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# SOME REMARKS ON RESULTS OF MORTICI

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Abstract. In this paper, we establish new generalizations of Turán-type inequality involving the Gamma and Polygamma functions.

## 1. Introduction

P. Turán ([1]) proved that the Legendre polynomials $P_n(x)$ satisfy the determinantal inequality

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \leq 0, \quad -1 \leq x \leq 1 \tag{1.1}$$

where $n = 0, 1, 2, \ldots$ and equality occurs only if $x = \pm 1$. This classical result has been extended in several directions: ultraspherical polynomials, Lagguerre and Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, etc. Today there is a huge literature on Turán inequalities, since they have important applications in complex analysis, number theory, combinatorics, theory of mean-values, or statistics and control theory.

Recently, Mortici ([2], [3]) proved some Turán-type inequalities for some special functions as well as the polygamma functions.

The aim of this paper is to prove new generalizations of Turán-type inequalities for the Gamma and Polygamma functions in ([2],[3]).

## 2. Main Results

**Lemma 2.1** (Hölder's inequality)**.** *If $p_1, \ldots, p_n > 0$ are such that $\sum_{i=1}^{n} p_i^{-1} = 1$ and $f_1, \ldots, f_n$ are non-negative functions such that these integrals exist, then the following*

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*inequality holds:*

$$\prod_{i=1}^{n}\left(\int_0^\infty f_i^{p_i}(t)dt\right)^{\frac{1}{p_i}} \geq \int_0^\infty \left(\prod_{i=1}^{n} f_i(t)\right) dt. \tag{2.1}$$

In what follows, we use the integral representations, for $x > 0$ and $n = 1, 2, \ldots$

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}t^{x-1}\log^n t dt, \tag{2.2}$$

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-tx}}{1-e^{-t}}dt, \tag{2.3}$$

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t - 1}dt, \quad x > 1, \tag{2.4}$$

and

$$\theta(x) = (-1)^n\int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{n-1}e^{-tx}dt, \tag{2.5}$$

where $\Gamma$ is the gamma function, $\psi^{(n)}$ is the $n$-th polygamma function and $\zeta$ is the Reimann-zeta function.

We first give a generalization of Mortici ([2, Theorem 2.1], [3, Theorem 2.2]).

**Theorem 2.1.** *Let $p_1, \ldots, p_n$ be conjugate parameters such that $p_i > 1,\ i = 1, \ldots, n$, and $\sum_{i=1}^n \frac{1}{p_i} = 1$, and let $m_1, \ldots, m_n \geq 1$ be integers such that $\sum_{i=1}^n \frac{m_i}{p_i}$ is an integer. Then the following inequality holds for $x_i > 0,\ i = 1, \ldots, n$:*

$$\prod_{i=1}^{n}\left|\psi^{(m_i)}(x_i)\right|^{\frac{1}{p_i}} \geq \left|\psi^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right)\right|. \tag{2.6}$$

*Proof.* By Hölder's inequality (2.1) and (2.3), we have

$$\begin{aligned}
\prod_{i=1}^{n}\left|\psi^{(m_i)}(x_i)\right|^{\frac{1}{p_i}} &= \prod_{i=1}^{n}\left(\int_0^\infty \frac{t^{m_i}e^{-tx_i}}{1-e^{-t}}dt\right)^{\frac{1}{p_i}} \\
&\geq \int_0^\infty\left(\prod_{i=1}^{n}\frac{t^{\frac{m_i}{p_i}}e^{\frac{-tx_i}{p_i}}}{(1-e^{-t})^{\frac{1}{p_i}}}\right)dt \\
&= \int_0^\infty \frac{t^{\sum_{i=1}^n \frac{m_i}{p_i}}e^{-t\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)}}{1-e^{-t}}dt \\
&= \left|\psi^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^{n}\frac{x_i}{p_i}\right)\right|.
\end{aligned}$$

Hence (2.6) is valid. □

The next result is a generalization of Mortici ([2, Theorem 2.2], [3, Theorem 2.1, 2.3]).

**Theorem 2.2.** *Let $p_1,\dots,p_n$ be conjugate parameters such that $p_i>1,\ i=1,\dots,n$, and $\sum_{i=1}^n \frac{1}{p_i}=1$, and let $m_1,\dots,m_n\ge 1$ be integers such that $\sum_{i=1}^n \frac{m_i}{p_i}$ is an integer. Then the following inequalities hold:*

$$\prod_{i=1}^n \left(\Gamma^{(m_i)}(x_i)\right)^{\frac{1}{p_i}} \ge \Gamma^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^n \frac{x_i}{p_i}\right),\quad x_i>0,\ i=1,\dots,n, \tag{2.7}$$

$$\prod_{i=1}^n (\zeta(x_i))^{\frac{1}{p_i}} \ge \frac{\Gamma\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)}{\prod_{i=1}^n (\Gamma(x_i))^{\frac{1}{p_i}}}\zeta\left(\sum_{i=1}^n \frac{x_i}{p_i}\right),\quad x_i>1,\ i=1,\dots,n, \tag{2.8}$$

$$\prod_{i=1}^n \left|\theta^{(m_i)}(x_i)\right|^{\frac{1}{p_i}} \ge \left|\theta^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)\right|,\quad x_i>0,\ i=1,\dots,n. \tag{2.9}$$

*Proof.* We just use Hölder's inequality (2.1) and the proofs are similar to the proof of Theorem 2.1. We omit the details. □

The next result is a generalization of Mortici ([2, Theorem 3.1]).

**Theorem 2.3.** *Let $p_1,\dots,p_n$ be conjugate parameters such that $p_i>1,\ i=1,\dots,n$, and $\sum_{i=1}^n \frac{1}{p_i}=1$, and let $m_1,\dots,m_n\ge 0$ be even integers such that $\sum_{j=1}^n \frac{m_j}{p_j}$ is even integers. Then the following inequality holds for $x_i>0$, $i=1,\dots,n$:*

$$\exp\Gamma^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^n \frac{x_i}{p_i}\right) \le \prod_{i=1}^n \left(\exp\Gamma^{(m_i)}(x_i)\right)^{\frac{1}{p_i}}. \tag{2.10}$$

*Proof.* Using (2.2) and Weighted AM-GM inequality

$$\begin{aligned}
&\sum_{i=1}^n \frac{\Gamma^{(m_i)}(x_i)}{p_i} - \Gamma^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)\\
&= \sum_{i=1}^n \frac{1}{p_i}\int_0^\infty e^{-t}t^{x_i-1}\log^{m_i} t dt - \int_0^\infty e^{-t}t^{\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)-1}\log^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)} tdt\\
&= \int_0^\infty \left(\sum_{i=1}^n \frac{1}{p_i}t^{x_i-\left(\sum_{j=1}^n \frac{x_j}{p_j}\right)}\log^{m_i-\left(\sum_{j=1}^n \frac{m_j}{p_j}\right)} t-1\right)e^{-t}t^{\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)-1}\log^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)} tdt\\
&\ge \int_0^\infty \left(\prod_{i=1}^n t^{\frac{x_i}{p_i}-\frac{1}{p_i}\left(\sum_{j=1}^n \frac{x_j}{p_j}\right)}\log^{\frac{m_i}{p_i}-\frac{1}{p_i}\left(\sum_{j=1}^n \frac{m_j}{p_j}\right)} t-1\right)e^{-t}t^{\left(\sum_{i=1}^n \frac{x_i}{p_i}\right)-1}\log^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)} tdt\\
&= 0.
\end{aligned}$$

The conclusion follows by exponentiating the inequality

$$\sum_{i=1}^n \frac{\Gamma^{(m_i)}(x_i)}{p_i} \ge \Gamma^{\left(\sum_{i=1}^n \frac{m_i}{p_i}\right)}\left(\sum_{i=1}^n \frac{x_i}{p_i}\right).$$

□

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