Kragujevac Journal of Mathematics
Volume 36 Number 1 (2012), Pages 163–174.

ON (p, q)-TH ORDER OF A FUNCTION OF TWO COMPLEX VARIABLES ANALYTIC IN THE UNIT POLYDISC

RATAN KUMAR DUTTA

Abstract. In this paper we study the maximum modulus and the coefficients of the power series expansion of a function of two complex variables analytic in the unit polydisc.

1. Introduction, Definitions and Notations

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M(r) = M(r, f)$ be the maximum of $|f(z)|$ on $|z| = r$.

In [9] Sons defined the order ρ and the lower order λ as

$$\rho = \limsup_{r\to 1} \frac{\log\ \log M(r, f)}{-\log(1 - r)}, \qquad \lambda = \liminf_{r\to 1} \frac{\log\ \log M(r, f)}{-\log(1 - r)}.$$

Maclane [7] and Kapoor [6] proved the following results which characterized the order and lower order of a function f analytic in U, in terms of the coefficients c_n.

Theorem 1.1. [7] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having order ρ $(0 \le \rho \le \infty)$. Then*

$$\frac{\rho}{1 + \rho} = \limsup_{n\to\infty} \frac{\log^+ \log^+ |c_n|}{\log n}.$$

Theorem 1.2. [6] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having lower order λ $(0 \le \lambda \le \infty)$. Then*

$$\frac{\lambda}{1 + \lambda} \ge \liminf_{n\to\infty} \frac{\log^+ \log^+ |c_n|}{\log\ n}.$$

In the paper we use the following definitions and notations.

Notation 1.1. [8] $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]}\ x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

Key words and phrases. Analytic function, (p, q)-th order, lower (p, q)-th order, maximum modulus, unit polydisc.

2010 *Mathematics Subject Classification.* 32A15.

Received: November 9, 2011.

Notation 1.2. [1] For $0 < x < \infty$ we write $\log^{*(0)} x = x$, $\log^{*(1)} x = \log(1+x)$, $\log^{*(2)} x = \log(1+\log(1+x))$, $\log^{*(3)} x = \log(1+\log(1+\log(1+x)))$ etc.

Definition 1.1. [5] If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic in U, its p-th order ρ_p and lower p-th order λ_p are defined as

$$\rho_p = \limsup_{r\to 1} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \qquad \lambda_p = \liminf_{r\to 1} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \qquad p \geq 2.$$

Using the definitions of p-th order and lower p-th order Banerjee, [1] generalized Theorem 1.1 and Theorem 1.2 in the following manner.

Theorem 1.3. [1] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U and have p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then*

$$\frac{\rho_p}{1+\rho_p} = \lim\sup_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log\ n}.$$

Theorem 1.4. [1] *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U and have lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then*

$$\frac{\lambda_p}{1+\lambda_p} \geq \lim\inf_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log n}.$$

Definition 1.2. [2] Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc

$$P : \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}.$$

Then order of f is denoted by ρ and defined by

$$\rho = \inf\left\{\mu > 0 : F(r_1, r_2) < \exp\left(\frac{1}{1-r_1}.\frac{1}{1-r_2}\right)^{\mu}; \text{for all } 0 < r_0(\mu) < r_1, r_2 < 1\right\}.$$

Equivalent formula for ρ is

$$\rho = \lim\sup_{r_1,r_2\to 1} \frac{\log\log\ F(r_1, r_2)}{-\ \log(1-r_1)(1-r_2)}.$$

Recently Banerjee and Dutta [3] introduced the definition of p-th order and lower p-th order of functions of two complex variables analytic in the unit polydisc and generalized the above results.

Definition 1.3. [3] Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U = \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

and let

$$F(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_j| \leq r_j; j = 1, 2\}$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\rho_p = \lim_{r_1, r_2 \to 1} \sup \frac{\log^{[p]} F(r_1, r_2)}{-\log(1 - r_1)(1 - r_2)},$$
$$\lambda_p = \lim_{r_1, r_2 \to 1} \inf \frac{\log^{[p]} F(r_1, r_2)}{-\log(1 - r_1)(1 - r_2)}, \quad p \geq 2.$$

Remark 1.1. When $p = 2$, Definition 1.3 coincides with Definition 1.2.

Theorem 1.5. [3] *Let $f(z_1, z_2)$ be analytic in U and have p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then*

$$\frac{\rho_p}{1 + \rho_p} = \lim \sup_{m,n \to \infty} \frac{\log^{+[p]} |c_{mn}|}{\log\ mn}.$$

Theorem 1.6. [3] *Let $f(z_1, z_2)$ be analytic in U and have lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then*

$$\frac{\lambda_p}{1 + \lambda_p} \geq \lim \inf_{m,n \to \infty} \frac{\log^{+[p]} |c_{mn}|}{\log\ mn}.$$

In this paper we introduce the following definitions of (p, q)-th order and lower (p, q)-th order of functions of two complex variables analytic in the unit polydisc and prove a similar analytic expression.

Definition 1.4. Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U = \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

and let

$$F(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_j| \leq r_j; j = 1, 2\}$$

be its maximum modulus. Then the (p, q)-th order ρ_q^p and the lower (p, q)-th order λ_q^p are define as

$$\rho_q^p = \lim_{r_1, r_2 \to 1} \sup \frac{\log^{[p]} F(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)}\right)},$$
$$\lambda_q^p = \lim_{r_1, r_2 \to 1} \inf \frac{\log^{[p]} F(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)}\right)}, \qquad p \geq q + 1 \geq 2.$$

Remark 1.2. When $q = 1$, Definition 1.4 corresponds to Definition 1.3.

Here $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ denotes a function of two complex variables analytic in the unit polydisc U. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as they are available in [4], [10] and [11].

2. Lemmas

The following lemmas will be needed in the rest of the paper.

Lemma 2.1. *Let the maximum modulus $F(r_1, r_2)$ of a function $f(z_1, z_2)$ analytic in U, satisfy*

$$\log^{[p-1]} F(r_1, r_2) < \left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^A, \tag{2.1}$$

$0 < A < \infty$ *for all* r_1, r_2 *such that* $r_0(A) < r_1, r_2 < 1$.
Then for all $m > m_0(A) > 1$ *and* $n > n_0(A) > 1$,

$$\log^{[p-1]} |c_{mn}| \leq [3 + O(1)](\log^{[q-1]} mn)^{\frac{A}{A+1}}.$$

Proof. Define two sequences $\{r_{1m}\}$ and $\{r_{2n}\}$ by

$$(1 - r_{1m})^{-1} = \exp^{[q-1]}\left\{\left(\log^{[q-1]} m\right)^{\frac{1}{2(A+1)}}\right\}$$

and

$$(1 - r_{2n})^{-1} = \exp^{[q-1]}\left\{\left(\log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right\}.$$

Then $r_{1m} \to 1$ and $r_{2n} \to 1$ as $m, n \to \infty$.
By Cauchy's inequality and (2.1) we have for all $m > m_0(A) > 1$ and $n > n_0(A) > 1$,

$$\begin{aligned}
& \log |c_{mn}| \\
\leq\ & \log F(r_{1m}, r_{2n}) - m \ \log \ r_{1m} - n \ \log \ r_{2n} \\
<\ & \exp^{[p-2]}\left\{\log^{[q-1]}\left(\frac{1}{1-r_{1m}}.\frac{1}{1-r_{2n}}\right)\right\}^A + [m(1-r_{1m}) + n(1-r_{2n})][1+O(1)] \\
=\ & \exp^{[p-2]}\left[\log^{[q-1]}\left\{\left(\exp^{[q-1]}\left(\log^{[q-1]} m\right)^{\frac{1}{2(A+1)}}\right)\left(\exp^{[q-1]}\left(\log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right)\right\}\right]^A \\
& + \left[\frac{m}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} m\right)^{\frac{1}{2(A+1)}}\right\}} + \frac{n}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right\}}\right][1+O(1)] \\
\leq\ & \exp^{[p-2]}\left[\log^{[q-1]}\left\{\exp^{[q-1]}\left(\log^{[q-1]} m \ \log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right\}\right]^A \\
& + \left[\frac{m}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} m\right)^{\frac{1}{2(A+1)}}\right\}} + \frac{n}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right\}}\right][1+O(1)]
\end{aligned}$$

$$
\begin{aligned}
&\le \exp^{[p-2]}\left(\log^{[q-1]} mn\right)^{\frac{A}{A+1}} \\
&\quad + \left[\frac{m}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} m\right)^{\frac{1}{2(A+1)}}\right\}} + \frac{n}{\exp^{[q-1]}\left\{\left(\log^{[q-1]} n\right)^{\frac{1}{2(A+1)}}\right\}}\right][1+O(1)] \\
&\le \left[\exp^{[p-2]}\left(\log^{[q-1]} mn\right)^{\frac{A}{A+1}}\right][3+O(1)] \\
&\le \exp^{[p-2]}\left\{[3+O(1)]\left(\log^{[q-1]} mn\right)^{\frac{A}{A+1}}\right\}.
\end{aligned}
$$

Therefore

$$\log^{[p-1]}|c_{mn}| \le [3+O(1)]\left(\log^{[q-1]} mn\right)^{\frac{A}{A+1}}.$$

This proves the lemma. □

Lemma 2.2. *Let $f(z_1, z_2)$ be analytic in U and satisfy*

$$\log^{[p-1]}|c_{mn}| < \left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{D}\right\}\right]\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{D}\right\}\right],$$

$0 < C < \infty,\ 0 < D < 1$, *for all $m > m_0(C,D)$ and $n > n_0(C,D)$. Then for all $r_1,\ r_2$ such that $r_1\ {}_0(C,D) < r_1 < 1$ and $r_2\ {}_0(C,D) < r_2 < 1$,*

$$\log^{[p-1]} F(r_1, r_2) < T(C,D)\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\frac{D}{1-D}},$$

where

$$T(C,D) = C^{\frac{2}{1-D}} D^{\frac{2D}{1-D}}[2+o(1)].$$

Proof. For all $m > m_0(C,D)$ and $n > n_0(C,D)$,

$$\log^{[p-1]}|c_{mn}| < \left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{D}\right\}\right]\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{D}\right\}\right].$$

Now for $|z_1| = r_1 < 1$ and $|z_2| = r_2 < 1$,

$$
\begin{aligned}
F(r_1, r_2) &< \sum_{m,\ n=0}^{\infty} |c_{mn}|\ r_1^m\ r_2^n \\
&< K(m_0, n_0) + \sum_{\substack{m=m_0+1\\ n=n_0+1}}^{\infty}\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{D}\right\}\right] \\
&\qquad \left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{D}\right\}\right] r_1^m\ r_2^n \\
&\le K(m_0, n_0) + \left[\sum_{m=m_0+1}^{\infty}\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\}\right] r_1^m\right] \\
&\qquad \left[\sum_{n=n_0+1}^{\infty}\left[\exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{\frac{B}{B+1}}\right\}\right] r_2^n\right],
\end{aligned}
$$

where $B = \frac{D}{1-D}$.
Choose

$$M = M(r_1) = \left[\exp^{[q-1]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_1})}\right)^{B+1}\right]$$

and

$$N = N(r_2) = \left[\exp^{[q-1]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_2})}\right)^{B+1}\right],$$

where $[x]$ denotes the greatest integer not greater then x.
Clearly $M(r_1) \to \infty$ and $N(r_2) \to \infty$ as $r_1,\ r_2 \to 1$.
The above estimate of $F(r_1, r_2)$ for all $r_1,\ r_2$ sufficiently close to 1 gives,

$$F(r_1, r_2) < K(m_0, n_0) + \left[M(r_1)H(r_1) + \sum_{m=M+1}^{\infty} r_1^{m/2}\right] \left[N(r_2)H(r_2) + \sum_{n=N+1}^{\infty} r_2^{n/2}\right] \tag{2.2}$$

where

$$H(r_1) = \max_m\ \exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\}\ r_1^m$$

and

$$H(r_2) = \max_n\ \exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{\frac{B}{B+1}}\right\}\ r_2^n,$$

for if $m \geq M+1$, then

$$m > \exp^{[q-1]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}\left(\log\frac{1}{r_1}\right)}\right)^{B+1}.$$

So

$$\begin{aligned} C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}} &< \frac{\log^{[q-1]} n}{2^{2p-3}}\ \log^{*(p-2)}\left(\log\frac{1}{r_1}\right) \\ &< \frac{n}{2^{2p-3}}\ \log^{*(p-2)}\left(\log\frac{1}{r_1}\right) \\ &= \log\left[1 + \log^{*(p-3)}\left(\log\frac{1}{r_1}\right)\right]^{\frac{n}{2^{2p-3}}} \\ &\leq \log\left[1 + \frac{n}{2^{2p-4}}\ \log^{*(p-3)}\left(\log\frac{1}{r_1}\right)\right]. \end{aligned}$$

Hence

$$\begin{aligned}\exp\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} &\leq 1+\frac{n}{2^{2p-4}}\ \log^{*(p-3)}\left(\log\frac{1}{r_1}\right)\\ &\leq \frac{n}{2^{2p-5}}\ \log^{*(p-3)}\left(\log\frac{1}{r_1}\right)\\ &\leq \log\left[1+\frac{n}{2^{2p-6}}\ \log^{*(p-4)}\left(\log\frac{1}{r_1}\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}\exp^{[2]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} &\leq 1+\frac{n}{2^{2p-6}}\ \log^{*(p-4)}\left(\log\frac{1}{r_1}\right)\\ &\leq \frac{n}{2^{2p-7}}\ \log^{*(p-4)}\left(\log\frac{1}{r_1}\right).\end{aligned}$$

Taking repeated exponential, we obtain

$$\begin{aligned}\exp^{[p-2]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} &< \frac{m}{2}\ \log\frac{1}{r_1}\\ \text{i.e.}\quad \exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} r_1^m &< r_1^{\frac{m}{2}}.\end{aligned}$$

The infinite series $\sum_{m=M+1}^{\infty} r_1^{\frac{m}{2}}$ in (2.2) is bounded by $r_1^{\frac{M+1}{2}}\left(\frac{1}{1-r_1^{\frac{1}{2}}}\right)$.

Since $B>0$, we have

$$\begin{aligned}&-\frac{M+1}{2}\ \log\frac{1}{r_1}-\log(1-r_1^{\frac{1}{2}})\\ \leq\ &-\frac{1}{2}\exp^{[q-1]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_1})}\right)^{B+1}\log\frac{1}{r_1}\\ &-\log(1-r_1)+\log(1+r_1^{\frac{1}{2}})\\ \leq\ &-\frac{1}{2}\exp^{[q-1]}\left(\frac{2^{2p-3}C}{\log\frac{1}{r_1}}\right)^{B+1}\log\frac{1}{r_1}-\log(1-r_1)+\log(1+r_1^{\frac{1}{2}})\\ \to\ &-\infty \text{ as } r_1\to 1.\end{aligned}$$

Thus for r_1 sufficiently close to 1,

$$\sum_{m=M+1}^{\infty} r_1^{\frac{m}{2}}=o(1).$$

Similarly for r_2 sufficiently close to 1,

$$\sum_{n=N+1}^{\infty} r_1^{\frac{n}{2}}=o(1).$$

The maximum of

$$\exp^{[p-1]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} r_1^m$$

is at the point

$$m = \exp^{[q-1]}\left\{\frac{BC}{B+1}\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B+1}{2}}$$

and $H(r_1)$ is given by

$$\begin{aligned}\log H(r_1) &= \exp^{[p-2]}\left\{C\left(\log^{[q-1]} m\right)^{\frac{B}{B+1}}\right\} + m\log r_1\\ &= \exp^{[p-2]}\left[\frac{C.B^B.C^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B}{2}}\right]\\ &\quad -\exp^{[q-1]}\left\{\frac{BC}{B+1}\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B+1}{2}}\log\frac{1}{r_1}\\ &\le \exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B}{2}}\right]. \qquad (2.3)\end{aligned}$$

Similarly, the maximum of

$$\exp^{[p-1]}\left\{C\left(\log^{[q-1]} n\right)^{\frac{B}{B+1}}\right\} r_2^n$$

is at the point

$$n = \exp^{[q-1]}\left\{\frac{BC}{B+1}\log^{[q-1]}\left(\frac{1}{1-r_2}\right)\right\}^{\frac{B+1}{2}}$$

and $H(r_2)$ is given by

$$\log H(r_2) \le \exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_2}\right)\right\}^{\frac{B}{2}}\right]. \qquad (2.4)$$

Also

$$\sum_{m=M+1}^{\infty} r_1^{\frac{m}{2}} = o(1), \quad \sum_{n=N+1}^{\infty} r_2^{\frac{n}{2}} = o(1).$$

Thus for $r_1,\ r_2$ sufficiently close to 1, from (2.2)

$$\begin{aligned}F(r_1,r_2) &\le [M(r_1)H(r_1)+o(1)][N(r_2)H(r_2)+o(1)]\\ &\quad\left[1+\frac{K(m,n)}{[M(r_1)H(r_1)+o(1)][N(r_2)H(r_2)+o(1)]}\right]\\ &= [M(r_1)H(r_1)+o(1)][N(r_2)H(r_2)+o(1)][1+O(1)].\end{aligned}$$

Therefore

$$\begin{aligned}
& \log F(r_1,r_2) \\
\leq\ & \log M(r_1) + \log H(r_1) + \log N(r_2) + \log H(r_2) + O(1) \\
\leq\ & \exp^{[q-2]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_1})}\right)^{B+1} \\
& + \exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B}{2}}\right] \\
& + \exp^{[q-2]}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}(\log\frac{1}{r_2})}\right)^{B+1} \\
& + \exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_2}\right)\right\}^{\frac{B}{2}}\right] + O(1) \\
& \qquad\qquad\qquad [\text{from } (2.3) \text{ and } (2.4)] \\
\leq\ & 2\exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\right\}^{\frac{B}{2}}\right] \\
& +2\exp^{[p-2]}\left[\frac{C^{B+1}.B^B}{(B+1)^B}\left\{\log^{[q-1]}\left(\frac{1}{1-r_2}\right)\right\}^{\frac{B}{2}}\right] + O(1) \\
\leq\ & \exp^{[p-2]}\left[\frac{C^{2(B+1)}.B^{2B}}{(B+1)^{2B}}\left\{\log^{[q-1]}\left(\frac{1}{1-r_1}\right)\log^{[q-1]}\left(\frac{1}{1-r_2}\right)\right\}^{\frac{B}{2}}\right] \\
& \qquad\qquad\qquad [2+o(1)] \\
\leq\ & \exp^{[p-2]}\left[\frac{C^{2(B+1)}.B^{2B}}{(B+1)^{2B}}\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{B}\right][2+o(1)].
\end{aligned}$$

$$\begin{aligned}
\text{i.e., } \log^{[p-1]} F(r_1,r_2) &\leq \frac{C^{2(B+1)}.B^{2B}}{(B+1)^{2B}}[2+o(1)]\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{B} \\
&= C^{\frac{2}{1-D}}D^{\frac{2D}{1-D}}[2+o(1)]\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\frac{D}{1-D}} \\
&= T(C,D)\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\frac{D}{1-D}},
\end{aligned}$$

where

$$T(C,D) = C^{\frac{2}{1-D}}D^{\frac{2D}{1-D}}[2+o(1)].$$

This proves the lemma. □

3. Theorem

In this section, we prove the following theorem.

Theorem 3.1. *Let $f(z_1, z_2)$ be analytic in U and have the (p,q)-th order ρ_q^p $(0 \leq \rho_q^p \leq \infty)$. Then*

$$\frac{\rho_q^p}{1+\rho_q^p} = \lim \sup_{m,\ n\to\infty} \frac{\log^{+[p]} |c_{mn}|}{\log^{[q]} mn}. \tag{3.1}$$

Proof. If $|c_{mn}|$ is bounded by K for all $m,\ n$, then $\sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ is bounded by $\frac{K}{(1-r_1)(1-r_2)}$.
Therefore

$$\begin{aligned} F(r_1, r_2) &\leq \sum_{m,n=0}^{\infty} |c_{mn}|\ r_1^m\ r_2^n \\ &\leq \frac{K}{(1-r_1)(1-r_2)} \\ &\leq \exp^{[p-1]}\left[\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)^{\epsilon}\right], \quad \text{for } p \geq q+1 \end{aligned}$$

for any $0 < \epsilon < 1$ and $r_1,\ r_2$ sufficiently close to 1.
Therefore

$$\rho_q^p = \lim \sup_{r_1,\ r_2 \to 1} \frac{\log^{[p]} F(r_1, r_2)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)} \leq \epsilon$$

since $0 < \epsilon < 1$ arbitrary, $\rho_q^p = 0$ and so (3.1) is satisfied.
Thus we need to consider only the case

$$\lim \sup_{m,\ n\to\infty} |c_{mn}| = \infty.$$

In this respect, all the $\log^+$ in (3.1) may be replaced by log. First let $0 < \rho_q^p < \infty$. Then for all $r_1,\ r_2$ sufficiently close to 1 and for arbitrary $\varepsilon > 0$, we get from the definition of (p,q)-th order,

$$\begin{aligned} \log^{[p-1]} F(r_1, r_2) &\leq \left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\rho_q^p+\varepsilon} \\ &= \left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\mu}, \end{aligned}$$

where $\mu = \rho_q^p + \varepsilon$.

Using Lemma 2.1 with $A = \mu$ it follows from the above inequality that for $m > m_0(\mu)$ and $n > n_0(\mu)$,

$$\begin{aligned} \log^{[p-1]} |c_{mn}| &\leq [3+O(1)](\log^{[q-1]} mn)^{\frac{\mu}{\mu+1}} \\ \log^{[p]} |c_{mn}| &\leq \log[3+O(1)] + \frac{\mu}{\mu+1} \log^{[q]} mn. \end{aligned}$$

Therefore,

$$\lim\sup_{m,\ n\to\infty}\frac{\log^{[p]}|c_{mn}|}{\log^{[q]}mn}\leq\frac{\mu}{1+\mu}.$$

Since $\varepsilon>0$ is arbitrary, it follows that

$$\lim\sup_{m,\ n\to\infty}\frac{\log^{[p]}|c_{mn}|}{\log^{[q]}mn}\leq\frac{\rho_q^p}{1+\rho_q^p}. \tag{3.2}$$

Since f is analytic in U, the above inequality is trivially true if $\rho_q^p=\infty$ and the right hand side is interpreted as 1 in this case.
Conversely, if

$$\theta=\lim\sup_{m,\ n\to\infty}\frac{\log^{[p]}|c_{mn}|}{\log^{[q]}mn}$$

then $0\leq\theta\leq 1$.
First let $\theta<1$ and choose $\theta<\theta'<1$.
Then for all sufficiently large m,n

$$\log^{[p-1]}|c_{mn}|\leq\left(\log^{[q-1]}mn\right)^{\theta'}.$$

Using Lemma 2.2 with $C=1,\ D=\theta'$, it follows from the above inequality that for all $r_1,\ r_2$ such that $r_0(\theta')<r_1,\ r_2<1$,

$$\log^{[p-1]}F(r_1,r_2)\leq\theta'^{\frac{2\theta'}{1-\theta'}}\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}^{\frac{\theta'}{1-\theta'}}[2+o(1)].$$

Therefore,

$$\log^{[p]}F(r_1,r_2)\leq\frac{2\theta'}{1-\theta'}\log(\theta')+\frac{\theta'}{1-\theta'}\log\left\{\log^{[q-1]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)\right\}+\log[2+o(1)]$$

$$\text{i.e.,}\quad\lim\sup_{r_1,\ r_2\to 1}\frac{\log^{[p]}F(r_1,r_2)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)}\leq\frac{\theta'}{1-\theta'}\lim\sup_{r_1,\ r_2\to 1}\frac{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)}{\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2)}\right)}.$$

Therefore,

$$\rho_q^p\leq\frac{\theta'}{1-\theta'}.$$

Since $\theta'>\theta$ is arbitrary, it follows that

$$\frac{\rho_q^p}{1+\rho_q^p}\leq\theta=\lim\sup_{m,\ n\to\infty}\frac{\log^{[p]}|c_{mn}|}{\log^{[q]}mn}. \tag{3.3}$$

If $\theta=1$, the above inequality is obviously true.
Inequalities (3.2) and (3.3) together give (3.1) when $\lim\sup_{m,\ n\to\infty}|c_{mn}|=\infty$.

This proves the theorem. □

Conjecture 3.1. *Is it possible to prove similar result for lower (p,q)-th order of a function analytic in a unit polydisc?*

References

[1] D. Banerjee, *On p-th order of a function analytic in the unit disc*, Proc. Nat. Acad. Sci. India, **75(A), IV** (2005), 249–253.

[2] D. Banerjee and R. K. Dutta, *Relative order of functions of two complex variables analytic in the unit disc*, J. Math., **1** (2008) 37–44.

[3] D. Banerjee and R. K. Dutta, *On p-th order of a function of two complex variables analytic in the unit polydisc*, Proc. Nat. Acad. Sci. India, (Section-A), **81, I** (2011), 49–58.

[4] W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.

[5] O. P. Juneja and G. P. Kapoor, *Analytic Functions-Growth Aspects*, Pitman Advanced Publishing Program, 1985.

[6] G. P. Kapoor, *On the lower order of functions analytic in the unit disc*, Math. Japon., **17** (1972), 49–54.

[7] G. R. Maclane, *Asymptotic Value of Holomorphic Functions*, Rice University Studies: Houston, 1963.

[8] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., **69** (1963), 411–414.

[9] L. R. Sons, *Regularity of growth and gaps*, J. Math. Anal. Appl., **24** (1968), 296–306.

[10] G. Valiron, *Lectures on the general theory of Integral functions*, Chelsea Publishing Company, 1949.

[11] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic ublishers and Science Press, Beijing, 2003.

Department of Mathematics
Patulia High School, Patulia
Kolkata-700119
West Bengal
India.
E-mail address: ratan_3128@yahoo.com