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TRIPARTITE MULTIDIGRAPHS AND IMBALANCES

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Abstract. A tripartite r-digraph ($r \geq 1$) is an orientation of a tripartite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. For any vertex x in a tripartite r-digraph $D(U, V, W)$, let d_x^+ and d_x^- denote the outdegree and indegree respectively of x. Define $a_{u_i} = d_{u_i}^+ - d_{u_i}^-$, $b_{v_j} = d_{v_j}^+ - d_{v_j}^-$ and $c_{w_k} = d_{w_k}^+ - d_{w_k}^-$ as the r-imbalances of the vertices $u_i \in U$, $v_j \in V$ and $w_k \in W$ respectively. We characterize r-imbalances in tripartite r-digraphs and obtain necessary and sufficient conditions for three sequences of integers to be r-imbalance sequences of some tripartite r-digraph.

1. Introduction

An *r-digraph* ($r \geq 1$) is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. So a 1-digraph is an oriented graph and a complete 1-digraph is a tournament.

The *imbalance* of a vertex v_i in an r-digraph is defined as

$$a_{v_i} = a_i = d_{v_i}^+ - d_{v_i}^-,$$

where d_i^+ and d_i^- denote respectively the outdegree and indegree of v_i. The imbalance sequence of an r-digraph is formed by listing the vertex imbalances in non-decreasing order or non-increasing order.

The following [6] is the combinatorial criterion for sequence of integers to be the imbalance sequence of some r-digraph. Of course this is the extension of a criterion for imbalances in simple digraphs found in [2].

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Theorem 1.1. [2] *A sequence $A = [a_i]_1^n$ of integers in non-increasing order is an imbalance sequence of an r-digraph if and only if*

$$\sum_{i=1}^{k} a_i \leq rk(n-k),$$

for $1 \leq k \leq n$, with equality when $k = n$.

A *bipartite r-digraph* ($r \geq 1$) is an orientation of a bipartite multigraph without loops and with at most r edges between any pair of vertices, one vertex from each part. The following characterization of imbalance sequences in bipartite r-digraphs can be seen in [7].

Theorem 1.2. *The sequences $A = [a_1, \cdots, a_p]$ and $B = [b_1, \cdots, b_q]$ of integers in non-increasing order are the imbalance sequences of some bipartite r-digraph if and only if*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \leq r[k(q-l) + l(p-k)],$$

for $1 \leq k \leq p$ and $1 \leq l \leq q$ with equality when $k = p$ and $l = q$.

Various results on imbalances in digraphs can be seen in [1, 5, 6], imbalances in oriented bipartite graphs can be seen in [10].

2. Imbalance sequences in tripartite r-digraphs

A *tripartite r-digraph* ($r \geq 1$) is an orientation of a tripartite multigraph that is without loops and contains at most r edges between any pair of vertices, each vertex from a distinct part. Clearly tripartite 1-digraph is an oriented tripartite graph and a complete tripartite 1-digraph is a tripartite tournament. Some results in these types of tripartite digraphs can be found in [8, 9].

If $D(U, V, W)$ is a tripartite r-partite with parts $U = \{u_1, \cdots, u_l\}$, $V = \{v_1, \cdots, v_m\}$ and $W = \{w_1, \cdots, w_n\}$, then $a_{u_i} = a_i = d^+_{u_i} - d^-_{u_i}$, $b_{v_j} = b_j = d^+_{v_j} - d^-_{v_j}$ and $c_{w_k} = c_k = d^+_{w_k} - d^-_{w_k}$ are respectively the imbalances of vertices $u_i \in U$, $v_j \in V$ and $w_k \in W$. The sequences $A = [a_1, \cdots, a_l]$, $B = [b_1, \cdots, b_m]$ and $C = [c_1, \cdots, c_n]$ in non-increasing or non-decreasing order are called the *imbalance sequences* of $D = (U, V, W)$.

In $D(U, V, W)$, if there are f arcs directed from u to v and g arcs directed from v to u with $0 \leq f, g, f + g \leq r$, we denote this by $u(f - g)v$.

An r-triple in D is an induced r-subdigraph of three vertices, one vertex from each part and is of the form $u(f_1 - f_2)v(g_1 - g_2)w(h_1 - h_2)u$ where $0 \leq f_1, f_2, g_1, g_2, h_1, h_2, f_1 + f_2, g_1 + g_2, h_1 + h_2 \leq r$. Also, an oriented triple (or 1-triple) in D is an induced 1-subdigraph of three vertices, one vertex from each part. An oriented triple is said to be *transitive* if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$ or $u(1-0)v(0-0)w(0-1)u$ or $u(1-0)v(0-0)w(0-0)u$ or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. A tripartite r-digraph is said to be *transitive* if every of its oriented triple is transitive. In particular, an r-triple C in a tripartite r-digraph is transitive if every oriented triple of C is transitive.

The following result shows that any two transitive r-digraphs with same set of imbalance sequences can be transformed from one another.

Theorem 2.1. *Let D and D' be two tripartite r-digraphs with same imbalance sequences. Then D can be transformed into D' by successively transforming*

(i) *appropriate oriented triples in one of the following ways, either*
   (a) *by changing an intransitive oriented triple $u(1-0)v(1-0)w(1-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$, which has the same imbalance sequences or vice-versa, or*
   (b) *by changing an intransitive oriented triple $u(1-0)v(1-0)w(0-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$, which has the same imbalance sequences, or vice versa, or*

(ii) *by changing the symmetric arcs $u(1-1)v$ to $u(0-0)v$, which has the same imbalance sequences, or vice versa.*

Proof. Let A, B and C be imbalance sequences of an $l \times m \times n$ tripartite r-digraph whose parts are U, V and W, and let D^* be the tripartite digraph having parts U^*, V^* and W^*. To establish the result, it is sufficient to show that D^* can be obtained from D by transforming oriented triples and symmetric arcs in any of the ways given in (i)(a), or (i)(b), or (ii).

We fix the orders of second and third part as m and n respectively and use induction on the order l of the first part. For $l = 1$, the result is trivial. Assume that the result is true when there are fewer than l vertices in the first part. Let p and q be such that for $i > p$ and $j > q$, $1 \le p < i \le m$ and $1 \le q < j \le n$ the corresponding arcs have like orientations in D and D^*. For p and q we have the following cases to consider.

(I) There are oriented arcs of the form
   (i) $u_l(1-0)v_p(1-0)w_q$ and $u_l^*(0-0)v_p^*(0-0)w_q^*$,
   (ii) $u_l(0-0)v_p(0-1)w_q$ and $u_l^*(1-0)v_p^*(0-0)w_q^*$,
   (iii) $u_l(1-0)v_p(0-0)w_q$ and $u_l^*(0-0)v_p^*(0-1)w_q^*$.

(II) There are oriented arcs of the form
   (iv) $u_l(0-0)v_p$ and $u_l^*(1-0)v_p^*$,
   (v) $u_l(0-1)v_p$ and $u_l^*(0-0)v_p^*$.

Since u_l and u_l^* have same imbalances, so

(i) $u_l(0-1)w_q$ and $u_l^*(0-0)w_q^*$ or $u_l(0-0)w_q$ and $u_l^*(1-0)w_q^*$. Therefore there is an oriented triple $u_l(1-0)v_p(1-0)w_q(1-0)u_l$, or $u_l(1-0)v_p(1-0)w_q(0-0)u_l$ in D and corresponding to this there is an oriented triple $u_l^*(0-0)v_p^*(0-0)w_q^*(0-0)u_l^*$ or $u_l^*(0-0)v_p^*(0-0)w_q^*(0-1)u_l^*$ respectively in D^*.

(ii) $u_l(1-0)w_q$ and $u_l^*(0-0)w_q^*$. So there is an oriented triple $u_l(0-0)v_p(0-1)w_q(0-1)u_l$ in D and the corresponding oriented triple $u_l^*(1-0)v_p^*(0-0)w_q^*(0-0)u_l^*$ in D^*.

(iii) $u_l(0-1)w_q$ and $u_l^*(0-0)w_q^*$. $u_l(1-0)v_p(0-0)w_q(1-0)u_l$ is an oriented triple in D and $u_l^*(0-0)v_p^*(0-1)w_q^*(0-0)u_l^*$ is the corresponding triple in D^*.

(iv) $u_l(0-0)v_p$ and $u_l^*(1-1)v_p^*$.

(v) $u_l(1-1)v_p$ and $u_l^*(0-0)v_p^*$.

It follows from (i)-(v) that a tripartite r-digraph D^* can be obtained from D by any one of the transformations (i)(a), or (i)(b), or (ii). Hence by induction the result follows. □

The following observation is a consequence of Theorem 2.1.

Corollary 2.1. *Among all tripartite r-digraphs with given imbalance sequences, those with the fewest arcs are transitive.*

A *transmitter* is a vertex with indegree zero. In a transitive tripartite r-digraph with imbalance sequences $A=[a_1,\cdots,a_l]$, $B=[b_1,\cdots,b_m]$ and $C=[c_1,\cdots,c_n]$, any of the vertex with imbalance a_l or b_m or c_n may act as a transmitter.

The next result provides a useful recursive test whether the given sequences of integers in a non-decreasing order are the imbalance sequences of a tripartite r-digraph.

Theorem 2.2. *Let $A=[a_1,\cdots,a_l]$, $B=[b_1,\cdots,b_m]$ and $C=[c_1,\cdots,c_n]$ be the sequences of integers in non-decreasing order with $a_l>0$, $b_m\le r(l+n)$ and $c_n\le r(l+m)$. Suppose A' is obtained from A by deleting one entry a_l, and B' and C' are obtained as follows. Choose g, $1\le g\le r$, such that $(g-1)(m+n)<a_l\le g(m+n)$, and increasing $a_l-(g-1)(m+n)$ smallest entries of B and C by g each and $g(m+n)-a_l$ remaining entries by $g-1$ each. Then A,B and C are imbalances sequences of some tripartite r-digraph if and only if A',B' and C' are imbalance sequences of some tripartite r-digraph.*

Proof. Suppose A',B' and C' are imbalance sequences of some tripartite r-digraph $D'=(U',V',W')$. Then a tripartite digraph D with imbalance sequences A,B and C can be obtained by adding a transmitter u_l in U' such that $u_l(g-0)x$ for those vertices x in V' and W' whose imbalance are increased by g in going from A,B and C to A',B' and C', and $u_l((g-1)-0)y$ for those vertices y in V' and W' whose imbalances are increased by $g-1$ in going from A,B and C to A',B' and C'.

Conversely, assume that A, B and C are the imbalance sequences of a tripartite r-digraph $D(U,V,W)$. By Corollary 2.1, suppose D is transitive. Then there is a vertex u_l in U with imbalance a_l (or a vertex v_m in V with imbalance b_m or a vertex w_n in W with imbalance c_n) which is a transmitter. Clearly, $a_l>0$ so that $d^+_{u_l}>0$ and $d^-_{u_l}=0$.

Let X be the set of $a_l-(g-1)(m+n)$ vertices of smallest imbalances in V and W and let $Y=(V\cup W)-X$. Construct $D(U,V,W)$ such that $u_l(g-0)x$ for all vertices x in X and $u_l((g-1)-0)y$ for all vertices y in Y. Clearly $D(U,V,W)-\{u_l\}$ realizes A', B' and C'. □

Theorem 2.2 provides an algorithm for determining whether the sequences A, B and C of integers in a non-decreasing order are the imbalance sequences and for constructing a corresponding tripartite digraph. Let $A = [a_1, \cdots, a_l]$, $B = [b_1, \cdots, b_m]$ and $C = [c_1, \cdots, c_n]$ be imbalance sequences of a tripartite r-digraph with parts $U = \{u_1, \cdots, u_l\}$, $V = \{v_1, \cdots, v_m\}$ and $W = \{w_1, \cdots, w_n\}$, where $a_l > 0$, $b_m \leq r(l+n)$ and $c_n \leq r(l+m)$. Choose g, $1 \leq g \leq r$, such that $(g-1)(m+n) < a_l \leq g(m+n)$. Delete a_l and performing operation of Theorem 2.2, we get B' and C'. If imbalances of the vertices x are increased by g in this process, then the construction yields $u_l(g-0)x$ and if these are increased by $g-1$, the construction yields $u_l((g-1)-0)y$. Note that if at least one of the conditions $a_l > 0$, $b_m \leq r(l+n)$ and $c_n \leq r(l+m)$ does not hold, then we delete b_m or c_n for which the conditions get satisfied and the same argument as in above is used for defining arcs. If this process is applied recursively, then it tests whether or not A, B and C are imbalance sequences, and if they are then a tripartite r-digraph with imbalance sequences A, B and C is constructed.

We illustrate this process for $r = 2$ as follows, beginning with sequences A_1, B_1 and C_1.

$A_1 = [-4, -2, -1, 7]$, $B_1 = [-4, 4]$, $C_1 = [-2, 0, 2]$
$A_2 = [-4, -2, -1]$, $B_2 = [-2, 5]$, $C_2 = [0, 1, 3]$
$u_4(2-0)v_1$, $u_4(2-0)w_1$, $u_4(1-0)v_2$, $u_4(1-0)w_2$, $u_4(1-0)w_3$
$A_3 = [-3, -1, 0]$, $B_3 = [-2]$, $C_3 = [1, 2, 3]$
$v_2(1-0)u_1$, $v_2(1-0)u_2$, $v_2(1-0)u_3$, $v_2(1-0)w_1$, $v_2(1-0)w_2$
$A_4 = [-2, 0, 0]$, $B_4 = [-1]$, $C_4 = [1, 2]$
$w_3(1-0)u_1$, $w_3(1-0)u_2$, $w_3(1-0)v_1$
$A_5 = [-1, 0, 0]$, $B_5 = [0]$, $C_5 = [1]$
$w_2(1-0)u_1$, $w_2(1-0)v_1$
$A_6 = [0, 0, 0]$, $B_6 = [0]$, $C_6 = \phi$
$w_1(1-0)u_1$.

Clearly, a tripartite 2-digraph with parts $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2\}$, $W = \{w_1, w_2, w_3\}$, in which $u_4(2-0)v_1$, $u_4(2-0)w_1$, $u_4(1-0)v_2$, $u_4(1-0)w_2$, $u_4(1-0)w_3$, $v_2(1-0)u_1$, $v_2(1-0)u_2$, $v_2(1-0)u_3$, $v_2(1-0)w_1$, $v_2(1-0)w_2$, $w_3(1-0)u_1$, $w_3(1-0)u_2$, $w_3(1-0)v_1$, $w_2(1-0)u_1$, $w_2(1-0)v_1$, $w_1(1-0)u_1$ are arcs, has imbalance sequences $[-4, -2, -1, 7]$, $[-4, 4]$ and $[-2, 0, 2]$.

The next result is a combinatorial criterion giving necessary and sufficient conditions for three sequences of integers in non-increasing order to be imbalance sequences of some tripartite r-digraph. This is analogous to a result on scores in oriented tripartite graphs [8].

Theorem 2.3. *The integer sequences $A = [a_1, \cdots, a_l]$, $B = [b_1, \cdots, b_m]$ and $C = [c_1, \cdots, c_n]$ in non-increasing order are imbalance sequences of some tripartite r-digraph if and only if*

$$(2.1)\quad \sum_{i=1}^{f} a_i + \sum_{j=1}^{g} b_j + \sum_{k=1}^{h} c_k \leq r[f(m-g)(n-h) + g(l-f)(n-h) + h(l-f)(m-g)]$$

for $1 \leq f \leq l$, $1 \leq g \leq m$, $1 \leq h \leq n$, *with equality when* $f = l$, $g = m$ *and* $h = n$.

Proof. The necessity follows from the fact that a directed tripartite subgraph of a tripartite r-digraph induced by f, g and h vertices respectively from the first, second and third part has a sum of imbalances zero, and these vertices can gather at most $r[f(m-g)(n-h)+g(l-f)(n-h)+h(l-f)(m-g)]$ imbalances from the remaining $l-f$, $m-g$ and $n-h$ vertices.

For sufficiency, assume that $A = [a_1, \cdots, a_l]$, $B = [b_1, \cdots, b_m]$ and $C = [c_1, \cdots, c_n]$ are the sequences of integers in non-increasing order satisfying conditions (2.1) but are not imbalance sequences of any tripartite r-digraph. Let these sequences be chosen in such a way that l, m and n are the smallest possible and a_l is least with that choice of l, m and n. We have two cases to consider.

Case (i) Equality in (2.1) holds for some $f < l$, $g < m$ and $h < n$. That is

$$\sum_{i=1}^{f} a_i + \sum_{j=1}^{g} b_j + \sum_{k=1}^{h} c_k = r[f(m-g)(n-h) + g(l-f)(n-h) + h(l-f)(m-g)].$$

Consider the sequences

$$\begin{aligned} A' &= [a'_i]_1^f \\ &= [(a_1 - r(m-g)(n-h)),\ (a_2 - r(m-g)(n-h)),\ \cdots, (a_f - r(m-g)(n-h))], \\ B' &= [b'_j]_1^g \\ &= [(b_1 - r(l-f)(n-h)),\ (b_2 - r(l-f)(n-h)),\ \cdots, (b_g - r(l-f)(n-h))] \end{aligned}$$

and

$$C' = [c'_k]_1^h = [(c_1 - r(l-f)(m-g)),\ (c_2 - r(l-f)(m-g)),\ \cdots, (c_h - r(l-f)(m-g))],$$

where for $1 \leq i \leq f$, $1 \leq j \leq g$ and $1 \leq k \leq h$,

$$a'_i = a_i - r(m-g)(n-h),\ b'_j = b_j - r(l-f)(n-h) \text{ and } c'_k = c_k - r(l-f)(m-g).$$

For $1 \leq s < f$, $1 \leq t < g$ and $1 \leq u < h$,

$$\begin{aligned} \sum_{i=1}^{s} a'_i + \sum_{j=1}^{t} b'_j + \sum_{k=1}^{u} c'_k &= \sum_{i=1}^{s}[a_i - r(m-g)(n-h)] + \sum_{j=1}^{t}[b_j - r(l-f)(n-h)] \\ &+ \sum_{k=1}^{u}[c_k - r(l-f)(m-g)] \\ &= \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j + \sum_{k=1}^{u} c_k \\ &- sr(m-g)(n-h) - tr(l-f)(n-h) - ur(l-f)(m-g) \\ &\leq r[s(m-t)(n-u) + t(l-s)(n-u) + u(l-s)(m-t)] \\ &- r[s(m-g)(n-h) + t(f-s)(h-u) + u(f-s)(g-t)] \\ &\leq r[s(g-t)(h-u) + u(f-s)(g-t) + u(f-s)(g-t)] \end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{f} a'_i + \sum_{j=1}^{g} b'_j + \sum_{k=1}^{h} c'_k &= \sum_{i=1}^{f}[a_i - r(m-g)(n-h)] + \sum_{j=1}^{g}[b_j - r(l-f)(n-h)] \\ &+ \sum_{k=1}^{h}[c_k - r(l-f)(m-g)] \\ &= \sum_{i=1}^{f} a_i + \sum_{j=1}^{g} b_j + \sum_{k=1}^{h} c_k \\ &- fr(m-g)(n-h) - gr(l-f)(n-h) - hr(l-f)(m-g) \\ &= r[f(m-g)(n-h) + g(l-f)(n-h) + h(l-f)(m-g)] \\ &- r[f(m-g)(n-h) + g(f-s)(h-u) + h(f-s)(g-t)] \\ &= 0.\end{aligned}$$

Thus the sequences $A' = [a'_i]_1^f$, $B' = [b'_j]_1^g$ and $C' = [c'_k]_1^h$ satisfy (2.1) and by the minimality of l, m and n the sequences A', B' and C' are the imbalance sequences of some tripartite r-digraph $D'(U' \cup V' \cup W', E')$.

Let

$$A'' = [(a_{f+1} - rgh),\ (a_{f+2} - rgh),\ \cdots,(a_l - rgh)],$$

$$B'' = [(b_{g+1} - rfh),\ (b_{g+2} - rfh),\ \cdots,(b_m - rfh)]$$

and

$$C'' = [(c_{h+1} - rfg),\ (c_{g+2} - rfg),\ \cdots,(c_n - rfg)].$$

We have for $1 \le s \le l-f$, $1 \le t \le m-g$ and $1 \le u \le n-h$,

$$\begin{aligned}&\sum_{i=1}^{s}[a_{f+i} + rgh] + \sum_{j=1}^{t}[b_{j+h} + rfh] + \sum_{k=1}^{u}[c_{h+k} + rfg] \\ &= \sum_{i=1}^{f+s} a_i + \sum_{j=1}^{g+t} b_j + \sum_{k=1}^{h+u} c_k - (\sum_{i=1}^{f} a_i + \sum_{j=1}^{g} b_j + \sum_{k=1}^{h} c_k) + srgh + trfh + urfg \\ &\le r(f+s)[m-(g+t)][n-(h+u)] + r(g+t)[l-(f+s)][n-(h+u)] \\ &+ r(h+u)[l-(f+s)][m-(g+t)] \\ &- r[f(m-g)(n-h) + g(l-f)(n-h) + h(l-f)(m-g)] + srgh + trfh + urfg \\ &\le r[s(m-g-t)(n-h-u) + t(l-f-s)(n-h-u) + u(l-f-s)(m-g-t)]\end{aligned}$$

with equality when $s = l-f$, $t = m-g$ and $u = n-h$. Therefore by the minimality for l, m and n, the sequences A'', B'' and C'' are the imbalance sequences of some tripartite r-digraph $D''(U'' \cup V'' \cup W'', E'')$. Now construct a tripartite r-digraph $D(U \cup V \cup W, E)$ as follows. Let $U = U' \cup U''$, $V = V' \cup V''$ and $W = W' \cup W''$ with $U' \cap U'' = \phi$, $V' \cap V'' = \phi$ $W' \cap W'' = \phi$. Let arc set E contain those arcs which are in $D'(U' \cup V' \cup W', E')$ and $D''(U'' \cup V'' \cup W'', E'')$ together with r arcs from each vertex of U', V' and W' to every vertex of U'', V'' and W''. The resulting tripartite r-digraph has imbalance sequences A, B and C, which is a contradiction.

Case (ii). Suppose strict inequality holds in (2.1) for some $f \neq l$, $g \neq m$ and $h \neq n$. That is,

$$\sum_{i=1}^{f} a_i + \sum_{j=1}^{g} b_j + \sum_{k=1}^{h} c_k < r[f(m-g)(n-h) + g(l-f)(n-h) + h(l-f)(m-g)],$$

for $1 \leq f < l$, $1 \leq g < m$, and $1 \leq h < n$. Let $A_1 = [a_1 + 1, a_2, \cdots, a_{l-1}, a_l - 1]$, $B_1 = [b_1, \cdots, b_m]$ and $C_1 = [c_1, \cdots, c_n]$ so that A_1, B_1 and C_1 satisfy conditions (2.1). By the minimality of a_l, the sequences A_1, B_1 and C_1 are the imbalance sequences of some tripartite r-digraph $D_1(U_1, V_1, W_1)$. Let $a_{u_1} = a_1 + 1$ and $a_{u_l} = a_l - 1$. Since $a_{u_1} > a_{u_l} + 1$, there exists a vertex x either in V_1, or in W_1 such that $u_1(0-0)x(1-0)u_l$, or $u_1(1-0)x(0-0)u_l$, or $u_l(1-0)x(1-0)u_1$, or $u_l(0-0)x(0-0)u_1$, is an oriented triple in $D_1(U_1, V_1, W_1)$ and if these are changed to $u_1(0-1)x(0-0)u_l$, or $u_1(0-0)x(1-0)u_l$, or $u_l(0-0)x(0-0)u_1$, or $u_l(0-1)x(0-1)u_1$ respectively, the result is a tripartite r-digraph with imbalance sequences A, B and C, again a contradiction. Hence the result follows. □

3. Imbalance sets in tripartite r-digraphs

The set of distinct imbalances of the vertices in a tripartite r-digraph is called its *imbalance set*. We can see existence of digraphs with given imbalance sets in [6]. The following result gives the existence of a given tripartite r-digraph with a given set of integers, which acts as an imbalance set. We note that (a, b) represents the greatest common divisor of a and b.

Theorem 3.1. *Let $P = \{p_1, p_2, \cdots, p_m\}$ and $G = \{-q_1, -q_2, \cdots, -q_n\}$ where $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$ are positive integers with $p_1 < p_2 < \cdots < p_m$ and $q_1 < q_2 < \cdots < q_n$ and $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$ where $1 \leq t \leq r$. Then there exists a tripartite r-digraph with imbalance set $P \cup Q$.*

Proof. Since $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$ where $1 \leq t \leq r$, then there exist positive integers $f_1, f_2, \cdots, f_m$ and $g_1, g_2, \cdots, g_n$ with $f_1 < f_2 < \cdots < f_m$ and $g_1 < g_2 < \cdots < g_n$ such that $p_i = tf_i$ and $q_i = tg_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Construct a tripartite r-digraph $D(U, V, W)$ as under. Let

$$U = U_1 \cup U_2 \cup \cdots \cup U_m \cup U^1 \cup U^2 \cup \cdots \cup U^n,$$
$$V = V_1' \cup V_1 \cup V_2 \cup \cdots \cup V_m \cup V^1 \cup V^2 \cup \cdots \cup V^n,$$
$$W = W_1,$$

with $U_i \cap U_j = \phi$, $U_i \cap U^j = \phi$, $U^i \cap U^j = \phi$, $V_i \cap V_j = \phi$ $V_1' \cap V_i = \phi$, $V^i \cap V^j = \phi$, $V_1' \cap V^i = \phi$, $V_i \cap V^j = \phi$ for $i \neq j$, $|U_i| = g_1$ for all $1 \leq i \leq m$, $|U^i| = g_i$ for all $1 \leq i \leq n$, $|V_1'| = g_1$, $|V_i| = f_i$ for all $1 \leq i \leq m$, $|V^i| = f_1$ for all $1 \leq i \leq n$ and $|W_1| = f_1$. Let there be t arcs directed from every vertex of U_i to every vertex of V_i

for all $1 \le i \le m$; t arcs directed from every vertex of U^i to each vertex of V^i for all $1 \le i \le n$ and t arcs directed from every vertex of V_1' to each vertex of W_1, so that we obtain a tripartite r-digraph $D(U,V,W)$ with the imbalances of vertices as follows.

For $1 \le i \le m$

$$a_{u_i} = t|V_i| - 0 = tf_i = p_i$$

for all $u_i \in U_i$,

$$b_{v_i} = 0 - t|U_i| = -tg_i = -q_i$$

for all $v_i \in V_i$.

For $1 \le i \le n$

$$a_{u^i} = t|V^i| - 0 = tf_i = p_i$$

for all $u^i \in U^i$,

$$b_{v^i} = 0 - t|U^i| = -tg_i = -q_i$$

for all $v^i \in V^i$,

$$b_{v_1'} = t|W_1| - 0 = tf_1 = p_1$$

for all $v_1' \in V_1'$;

and

$$c_{w_1} = 0 - t|W_1'| - 0 = -tg_1 = -q_1$$

for all $w_1 \in W_1$.

Therefore imbalance set of $D(U,V,W)$ is $P \cup Q$. □

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