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FIXED POINT THEOREMS FOR T-ZAMFIRESCU OPERATORS

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Abstract. In this paper, we introduce two new iteration schemes, namely T-Mann iteration and T-Ishikawa iteration and study the convergence of these iterations for the class of T-Zamfirescu operators in real Banach spaces. Our result in this paper improves the corresponding result proved by Morales and Rojas [8].

1. Introduction

The first important result on fixed points for contractive type mapping was the well-known Banach's contraction principle which was published in 1922. After this classical result, Kannan in [5] analyzed a new type of contractive condition. Since then there have been many theorems dealing with mappings satisfying various types of contractive inequalities. Another important result on fixed points for contractive type mapping in the setting of compact metric space is generally attributed to Edelstein [4].

In 2009, Beiranvand et al. [1], introduced the notions of T-Banach contraction and T-contractive mapping and extended the Banach contraction principle and Edelstein's fixed point theorem. In the same year, Moradi [6] introduced the T-Kannan contractive type mappings, extending in this way the well-known Kannan's fixed point theorem given in [5]. Followed by this, Morales and Rojas [7] introduced the notion of T-Chatterjea mapping and obtained sufficient conditions for the existence of a unique fixed point of these mappings in the frame work of complete cone metric spaces. The same authors [9], then introduced the concept of T-Zamfirescu operators and obtained sufficient conditions for the existence of a unique fixed point of T-Zamfirescu operators in the setting of complete cone metric spaces. In [8], they studied the existence

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of fixed points for T-Zamfirescu operators in complete metric spaces and proved a convergence theorem of T-Picard iteration for the class of T-Zamfirescu operators.

In this paper, we introduce two new iteration schemes which are named as T-Mann iteration scheme and T-Ishikawa iteration scheme and establish strong convergence theorems of these iteration schemes to the fixed point of T-Zamfirescu operators in real Banach spaces.

2. Preliminaries

We need the following definitions [8] to prove our main results:

Definition 2.1. Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T**-Banach contraction** (TB contraction) if there is $a \in [0, 1)$ such that

$$d(TSx, TSy) \leq ad(Tx, Ty), \qquad \text{for all } x, y \in M.$$

In the above definition, if we take $T = I$, the identity map, then we obtain the definition of *Banach's contraction* [2].

Definition 2.2. Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T**-Kannan contraction** (TK contraction) if there is $b \in [0, \frac{1}{2})$ such that

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \qquad \text{for all } x, y \in M.$$

In Definition 2.2, when $T = I$, the identity map, we get the definition of *Kannan operator* which is given in [5].

Definition 2.3. Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T**-Chatterjea contraction** (TC contraction) if there is $c \in [0, \frac{1}{2})$ such that

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)], \qquad \text{for all } x, y \in M.$$

When $T = I$, the identity map, in the above definition, it becomes *Chatterjea operator* which is given in [3].

Definition 2.4. Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T**-Zamfirescu operator** (TZ operator) if there are real numbers $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$ such that for all $x, y \in M$ at least one of the conditions is true:
(TZ_1) : $d(TSx, TSy) \leq ad(Tx, Ty)$,
(TZ_2) : $d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)]$,
(TZ_3) : $d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)]$.

Here when the function T is equated to I, the identity map, then we obtain definition of *Zamfirescu operator* introduced in [10].

Definition 2.5. Let (M,d) be a metric space and $T : M \to M$.

(1) A function T is said to be **sequentially convergent**, if we have for every sequence $\{y_n\}$, $\{Ty_n\}$ is convergent implies that $\{y_n\}$ is also convergent.
(2) A function T is said to be **subsequentially convergent**, if we have for every sequence $\{y_n\}$, $\{Ty_n\}$ is convergent implies that $\{y_n\}$ has a convergent subsequence.

In 2009, Morales and Rojas introduced a new iteration scheme, namely T-Picard iteration. The scheme is defined as follows:

Definition 2.6. Let (M,d) be a metric space, $x_0 \in M$ be arbitrary and $T, S : M \to M$ be two mappings. The sequence $\{Tx_n\} \in M$ defined by

$$Tx_{n+1} = TSx_n = TS^n x_0, \qquad n = 0, 1, 2, \ldots,$$

is called the **T-Picard iteration** associated to S [8].

Same authors obtained sufficient conditions for the existence of a unique fixed point of T-Zamfirescu operators in the setting of complete metric spaces. They also studied the strong convergence of the above iteration scheme under certain conditions and proved the following result:

Theorem 2.1. [8] *Let (M,d) be a complete metric space and $T, S : M \to M$ be two mappings such that T is continuous, one-to-one and subsequentially convergent. If S is a TZ operator, then S has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in M$ the T-Picard iteration associated to S, $TS^n x_0$ converges to Tx^* where x^* is the fixed point of S.*

Inspired and motivated by these facts, we introduce two new iteration schemes in this paper and study the convergence of these iteration schemes to the fixed point of T-Zamfirescu operators in real Banach spaces. The new schemes are defined as follows:

Definition 2.7. Let E be a Banach space, $x_0 \in E$ and $T, S : E \to E$ be two mappings. The sequence $\{Tx_n\} \in E$ defined by

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n TSx_n, \qquad n = 1, 2, 3, \ldots, \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ is called the **T-Mann iteration** associated to S.

When $\alpha_n = 1$, the iteration defined by (2.1) reduces to T-Picard iteration associated to S. When we substitute $T = I$, the identity map, the iteration given by (2.1) reduces to

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n, \qquad n = 0, 1, 2, \ldots,$$

which is Mann iteration. If we take $T = I$ and $\alpha_n = \lambda$, where $\lambda \in [0,1]$, in the iteration defined by (2.1), then we get

$$x_{n+1} = (1-\lambda)x_n + \lambda Sx_n, \qquad n = 0, 1, 2, \ldots,$$

which is Krasnoselskij iteration.

Definition 2.8. Let E be a Banach space, $x_0 \in E$ and $T, S : E \to E$ be two mappings. The sequence $\{Tx_n\} \in E$ defined by

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n STy_n, \tag{2.2}$$
$$Ty_n = (1-\beta_n)Tx_n + \beta_n TSx_n, \qquad n = 1, 2, 3, \dots,$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ is called the T-Ishikawa iteration associated to S.

When $T = I$, the iteration defined by (2.2) reduces to

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n Sx_n, \qquad n = 1, 2, 3, \dots,$$

which is Ishikawa iteration.

We need the following lemmas to prove our main results:

Lemma 2.1. [8] *Let (M, d) be a complete metric space and $T, S : M \to M$ be two functions. If S is a TZ-operator, then there is $0 \le \delta < 1$ such that*

$$d(TSx, TSy) \le \delta d(Tx, Ty) + 2\delta d(Tx, TSx), \qquad \text{for all } x, y \in M.$$

Lemma 2.2. [2] *Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be sequences of nonnegative numbers satisfying the inequality*

$$r_{n+1} \le (1-s_n)r_n + s_n t_n, \qquad \text{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} r_n = 0$.

3. Main Results

Theorem 3.1. *Let E be a real Banach space, K be a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TZ operator. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined as in (2.1) where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Proof. By Theorem 2.1, we get that S has a unique fixed point, say x^* in K. Since S is a TZ-operator, by applying Lemma 2.1, we have $0 \le \delta < 1$ such that

$$\|TSx - TSy\| \le \delta \|Tx - Ty\| + 2\delta \|Tx - TSx\|, \qquad \text{for all } x, y \in K. \tag{3.1}$$

Let $\{Tx_n\}_{n=0}^{\infty} \in K$ be the T-Mann iteration associated to S defined by (2.1) and $x_0 \in K$. Then

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &= \|(1-\alpha_n)Tx_n + \alpha_n TSx_n - Tx^*\| \\ &= \|(1-\alpha_n)(Tx_n - Tx^*) + \alpha_n(TSx_n - Tx^*)\|, \end{aligned}$$

which gives

$$\|Tx_{n+1} - Tx^*\| \le (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n \|TSx_n - Tx^*\|. \tag{3.2}$$

Taking $x = x^*$ and $y = x_n$ in (3.1), we get

$$\|TSx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\| + 2\delta \|Tx^* - TSx^*\|,$$

which implies

$$\|Tx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\|. \tag{3.3}$$

Using (3.3) in (3.2) we obtain,

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &\leq (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n\delta \|Tx^* - Tx_n\| \\ &= (1-\alpha_n + \alpha_n\delta)\|Tx_n - Tx^*\| \\ &= [1-\alpha_n(1-\delta)]\|Tx_n - Tx^*\|. \end{aligned}$$

Since $0 \leq \delta < 1$, $\alpha_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, by setting $s_n = (1-\delta)\alpha_n$, $r_n = \|Tx_n - Tx^*\|$ and by applying Lemma 2.2, we get that

$$\lim_{n\to\infty} \|Tx_{n+1} - Tx^*\| = 0.$$

Hence $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S. □

Corollary 3.1. [8, Theorem 3.1] *Let (M, d) be a complete metric space and $T, S : M \to M$ be two mappings such that T is continuous, one-to-one and subsequentially convergent. If S is a TZ operator, then S has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in M$, the T-Picard iteration associated to S, $TS^n x_0$ converges to Tx^* where x^* is the fixed point of S.*

Since the T-Kannan and T-Chatterjea contractive conditions are both included in the class of TZ- operators, by Theorem 3.1, we obtain the corresponding convergence theorems for T-Mann iteration in these classes of operators.

Corollary 3.2. *Let E be a real Banach space, K be a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TK-contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined as in (2.1) where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Corollary 3.3. *Let E be a real Banach space, K be a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TC-contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (2.1) where $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Theorem 3.2. *Let E be a real Banach space, K be a nonempty, closed, convex subset of E and $T, S : K \to K$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TZ operator. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (2.2) where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Proof. By Theorem 2.1, we get that S has a unique fixed point, say x^* in K. Since S is a TZ-operator, by using Lemma 2.1, we have $0 \leq \delta < 1$ such that

$$\|TSx - TSy\| \leq \delta \|Tx - Ty\| + 2\delta \|Tx - TSx\|, \qquad \text{for all } x, y \in K. \tag{3.4}$$

Let $\{Tx_n\}_{n=0}^{\infty} \in K$ be the T-Ishikawa iteration associated to S defined by (2.2) and $x_0 \in K$. Then

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &= \|(1-\alpha_n)Tx_n + \alpha_n STy_n - Tx^*\| \\ &= \|(1-\alpha_n)(Tx_n - Tx^*) + \alpha_n(STy_n - Tx^*)\| \\ &\leq (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n \|STy_n - Tx^*\|. \end{aligned} \tag{3.5}$$

Taking $x = x^*$ and $y = y_n$ in (3.4), we get

$$\|TSx^* - TSy_n\| \leq \delta \|Tx^* - Ty_n\| + 2\delta \|Tx^* - TSx^*\|,$$

which implies

$$\|Tx^* - TSy_n\| \leq \delta \|Tx^* - Ty_n\|.$$

Since S and T are commuting mappings, the above inequality gives

$$\|Tx^* - STy_n\| \leq \delta \|Tx^* - Ty_n\|. \tag{3.6}$$

Using (3.6) in (3.5), we obtain

$$\|Tx_{n+1} - Tx^*\| \leq (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n\delta \|Tx^* - Ty_n\|. \tag{3.7}$$

Now,

$$\begin{aligned} \|Ty_n - Tx^*\| &= \|(1-\beta_n)Tx_n + \beta_n TSx_n - Tx^*\| \\ &= \|(1-\beta_n)(Tx_n - Tx^*) + \beta_n(TSx_n - Tx^*)\| \\ &\leq (1-\beta_n)\|(Tx_n - Tx^*)\| + \beta_n \|(TSx_n - Tx^*)\|. \end{aligned} \tag{3.8}$$

Taking $x = x^*$ and $y = x_n$ in (3.4), we get

$$\|TSx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\| + 2\delta \|Tx^* - TSx^*\|.$$

Thus we have the inequality,

$$\|Tx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\|. \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$\begin{aligned} \|Ty_n - Tx^*\| &\leq (1-\beta_n)\|Tx_n - Tx^*\| + \beta_n\delta \|Tx_n - Tx^*\| \\ &= (1-\beta_n + \beta_n\delta)\|Tx_n - Tx^*\|. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.7), we get

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &\leq (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n\delta(1-\beta_n+\beta_n\delta)\|Tx_n - Tx^*\| \\ &= [1-\alpha_n + \alpha_n\delta(1-\beta_n+\beta_n\delta)]\|Tx_n - Tx^*\|. \end{aligned} \tag{3.11}$$

Now,

$$1 - \alpha_n + \alpha_n\delta(1-\beta_n+\beta_n\delta) = 1 - [\alpha_n(1-\delta)(1+\beta_n\delta)].$$

Since $(1 + \beta_n\delta) \geq (1 - \delta)$, we have

$$1 - \alpha_n + \alpha_n\delta(1 - \beta_n + \beta_n\delta) \leq 1 - (1 - \delta)^2\alpha_n. \tag{3.12}$$

Using (3.12) in (3.11), we obtain

$$\|Tx_{n+1} - Tx^*\| \leq [1 - (1 - \delta)^2\alpha_n] \|Tx_n - Tx^*\|.$$

Since $0 \leq \delta < 1$, $\alpha_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, by setting $s_n = (1 - \delta)^2\alpha_n$, $r_n = \|Tx_n - Tx^*\|$ and by applying Lemma 2.2, we get that

$$\lim_{n\to\infty} \|Tx_{n+1} - Tx^*\| = 0.$$

Therefore, $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S. Hence the proof follows. □

By Theorem 3.2, we obtain the corresponding convergence theorems for T-Ishikawa iteration in the classes of TK-contraction and TC-contraction as these two classes are included in the class of TZ-operators.

Corollary 3.4. *Let E be a real Banach space, K be a nonempty, closed, convex subset of E and $T, S : K \to K$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TK contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined as in* (2.2) *where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Corollary 3.5. *Let E be a real Banach space, K be a nonempty, closed, convex subset of E and $T, S : K \to K$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and S is a TC contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined as in* (2.2) *where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

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