}$$

where R_1, R_2 are as defined in Lemma 2.2 above.

Proof. Since $f(z) \in C^*[A, B]$, we can write

$$\frac{(zf'(z))'}{g'(z)} = h(z) \prec \frac{1+Az}{1+Bz}, \qquad g(z) \in \mathcal{K}.$$

Hence $F(z) = zf'(z)$. So

$$zF'(z) = z(zf'(z))' = zg'(z)\frac{(zf'(z))'}{g'(z)} = zg'(z)h(z).$$

Now

$$\frac{(zF'(z))'}{F'(z)} = \frac{(zg'(z))'h(z) + zg'(z)h'(z)}{g'(z)h(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)}. \tag{3.12}$$

Further $g(z)$ is a convex function, which is sufficient to show

$$\frac{(zg'(z))'}{g'(z)} \prec \frac{1+z}{1-z}. \tag{3.13}$$

It is well known that the images of closed disc $|z| \leqslant r$ under all the transformation $\omega = p(z)$ where $p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}$ with $\omega(z)$ regular in U, $\omega(0) = 0$ and $|\omega(z)| < 1$ in U, are contained in the closed disc with the centre C and the radius P where

$$C = \frac{1-ABr^2}{1-B^2r^2}, \qquad P = \frac{(A-B)r}{1-B^2r^2}.$$

So following (3.13), we have

$$\left|\frac{(zg'(z))'}{g'(z)} - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2}.$$

Thus

$$\Re e\frac{(zg'(z))'}{g'(z)} \geqslant \frac{1-r}{1+r}. \tag{3.14}$$

From Lemma 2.2 and $h(z) \in P[A,B]$, we obtain

$$\Re e\frac{zh'(z)}{h(z)} \geqslant \begin{cases} -\frac{(A-B)r}{(1-Ar)(1-Br)}, & R_1 \leqslant R_2, \\ 2\frac{\sqrt{(1-B)(1-A)(1+Ar^2)(1+Br^2)}-(1-ABr^2)}{(A-B)(1-r^2)} + \frac{A+B}{A-B}, & R_1 \geqslant R_2, \end{cases} \tag{3.15}$$

where R_1, R_2 are as defined in Lemma 2.2 above. Combining (3.14) and (3.15) with (3.12), we get the required result. □

Theorem 3.5. *If $F(z) = zf'(z)$ and $f(z) \in C^*[A,B]$, then*

(I) *For $\frac{3-\sqrt{5}}{2} \leqslant A \leqslant 1$, $F(z)$ is a convex function in $|z| < r_0$, where r_0 is the smallest positive root of*

$$ABr^3 - B(2+A)r^2 + (2A+1)r - 1 = 0. \tag{3.16}$$

(II) *For $-1 < A \leqslant \frac{3-\sqrt{5}}{2}$, $F(z)$ is a convex function in $|z| < r_1$, where r_1 is the smallest positive root of*

$$B(1-A)r^4 - 2B(1-A)r^3 + [1-AB-2(A-B)]r^2 - 2(1-A)r + 1 - A = 0. \tag{3.17}$$

Proof. Using Theorem 3.4, the bound of convexity for $F(z)$ is determined either from the equation

$$ABr^3 - B(2+A)r^2 + (2A+1)r - 1 = 0,$$

or from the equation

$$B(1-A)r^4 - 2B(1-A)r^3 + [1-AB-2(A-B)]r^2 - 2(1-A)r + 1 - A = 0.$$

Taking $R_1 = R_2$ in Theorem 3.4, it yields

$$ABr^4 - 2ABr^3 + (2A+2B-AB-1)r^2 - 2r + 1 = 0. \tag{3.18}$$

Eliminating r from (3.16) and (3.18), we obtain

$$(1+B)(BA^3 - 2BA^2 + 2A - 1) = 0, \tag{3.19}$$

which determines the transition from r_0 to r_1. If $B \neq -1$, then we have

$$B = \frac{2A-1}{A^2(2-A)}, \qquad A \neq 1.$$

In fact, $B < A$ implies that $0 < (1-A)^3(1+A)$ holds. On the other hand, $B = \frac{2A-1}{A^2(2-A)} > -1$ implies that $A < \frac{3-\sqrt{5}}{2} < 1$. If $B = -1$, eliminating r from (3.16) and (3.18), we obtain $A^2 - 3A + 1 = 0$, so $A = \frac{3-\sqrt{5}}{2}$. □

Corollary 3.1. *Let $F(z) = zf'(z)$, where $f(z) \in C^*(0)$, then*

$$\mathcal{R}e\frac{(zF'(z))'}{F'(z)} \geqslant \frac{r^2 - 2r - 1}{1 - r^2}$$

and $F(z)$ can not be a convex function in U.

Proof. By setting $A = 1, B = -1$ in Theorem 3.4 and Theorem 3.5 above, Corollary 3.1 is obtained. □

Corollary 3.2. *Let $F(z) = zf'(z)$, where $f(z) \in C^*[1, \frac{1}{2}]$, then*

$$\mathcal{R}e\frac{(zF'(z))'}{F'(z)} \geqslant \frac{-r^3 + 3r^2 - 6r + 1}{(1+r)(1-r)(2-r)} \tag{3.20}$$

and $F(z)$ is a convex function in $|z| < r_0$, where r_0 is the unique root in the interval $(0,1]$ of the equation

$$-r^3 + 3r^2 - 6r + 1 = 0.$$

Proof. The (3.20) is generated by putting $A = 1, B = \frac{1}{2}$ in Theorem 3.4 and Theorem 3.5. Furthermore, by letting

$$L(r) = -r^3 + 3r^2 - 6r + 1,$$

we deduce that

$$L(0) = 1 > 0, \quad L(1) = -3 < 0, \quad L'(r) < 0 \qquad (r \in (0,1])$$

which completes the proof of Corollary 3.2. □

Theorem 3.6. *Let $f(z) \in C^*[A, B]$ with respect to the convex function $g(z) \in \mathcal{K}$, and let*

$$F(z) = \int_0^z \frac{f(t)}{t}dt, \qquad G(z) = \int_0^z \frac{g(t)}{t}dt.$$

Then $F(z) \in C^*[A, B]$ with respect to $G(z)$.

Proof. Since Let $f(z) \in C^*[A, B]$ with respect to the convex function $g(z) \in \mathcal{K}$, so

$$\frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}. \tag{3.21}$$

From Lemma 2.4, it is easily seen that $G(z) \in \mathcal{K}$. Furthermore, we have

$$\frac{(zF'(z))'}{G'(z)} = \frac{zf'(z)}{g(z)}. \tag{3.22}$$

Following (3.21), (3.22), and Lemma 2.3, it gives

$$\frac{(zF'(z))'}{G'(z)} \prec \frac{1+Az}{1+Bz},$$

which completes the proof. □

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