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ON A CONJECTURE ON THE DIAMETER OF LINE GRAPHS OF GRAPHS OF DIAMETER TWO

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ABSTRACT. Let F_1 be the 5-vertex path, F_2 the graph obtained by identifying a vertex of a triangle with one end vertex of the 3-vertex path and F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle. Let $diam(G)$ be the diameter of the graph G. In the paper [H. S. Ramane and I. Gutman, *Counterexamples for properties of line graphs of graphs of diameter two*, Kragujevac J. Math. **34** (2010), 147–150] it is conjectured that if $diam(G) \leq 2$ and if none of the F_i, $i = 1, 2, 3$, is an induced subgraph of G, then $diam(L^k(G)) > 2$ for some $k \geq 2$. In this paper we prove this conjecture.

1. INTRODUCTION

Let G be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance* between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j. The shortest $v_i - v_j$ path is often called a *geodesic*. The *diameter* of a connected graph G, denoted by $diam(G)$, is the length of any longest geodesic in G.

The *line graph* of G, denoted by $L(G)$ is the graph whose vertex set is in a one-to-one correspondence with the edge set of the graph G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. For $k = 1, 2, \ldots$, the k-th iterated line graph of G is $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$.

For any subset S of vertices of G, the subgraph induced by S is the maximal subgraph of G with vertex set S. are to such a graph

Let F_1 be the 5-vertex path, F_2 the graph obtained by identifying a vertex of a triangle with one end vertex of the 3-vertex path, F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle and F_4 be the graph obtained

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by identifying one end vertex of a 4-vertex star with a middle vertex of a 3-vertex path, see Figure 1.

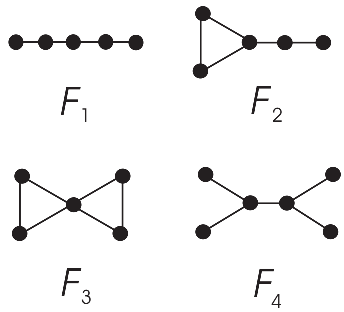

FIGURE 1. The graphs mentioned in Theorems 1.1–1.3.

In [2], the following Theorems 1.1, 1.2, and 1.3 were stated, of which Theorem 1.1 was correct whereas Theorems 1.2 and 1.3 were eventually found to be erroneous [1].

Theorem 1.1. *If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3 depicted in Figure 1 is an induced subgraph of G, then $diam(L(G)) \leq 2$.*

Theorem 1.2. *If $diam(G) \leq 2$ and if none of the four graphs depicted in Figure 1 is an induced subgraph of G, then none of these four graphs is an induced subgraph of $L(G)$.*

Theorem 1.3. *If $diam(G) \leq 2$ and if none of the four graphs depicted in Figure 1 is an induced subgraph of G, then for $k \geq 1$,*

(i) *$diam(L^k(G)) \leq 2$ and*
(ii) *none of the four graphs from Figure 1 is an induced subgraph of $L^k(G)$.*

Counterexamples for the Theorems 1.2 and 1.3 are found in [1]. The graph H given in Figure 2 does not contain F_i, $i = 1, 2, 3, 4$, as an induced subgraph, but its line graph $L(H)$ contains F_1 as induced subgraph. Therefore $diam(L^2(H)) > 2$. Another counterexample is K_4, the complete graph on 4 vertices. The second line graph of K_4 contains F_1 and F_2 as induced subgraphs, implying that $diam(L^3(K_4)) > 2$.

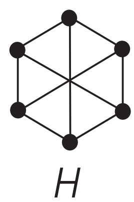

FIGURE 2. A counterexample for Theorem 1.2.

Theorem 1.1 is correct. Theorem 1.2 is erroneous. Because Theorem 1.3 is obtained by combining Theorems 1.1 and 1.2, it is also not generally valid. In fact Theorem 1.3 holds for $k = 1$ and is not true for $k > 1$.

In view of the counterexamples discovered in [1], the following conjecture was formulated:

Conjecture 1.1. *If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3 from Figure 1 is an induced subgraph of G, then $diam(L^k(G)) > 2$ for some $k \geq 2$.*

In what follows we prove this conjecture.

2. Proof of Conjecture 1.1

Theorem 2.1. *If G_1 is an induced subgraph of G then $L(G_1)$ is an induced subgraph of $L(G)$.*

Proof. Let $e_1, e_2, \ldots, e_k$ be the edges of an induced subgraph G_1 of G. Thus $e_1, e_2, \ldots, e_k$ are also the edges of G. If the edges e_i and e_j do not have a common vertex in G_1 then the vertices e_i and e_j are not adjacent in $L(G_1)$. Hence e_i and e_j are also not adjacent in $L(G)$. Similarly if the edges e_i and e_j are incident to a common vertex in G_1, then the vertices e_i and e_j are adjacent in $L(G_1)$. Hence e_i and e_j are adjacent in $L(G)$ also for $i, j = 1, 2, \ldots, k$. So the subgraph of $L(G)$ induced by the vertices $e_1, e_2, \ldots, e_k$ is isomorphic to $L(G_1)$. Hence $L(G_1)$ is an induced subgraph of $L(G)$. □

Let C_n, P_n, and $S_n = K_{1,n-1}$ be, respectively, the cycle, the path, and the star on n vertices. We restate Conjecture 1.1 as follows:

Conjecture 2.1. *Let G be a graph with $n \geq 4$ vertices and $G \neq C_4, C_5, K_{1,3}, P_4$. If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3 from Figure 1 are induced subgraphs of G, then $diam(L^k(G)) > 2$ for some $k \geq 2$.*

We now prove Conjecture 2.1.

Proof. Since $diam(G) \leq 2$, since $G \neq C_4, C_5, K_{1,3}, P_4$, and since F_1, F_2, and F_3 are not induced subgraphs of G, it follows that G has one of the graphs depicted in Figure 3 as an induced subgraph.

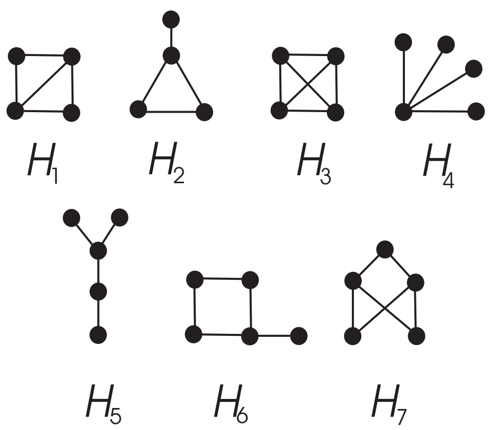

Figure 3. Graphs used in the proof of Conjecture 2.1.

If there would be any induced subgraph of G other than those shown in Figure 3, then that induced subgraph would contain one of the graphs shown in Figure 3 as induced subgraph.

By direct checking the following can be verified.

(i) $diam(L^4(H_1)) > 2$.
(ii) $L(H_2) = H_1$, so $diam(L^5(H_2)) = diam(L^4(H_1)) > 2$.
(iii) $diam(L^3(H_3)) > 2$.
(iv) $L(H_4) = H_3$, so $diam(L^4(H_4)) = diam(L^3(H_3)) > 2$.
(v) $L(H_5) = H_2$, so $diam(L^6(H_5)) = diam(L^5(H_2)) > 2$.
(vi) $diam(L^3(H_6)) > 2$.
(vii) $diam(L^3(H_7)) > 2$.

Therefore $diam(L^k(H_i)) > 2$, for some $k \geq 2$, $i = 1, 2, \ldots, 7$. Hence the proof follows from Theorem 4. □

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