Kragujevac Journal of Mathematics
Volume 36 Number 1 (2012), Pages 51–57.

HOSOYA POLYNOMIAL OF HANOI GRAPHS

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Abstract. A recursive method for the calculation of the Hosoya polynomial of Hanoi graph is designed. This make it possible to compute various distance–based invariants of Hanoi graphs.

1. Introduction

Let G be a connected graph and $V(G) = \{v_1, v_2, \dots, v_p\}$ its vertex set. The distance of the vertices v_i and v_j, denoted by $d(v_i, v_j|G)$, is the length of (= number of edges in) a shortest path that connects v_i and v_j [2]. If $i = j$, then $d(v_i, v_j|G) = 0$.

A large number of distance–based graph invariants has been studied both in the last 50–60 years and in the recent past (see [4, 5, 3, 1] and the references cited therein). Of these we mention here the Wiener index (W), the hyper–Wiener index (WW), the Harary index (Ha), the reciprocal Wiener index (RW), and the distance moments (W_ρ). These are defined as

$$\begin{aligned} W = W(G) &= \sum_{i<j} d(v_i, v_j|G), \\ WW = WW(G) &= \frac{1}{2} \sum_{i<j} \Big[d(v_i, v_j|G)^2 + d(v_i, v_j) \Big], \\ Ha = Ha(G) &= \sum_{i<j} \frac{1}{d(v_i, v_j|G)^2}, \\ RW = RW(G) &= \sum_{i<j} \frac{1}{d(v_i, v_j|G)}, \\ W_\rho = W_\rho(G) &= \sum_{i<j} d(v_i, v_j|G)^\rho. \end{aligned}$$

Key words and phrases. Distance (in graph), Hosoya polynomial, Hanoi graph.
2010 *Mathematics Subject Classification.* Primary: 05C12, Secondary: 05C90.
Received: January 8, 2012.
Revised: February 21, 2012.

In the study of these and other distance–based graph invariants it is often convenient to work with the numbers $d(G,k)$ which are equal to the number of pairs of vertices of the graph G that are at distance k. With this notation, we have

$$W = W(G) = \sum_{k\geq 1} k\, d(G,k), \tag{1.1}$$

$$WW = WW(G) = \frac{1}{2}\sum_{k\geq 1}(k^2+k)\, d(G,k), \tag{1.2}$$

$$Ha = Ha(G) = \sum_{k\geq 1}\frac{1}{k^2}\, d(G,k), \tag{1.3}$$

$$RW = RW(G) = \sum_{k\geq 1}\frac{1}{k}\, d(G,k), \tag{1.4}$$

$$W_\rho = W_\rho(G) = \sum_{k\geq 1} d(G,k)\, k^\rho. \tag{1.5}$$

One of the advantages of the latter formulas is that these can be applied also in the case of graphs that are not connected.

In the 1980s Hosoya [8] conceived a graph polynomial which he named "*Wiener polynomial*", but which most contemporary authors call the "*Hosoya polynomial*". It is defined as

$$H(G;\lambda) = \sum_{k\geq 1} d(G,k)\, \lambda^k. \tag{1.6}$$

The maximal distance between two vertices of the graph G is referred to as the *diameter* of G. It will be denoted by $d(G)$. Then $d(G,k) > 0$ holds for $1 \geq k \geq d(G)$, whereas $d(G,k) = 0$ holds for $k > d(G)$. In view of this, we see that $H(G;\lambda)$ is a polynomial of degree $d(G)$.

The theory of the Hosoya polynomial is nowadays well elaborated; its details and additional references can be found in the recent survey [6]. By comparing Eq. (1.6) with Eqs. (1.1)–(1.5), it should be evident that if one knows the Hosoya polynomial of a graph, then it is elementary to compute any of its distance–based invariants. The respective formulas can be found elsewhere [6]. The simplest such formula is

$$W(G) = H'(G;1)$$

where $H'(G;\lambda)$ denotes the first derivative of $H(G;\lambda)$. This result motivated Hosoya to call $H(G;\lambda)$ the Wiener polynomial [8].

2. The Hanoi graph

The tower of Hanoi puzzle, invented in 1883 by the French mathematician, Edouard Lucas, has become a classic example in the analysis of algorithms and discrete mathematics [9, 7].

The puzzle consists of n discs, no two of the same size, stacked on three vertical pegs, in such a way that no disc lies on top of a smaller disc. A permissible move is to take a top disc from one of the pegs and move it to one of the other pegs as long as it is not placed on top of a smaller disc. The set of configurations of the puzzle, together with the permissible moves, thus forms a graph in a natural way. The number of vertices in the n-disc Hanoi graph H_n is 3^n and the number of edges is $3(3^n - 1)/2$. One can verify that each Hanoi graph has a unique Hamiltonian cycle and 3^{n-1} small triangles.

The following figure shows the Hanoi graphs for $n = 1, 2, 3$.

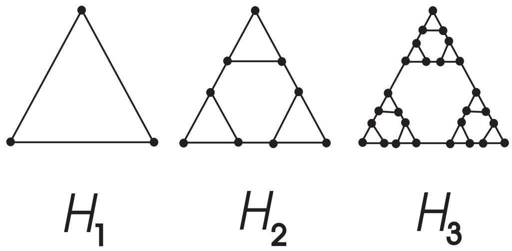

FIGURE 1. The first three Hanoi graphs.

The Hanoi graph H_n can be constructed by taking the vertices to be the odd binomial coefficients of the Pascal triangle, computed on the integers from 0 to $2^n - 1$ and drawing an edge whenever the coefficients are adjacent diagonally or horizontally [9]. More information on shortest paths in the tower of Hanoi graph and finite automata can be found in [10].

The main properties of Hanoi graphs are summarized in the following proposition:

Proposition 2.1. [9] *For all positive integers n, H_n is a planar 2-connected Hamiltonian graph of order 3^n and diameter $d = 2^n - 1$, with exactly three vertices of degree 2 and all other vertices of degree 3.*

In the subsequent section we show how the Hosoya polynomial of the Hanoi graphs can be recursively evaluated.

3. Hosoya polynomial of Hanoi graphs

The Hosoya polynomial of the Hanoi graph H_n is of the form

$$H(H_n; \lambda) = d(H_n, 1)\,\lambda + d(H_n, 2)\,\lambda^2 + \cdots + d(H_n, d)\,\lambda^d$$

where $d = 2^n - 1$ is the diameter of H_n.

Theorem 3.1. *Let $n \geq 1$ and let the ordered (2^{n-1})-tuple of integers S_n be recursively defined as:*

$$S_1 = \{1\},$$

$$S_2 = \{1, 2\} = \{S_1, 2S_1\}, \quad (3.1)$$

$$S_3 = \{1, 2, 2, 4\} = \{S_2, 2S_2\}, \quad (3.2)$$

$$S_4 = \{1,2,2,4,2,4,4,8\} = \{S_3, 2S_3\}, \quad (3.3)$$

$$\dots$$

$$S_n = \{S_{n-1}, 2S_{n-1}\}. \quad (3.4)$$

Denote the elements of S_n by ℓ_i, $i = 0,1,2,\dots,2^{n-1}-1$ so that

$$S_n = \{\ell_0, \ell_1, \ell_2, \dots, \ell_{2^{n-1}-1}\}$$

and assume that $\ell_t = 0$ for $t < 0$ and $t > 2^{n-1}-1$. For $k = 1,2,\dots,2^n-1$ define,

$$b(n,k) = \sum_{i=0}^{k} \ell_i \, \ell_{k-i-1}. \quad (3.5)$$

Then the coefficients of the Hosoya polynomial of the Hanoi graph H_n satisfy the recursion relation:

$$d(H_n,k) = 3[d(H_{n-1},k) + b(n,k)] \quad (3.6)$$

for $k = 1,2,\dots,2^n-1$.

Proof. Let H_n be the Hanoi graph with the vertex set $V(H_n)$ partitioned into disjoint subsets A, B, and C as shown in Figure 2

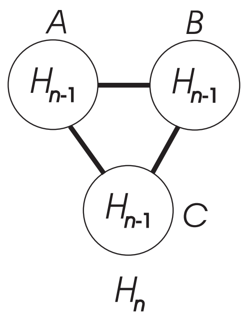

FIGURE 2. Partitioning of the vertices of the Hanoi graph H_n.

For $X,Y = A,B,C$, denote by $d_{XY}(H_n,k)$ the number of vertex pairs (v_i,v_j) of the Hanoi graph H_n that are at distance k, so that $v_i \in X$ and $v_j \in Y$. Then,

$$\begin{aligned} d(H_n,k) &= d_{AA}(H_n,k) + d_{BB}(H_n,k) + d_{CC}(H_n,k) \\ &+ d_{AB}(H_n,k) + d_{AC}(H_n,k) + d_{BC}(H_n,k). \end{aligned} \quad (3.7)$$

In view of the notation described in Figure 2,

$$d_{AA}(H_n,k) = d_{BB}(H_n,k) = d_{CC}(H_n,k) = d(H_{n-1},k). \quad (3.8)$$

In view of the symmetry of the Hanoi graphs (cf. Figures 1 and 2),

$$d_{AB}(H_n,k) = d_{AC}(H_n,k) = d_{BC}(H_n,k).$$

Therefore, we only need to compute $d_{AB}(H_n,k)$.

Let G_{AB} be the subgraph induced by the sets A and B as shown in Figure 3.

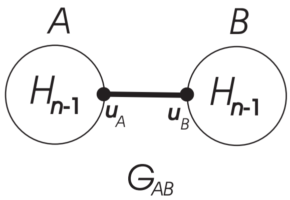

FIGURE 3. The subgraph induced by the vertex sets A and B of the Hanoi graph H_n.

Evidently, $d_{AB}(H_n, k)$ is equal to the number of vertex pairs of G_{AB} whose distance is k, such that one vertex belongs to A and the other to B. From Fig. 3 we see that the number of such vertex pairs is equal to the sum over $i = 0, 1, \ldots, k$, of the product of the number of vertices in A whose distance to vertex u_A is i and the number of vertices in B whose distance to vertex u_B is $k - i - 1$. If these numbers are denoted by ℓ_i and ℓ_{k-i-1}, then we have

$$d_{AB}(H_n, k) = \sum_{i=0}^{k} \ell_i \, \ell_{k-i-1} \equiv b(n, k) \tag{3.9}$$

as well as

$$d_{AC}(H_n, k) = d_{BC}(H_n, k) = b(n, k). \tag{3.10}$$

In order to complete the proof of Theorem 3.1, it only remains to verify that ℓ_i counts the number of vertices in H_{n-1} whose distance to vertex u_A (or more generally, to a vertex of degree 2) is equal to i.

That this indeed is the case can be directly checked for H_1, H_2, and H_3, resulting in the sequences (3.1), (3.2), and (3.3), respectively. That the general form of such a sequence is (3.4) is now easily proved by induction on n.

Thus Eq. (3.9) is valid, and therefore also Eq. (3.10) holds. Substituting Eqs. (3.9), (3.10), and (3.8) back into Eq. (3.7), we arrive at the recursive formula (3.6), by which the proof of Theorem 3.1 is completed. □

4. EXAMPLES

As illustrative examples of the above result, we compute $H(H_2; \lambda)$ and $H(H_3; \lambda)$, which are polynomials in λ of degree $2^2 - 1 = 3$ and $2^3 - 1 = 7$, respectively.

As easily seen, $H(H_1; \lambda) = 3\,\lambda$, i. e., $d(H_1, 1) = 3$ and $d(H_1, k) = 0$ for $k \geq 2$.

We first need $S_2 = \{1, 2\}$, which means that

$$\begin{aligned} \ell_0 &= 1 \\ \ell_1 &= 2 \end{aligned}$$

Now, by using formula (3.5),

$$\begin{aligned} b(2,1) &= \ell_0 \, \ell_0 = 1 \cdot 1 = 1, \\ b(2,2) &= \ell_0 \, \ell_1 + \ell_1 \, \ell_0 = 1 \cdot 2 + 2 \cdot 1 = 4, \\ b(2,3) &= \ell_0 \, \ell_2 + \ell_1 \, \ell_1 + \ell_2 \, \ell_0 = 1 \cdot 0 + 2 \cdot 2 + 0 \cdot 1 = 4. \end{aligned}$$

From Eq. (3.6) we now get

$$\begin{aligned} d(H_2,1) &= 3[d(H_1,1)+b(2,1)] = 3[3+1] = 12, \\ d(H_2,2) &= 3[d(H_1,2)+b(2,2)] = 3[0+4] = 12, \\ d(H_2,3) &= 3[d(H_1,3)+b(2,3)] = 3[0+4] = 12 \end{aligned}$$

which finally gives

$$H(H_2;\lambda) = 12\,\lambda + 12\,\lambda^2 + 12\,\lambda^3.$$

Repeating the calculation for $n=3$, we start with $S_3 = \{1,2,2,4\}$, which means that

$$\begin{aligned} \ell_0 &= 1, \\ \ell_1 &= 2, \\ \ell_2 &= 2, \\ \ell_3 &= 4. \end{aligned}$$

By formula (3.5),

$$\begin{aligned} b(3,1) &= \ell_0\,\ell_0 = 1\cdot 1 = 1, \\ b(3,2) &= \ell_0\,\ell_1 + \ell_1\,\ell_0 = 1\cdot 2 + 2\cdot 1 = 4, \\ b(3,3) &= \ell_0\,\ell_2 + \ell_1\,\ell_1 + \ell_2\,\ell_0 = 1\cdot 2 + 2\cdot 2 + 2\cdot 1 = 8, \\ b(3,4) &= \ell_0\,\ell_3 + \ell_1\,\ell_2 + \ell_2\,\ell_1 + \ell_3\,\ell_0 = 1\cdot 4 + 2\cdot 2 + 2\cdot 2 + 4\cdot 1 = 16, \\ b(3,5) &= \ell_0\,\ell_4 + \ell_1\,\ell_3 + \ell_2\,\ell_2 + \ell_3\,\ell_1 + \ell_4\,\ell_0 = 1\cdot 0 + 2\cdot 4 + 2\cdot 2 + 4\cdot 2 + 0\cdot 1 = 20, \\ b(3,6) &= \ell_0\,\ell_5 + \ell_1\,\ell_4 + \ell_2\,\ell_3 + \ell_3\,\ell_2 + \ell_4\,\ell_1 + \ell_5\,\ell_0 \\ &= 1\cdot 0 + 2\cdot 0 + 2\cdot 4 + 4\cdot 2 + 0\cdot 2 + 0\cdot 1 = 16, \\ b(3,7) &= \ell_0\,\ell_6 + \ell_1\,\ell_5 + \ell_2\,\ell_4 + \ell_3\,\ell_3 + \ell_4\,\ell_2 + \ell_5\,\ell_1 + \ell_6\,\ell_0 \\ &= 1\cdot 0 + 2\cdot 0 + 2\cdot 0 + 4\cdot 4 + 0\cdot 2 + 0\cdot 2 + 0\cdot 1 = 16 \end{aligned}$$

which together with Eq. (3.6) yields

$$\begin{aligned} d(H_3,1) &= 3[d(H_2,1)+b(3,1)] = 3[12+1] = 39, \\ d(H_3,2) &= 3[d(H_2,2)+b(3,2)] = 3[12+4] = 48, \\ d(H_3,3) &= 3[d(H_2,3)+b(3,3)] = 3[12+8] = 60, \\ d(H_3,4) &= 3[d(H_2,4)+b(3,4)] = 3[0+16] = 48, \\ d(H_3,5) &= 3[d(H_2,5)+b(3,5)] = 3[0+20] = 60, \\ d(H_3,6) &= 3[d(H_2,6)+b(3,6)] = 3[0+16] = 48, \\ d(H_3,7) &= 3[d(H_2,7)+b(3,7)] = 3[0+16] = 48 \end{aligned}$$

and finally,

$$H(H_3;\lambda) = 39\,\lambda + 48\,\lambda^2 + 60\,\lambda^3 + 48\,\lambda^4 + 60\,\lambda^5 + 48\,\lambda^6 + 48\,\lambda^7.$$

Acknowledgement: Kishori P. Narayankar and Lokesh S. B. thank the UGC's Project No. 39–36/2010 (SR), MRP. I. G. thanks the Serbian Ministry of Science and Education for support through Grant No. 174033.

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