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AN EXPLICIT FORMULA OF HESSIAN DETERMINANTS OF COMPOSITE FUNCTIONS AND ITS APPLICATIONS

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Abstract. The determinants of Hessian matrices of differentiable functions play important roles in many areas in mathematics. In practice it can be difficult to compute the Hessian determinants for functions with many variables.

In this article we derive a very simple explicit formula for the Hessian determinants of composite functions of the form:

$$f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n)).$$

Several applications of the Hessian determinant formula to production functions in microeconomics are also given in this article.

1. Introduction

The Hessian matrix $H(f)$ (or simply the Hessian) is the square matrix (f_{ij}) of second-order partial derivatives of a function f. If the second derivatives of f are all continuous in a neighborhood D, then the Hessian of f is a symmetric matrix throughout D. Because f is often clear from context, $H(f)(\mathbf{x})$ is frequently shortened to simply $H(\mathbf{x})$.

Hessian matrices of functions play important roles in many areas in mathematics. For instance, Hessian matrices are used in large-scale optimization problems within Newton-type methods because they are the coefficient of the quadratic term of a local Taylor expansion of a function, i.e.,

$$f(\mathbf{x} + \Delta\mathbf{x}) \approx f(\mathbf{x}) + J(\mathbf{x})\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T H(\mathbf{x})\Delta\mathbf{x},$$

where J is the Jacobian matrix, which is a vector (the gradient for scalar-valued functions). Another example is the application of Hessian matrices to Morse theory

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[15]. If the gradient of f on a manifold is zero at some point $\mathbf{x}$, then f has a critical point at $\mathbf{x}$. The determinant of the Hessian matrix at $\mathbf{x}$ is then called the *discriminant*. If this discriminant is zero, then $\mathbf{x}$ is called a degenerate critical point of f, this is also called a non-Morse critical point of f. Otherwise it is non-degenerate, this is called a Morse critical point of f. Morse theory studies differentiable functions on manifolds via Morse critical points. The theory gives a very direct way of analyzing the topology of a manifold. Morse theory via Hessian matrices allows one to find CW structures and handle decompositions on manifolds and to obtain substantial information about their homology.

Morse theory also have direct important applications to the theory of geodesics (critical points of the energy functional on paths) in differential geometry. These techniques were also used in R. Bott's proof of his celebrated periodicity theorem of homotopy groups of classical groups [3].

In practice it can be difficult to compute explicitly the determinants of large size matrices (see, e.g. [14, 16, 18, 19]).

In this article, we derive a very simple explicit formula for the determinants of the Hessian matrices of composite functions of the form:

$$f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n)). \tag{1.1}$$

Several applications of the Hessian determinant formula to production functions in microeconomics are also given in this article.

2. CD and CES production functions

In microeconomics, a *production function* is a positive non-constant function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

In 1928, C. W. Cobb and P. H. Douglas introduced in [9] a famous two-factor production function

$$Y = bL^kC^{1-k}, \tag{2.1}$$

where b represents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called Cobb-Douglas production function.

The Cobb-Douglas function is widely used in economics to represent the relationship of an output to inputs. Later work in the 1940s prompted them to allow for the exponents on C and L vary, which resulting in estimates that subsequently proved to be very close to improved measure of productivity developed at that time (cf. [10, 11]).

In its generalized form the Cobb-Douglas (CD) production function may be expressed as

$$f = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{2.2}$$

where γ is a positive constant and $\alpha_1, \dots, \alpha_n$ are nonzero constants.

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [2] introduced another two-factor production function given by

$$Q = F \cdot (aK^r + (1-a)L^r)^{\frac{1}{r}}, \tag{2.3}$$

where Q is the output, F the factor productivity, a the share parameter, K and L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution.

The generalized form of ACMS production function is given by

$$f = \gamma \Big(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho} \Big)^{\frac{h}{\rho}}, \tag{2.4}$$

where a_i, h, γ, ρ are constants with $a_i, \gamma, h \neq 0$ and $\rho \neq 0, 1$.

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. The elasticity of substitution was originally introduced by J. R. Hicks [12] in case of two inputs for the purpose of analyzing changes in the income shares of labor and capital. Elasticity of substitution is the elasticity of the ratio of two inputs to a production function with respect to the ratio of their marginal products. It tells how easy it is to substitute one input for the other.

R. G. Allen and J. R. Hicks suggested in [1] a generalization of Hicks original two variable elasticity. Let f be a twice differentiable production function with nonvanishing first partial derivatives. Put

$$H_{ij}(\mathbf{x}) = \frac{\dfrac{1}{x_i f_i} + \dfrac{1}{x_j f_j}}{-\dfrac{f_{ii}}{f_i^2} + \dfrac{2f_{ij}}{f_i f_j} - \dfrac{f_{jj}}{f_j^2}}, \quad 1 \leq i \neq j \leq n, \tag{2.5}$$

for $\mathbf{x} \in \mathbb{R}_+^n$, where the denominator is assumed to be different from zero.

The H_{ij} is called the *Hicks elasticity of substitution* of the i-th production variable with respect to the j-th production variable.

A twice differentiable production function f with nowhere zero first partial derivatives is said to satisfy the CES (constant elasticity of substitution) property if there is a nonzero constant $\sigma \in \mathbb{R}$ such that

$$H_{ij}(\mathbf{x}) = \sigma \ \text{ for } \mathbf{x} \in \mathbb{R}_+^n \text{ and } 1 \leq i \neq j \leq n. \tag{2.6}$$

It is easy to verify that the generalized ACMS production function satisfy the CES property with

$$H_{ij}(\mathbf{x}) = \frac{1}{\rho}$$

if $\rho \neq 1$. For $\rho = 1$ the denominator of H_{ij} is zero, hence it is not defined. For this reason, the generalized ACMS production function is also known as the *generalized CES production function*.

The same functional form arises as a utility function in consumer theory. For example, if there exist n types of consumption goods x_i, then aggregate consumption C could be defined using the CES aggregator as

$$C = \left(\sum_{i=1}^{n} a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}}, \tag{2.7}$$

where the coefficients a_i are share parameters, and s is the elasticity of substitution.

A production function $f = f(x_1, \cdots, x_n)$ is called *h-homogeneous* or *homogeneous of degree* h, if given any positive constant t,

$$f(tx_1, \ldots, tx_n) = t^h f(x_1, \ldots, x_n) \tag{2.8}$$

for some constant h. If $h > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $h < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale.

The presence of increasing returns means that a one percent increase in the usage levels of all inputs would result in a greater than one percent increase in output; the presence of decreasing returns means that it would result in a less than one percent increase in output. Constant returns to scale is the in-between case.

It is easy to see the generalized CD production function (2.2) is a homogeneous function with the *degree of homogeneity* given by $\sum_{i=1}^{n} \alpha_i$. And the generalized CES production function (2.4) is a homogeneous function with h as its degree of homogeneity.

The author has completely classified h-homogeneous production functions which satisfy the CES property in [8]. More precisely, he proved the following.

Theorem 2.1. *Let f be a twice differentiable n-factors h-homogeneous production function with non-vanishing first partial derivatives. If f satisfies the CES property, then it is either the generalized Cobb-Douglas production function or the generalized CES production function.*

Remark 2.1. When $n = 2$, Theorem 2.1 is due to Losonczi [13].

3. A Hessian determinant formula

Throughout this article, we assume that $h_1(x_1), \ldots, h_n(x_n)$ and $F(u)$ are twice differentiable functions with $F'(u) \neq 0$ and $n \geq 2$.

The following provides a very simple explicit formula for the Hessian determinant of a composite function of the form (1.1).

Hessian Determinant Formula. *The determinant of the Hessian matrix $H(f)$ of the composite function $f = F(h_1(x_1) + \cdots + h_n(x_n))$ is given by*

$$\det(H(f)) = (F')^n h_1'' \cdots h_n'' + (F')^{n-1} F'' \sum_{j=1}^{n} h_1'' \cdots h_{j-1}'' h_j'^2 h_{j+1}'' \cdots h_n'', \tag{3.1}$$

where

$$F' = F'(u),\ F'' = F''(u),\ h_j' = \frac{dh_j}{dx_j},\ h_j'' = \frac{d^2 h_j}{dx_j^2}$$

with $u = h_1(x_1) + \cdots + h_n(x_n)$.

Proof. Let f be a twice differentiable composite function given by

$$f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n)). \tag{3.2}$$

It follows from (3.2) that

$$f_{x_i x_i} = F' h_i'' + F'' h_i'^2, \quad f_{x_i x_j} = F'' h_i' h_j', \ \ 1 \leq i \neq j \leq n. \tag{3.3}$$

If $n = 2$, it is easy to verify from (3.3) that the Hessian determinant of the composite function is given by

$$\det(H(f)) = F'^2 h_1'' h_2'' + F' F'' (h_1'^2 h_2'' + h_1'' h_2'^2). \tag{3.4}$$

Therefore, we have formula (3.1) for $n = 2$.

Now we want to prove formula (3.1) for any integer $n \geq 3$ by applying mathematical induction. Thus, let us assume that (3.1) holds for $n = k$ with $k \geq 2$. We claim that it also holds for $n = k + 1$.

Case (a): *At least one of $h_1', \ldots, h_{k+1}'$ vanishes.* Without loss of generality, we may assume that $h_1' = 0$. Then we get

$$f_{11} = f_{12} = \cdots = f_{1k+1} = 0,$$

which implies that the first row of the Hessian matrix $H(f)$ vanishes. Therefore we have $\det(H(f)) = 0$.

On the other hand, the right-hand-side of (3.1) also vanishes by $h_1' = 0$. Thus, formula (3.1) holds trivially for $n = k + 1$ in this case.

Case (b): $h_1', \ldots, h_{k+1}' \neq 0$. It follows from (3.3) that the Hessian matrix $H(f)$ with $n = k + 1$ is given by

$$H(f) = \begin{pmatrix} F'h_1'' + F''h_1'^2 & F''h_1'h_2' & F''h_1'h_3' & \cdots & F''h_1'h_{k+1}' \\ F''h_1'h_2' & F'h_2'' + F''h_2'^2 & F''h_2'h_3' & \cdots & F''h_2'h_{k+1}' \\ F''h_1'h_3' & F''h_2'h_3' & F'h_3'' + F''h_3'^2 & \cdots & F''h_3'h_{k+1}' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F''h_1'h_{k+1}' & F''h_2'h_{k+1}' & F''h_3'h_{k+1}' & \cdots & F'h_{k+1}'' + F''h_{k+1}'^2 \end{pmatrix}. \tag{3.5}$$

Therefore the Hessian determinant of f satisfies

$$\det(H(f)) = \tag{3.6}$$
$$\begin{vmatrix} F'h_1'' & -\dfrac{F'h_1'h_2''}{h_2'} & 0 & \cdots & 0 \\ F''h_1'h_2' & F'h_2''+F''h_2'^2 & F''h_2'h_3' & \cdots & F''h_2'h_{k+1}' \\ F''h_1'h_3' & F''h_2'h_3' & F'h_3''+F''h_3'^2 & \cdots & F''h_3'h_{k+1}' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F''h_1'h_{k+1}' & F''h_2'h_{k+1}' & F''h_3'h_{k+1}' & \cdots & F'h_{k+1}''+F''h_{k+1}'^2 \end{vmatrix}.$$

Equation (3.6) is obtained from (3.5); replacing the first row of $H(f)$ by the first row minus h_1'/h_2' times the second row of $H(f)$. Hence we find

$$\det(H(f)) = \tag{3.7}$$
$$F'h_1''\begin{vmatrix} F'h_2''+F''h_2'^2 & F''h_2'h_3' & \cdots & F''h_2'h_{k+1}' \\ F''h_2'h_3' & F'h_3''+F''h_3'^2 & \cdots & F''h_3'h_{k+1}' \\ \vdots & \vdots & \vdots & \vdots \\ F''h_2'h_{k+1}' & F''h_3'h_{k+1}' & \cdots & F'h_{k+1}''+F''h_{k+1}'^2 \end{vmatrix}$$

$$+F'F''h_1'^2h_2''\begin{vmatrix} 1 & F''h_3' & \cdots & F''h_{k+1}' \\ h_3' & F'h_3''+F''h_3'^2 & \cdots & F''h_3'h_{k+1}' \\ \vdots & \vdots & \vdots & \vdots \\ h_{k+1}' & F''h_3'h_{k+1}' & \cdots & F'h_{k+1}''+F''h_{k+1}'^2 \end{vmatrix}.$$

Now, after applying the assumption that formula (3.1) holds for $n = k$, we derive from (3.1) and (3.7) that

$$\begin{aligned} \det(H(f)) &= (F')^{k+1}h_1''\cdots h_{k+1}'' \\ &+ (F')^kF''h_1''\sum_{j=2}^{k+1} h_2''\cdots h_{j-1}''h_j'^2h_{j+1}''\cdots h_{k+1}'') \\ &+F'F''h_1'^2h_2''\begin{vmatrix} 1 & F''h_3' & \cdots & F''h_{k+1}' \\ h_3' & F'h_3''+F''h_3'^2 & \cdots & F''h_3'h_{k+1}' \\ \vdots & \vdots & \vdots & \vdots \\ h_{k+1}' & F''h_3'h_{k+1}' & \cdots & F'h_{k+1}''+F''h_{k+1}'^2 \end{vmatrix}. \end{aligned} \tag{3.8}$$

From (3.8) we obtain

$$
\begin{aligned}
\det(H(f)) &= (F')^{k+1} h_1'' \cdots h_{k+1}'' \\
&\quad + (F')^k F'' h_1'' \sum_{j=2}^{k+1} h_2'' \cdots h_{j-1}'' {h_j'}^2 h_{j+1}'' \cdots h_{k+1}'') \\
&\quad + F' F'' {h_1'}^2 h_2'' \begin{vmatrix} 1 & F''h_3' & F''h_3' & \cdots & F''h_{k+1}' \\ 0 & F'h_3'' & 0 & \cdots & 0 \\ 0 & 0 & F'h_4'' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F'h_{k+1}'' \end{vmatrix} \\
&= (F')^{k+1} h_1'' \cdots h_{k+1}'' + (F')^k F'' \sum_{j=1}^{k+1} h_1'' \cdots h_{j-1}'' {h_j'}^2 h_{j+1}'' \cdots h_{k+1}'',
\end{aligned} \tag{3.9}
$$

which gives formula (3.1) for $n = k+1$. Consequently, we obtain formula (3.1) for any integer $n \geq 2$ by mathematical induction. □

4. A geometric interpretation of Hessian determinants

Let M be a hypersurface of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. The *Gauss map* $\nu : M \to S^{n+1}$ maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $\nu(p)$ is a unit normal vector $\xi(p)$ of M at $p \in M$. The Gauss map can always be defined locally, i.e., on a small piece of the hypersurface. It can be defined globally if the hypersurface is orientable.

The differential $d\nu$ of the Gauss map ν can be used to define a type of extrinsic quantity, known as the *shape operator*. Since each tangent space T_pM of M is an inner product space, the shape operator S_p can be defined as a linear operator on T_pM by

$$g(S_p v, w) = g(d\nu(v), w) \tag{4.1}$$

for $v, w \in T_pM$, where g is the induced metric on M. The eigenvalues of the shape operator are called principal curvatures. The determinant of the shape operator S_p, denoted by $G(p)$, is called the *Gauss-Kronecker curvature* (cf. [4, 5]). When $n = 2$, the Gauss-Kronecker curvature is called the *Gauss curvature*.

For a given function $f = f(x_1, \ldots, x_n)$, the *graph* of f is the non-parametric hypersurface of $\mathbb{E}^{n+1}$ defined by

$$\psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)). \tag{4.2}$$

It is well-known that the Gauss-Kronecker curvature of the graph (3.1) of f is given by (see for instance [6, 7, 17])

$$G = \frac{\det(H(f))}{w^{n+2}} \tag{4.3}$$

with $w = \sqrt{1 + \sum_{i=1}^n f_i^2}$.

Consequently we have the following.

Lemma 4.1. *Let $f = f(x_1, \dots, x_n)$ be a twice differentiable function of n variables. Then the Hessian matrix $H(f)$ is singular if and only if the graph of f in $\mathbb{E}^{n+1}$ has null Gauss-Kronecker curvature.*

5. Characterizations of CES production functions

By applying the Hessian Determinant Formula, we have the following.

Theorem 5.1. *Let $F(u)$ be a twice differentiable function with $F'(u) \neq 0$ and let*

$$f = F(a_1 x_1^{d_1} + a_2 x_2^{d_2} + \cdots + a_n x_n^{d_n}) \tag{5.1}$$

be the composite of F and $p(\mathbf{x}) = a_1 x_1^{d_1} + a_2 x_2^{d_2} + \cdots + a_n x_n^{d_n}$, where $a_1, \dots, a_n$ are nonzero constants. Then the Hessian matrix $H(f)$ of f is invertible unless one of the following four cases occur:

(i) *At least one of $d_1, \dots, d_n$ vanishes.*

(ii) *At least two of $d_1, \dots, d_n$ are equal to one.*

(iii) *F is a linear function and one of $d_1, \dots, d_n$ is equal to one.*

(iv) *$f = \gamma \left(a_1 x_1^d + a_2 x_2^d + \cdots + a_n x_n^d \right)^{\frac{1}{d}} + c$, for some constants γ, a_i, c, d with $a_i, \gamma \neq 0$ and $d \neq 0, 1$.*

Proof. Under the hypothesis of the theorem, we have $h_i(x_i) = a_i x_i^{d_i}$. Thus we get

$$h'(x_i) = a_i d_i x_i^{d_i - 1}, \quad h''(x_i) = a_i d_i (d_1 - 1) x_i^{d_i - 2},$$

for $i = 1, \dots, n$. Hence, after applying the Hessian Determinant Formula, we have

$$\begin{aligned} \det(H(f)) = \Big(\prod_{j=1}^{n} a_j d_j x_j^{d_j - 2} \Big) (F'(u))^{n-1} \Big\{ F'(u) \prod_{i=1}^{n} (d_i - 1) \\ + F''(u) \sum_{i=1}^{n} a_i (d_1 - 1) \cdots (d_{i-1} - 1) d_i (d_{i+1} - 1) \cdots (d_n - 1) x_i^{d_i} \Big\} \end{aligned} \tag{5.2}$$

with $u = p(\mathbf{x})$.

Since $a_1, \dots, a_n$ are nonzero constants and $F'(u) \neq 0$, we conclude from (5.2) that the Hessian matrix $H(f)$ is singular if and only if one of the following two cases occurs:

(a) At least one of $d_1, \dots, d_n$ is zero, or

(b) $d_1, \dots, d_n$ are nonzero constants and

$$\begin{aligned} 0 = F'(u) \prod_{i=1}^{n} (d_i - 1) \\ + F''(u) \sum_{i=1}^{n} a_i (d_1 - 1) \cdots (d_{i-1} - 1) d_i (d_{i+1} - 1) \cdots (d_n - 1) x_i^{d_i} \end{aligned} \tag{5.3}$$

holds.

Now, let us assume that case (b) occurs. Then $d_1, \dots, d_n$ are nonzero constants and (5.3) holds.

We divide this into three cases.

Case: (b.1) *At least two of* $d_1, \dots, d_n$ *are equal to one.* In this case, equality (5.3) holds trivially.

Case: (b.2) *Exactly one of* $d_1, \dots, d_n$ *is equal to one.* Without loss of generality, we may assume $d_1 = 1$ and $d_2, \dots, d_n \neq 1$. Then (5.3) implies that $F'' = 0$. Hence, F is a non-constant linear function. Conversely, if F is a linear function with $F' \neq 0$, then (5.3) yields $\prod_{i=1}^{n}(d_i - 1) = 0$. Thus, at least one of $d_1, \dots, d_n$ is equal to one.

Case: (b.3) $F'' \neq 0$ *and* $d_1, \dots, d_n \neq 0, 1$. In this case, equation (5.3) can be simply expressed as

$$\sum_{i=1}^{n} \frac{a_i d_i}{d_i - 1} x_i^{d_i} + \frac{F'(u)}{F''(u)} = 0. \tag{5.4}$$

After taking the partial derivative of (5.4) with respect to x_j for a given $j \in \{1, \dots, n\}$, we find

$$\frac{F'(u)F'''(u)}{F''(u)^2} = \frac{2d_j - 1}{d_j - 1}, \quad j = 1, \dots, n. \tag{5.5}$$

We conclude from (5.5) that $d_1 = \cdots = d_n = d$ for some constant $d \neq 0, 1$. Thus equation (5.5) becomes

$$(d-1)F'(u)F'''(u) = (2d-1)F''(u)^2. \tag{5.6}$$

Now, by solving the differential equation (5.6) for $F'(u)$ we find

$$F'(u) = a(u+b)^{\frac{1}{d}-1}, \quad F''(u) = a\left(\frac{1-d}{d}\right)(u+b)^{\frac{1}{d}-2} \tag{5.7}$$

for some constants a, b with $a \neq 0$.

After applying $d_1 = \cdots = d_n = d$ and after substituting (5.7) into (5.3) we derive that

$$0 = ab(d-1)^n (u+b)^{\frac{1}{d}-2}, \tag{5.8}$$

which implies $b = 0$. Hence we get $F'(u) = au^{\frac{1}{d}-1}$. Therefore

$$F(u) = \gamma u^{\frac{1}{d}} + c, \tag{5.9}$$

where $\gamma = ad \neq 0$ and c is a constant.

Now, by combining (5.9) with (5.1) and by using $d_1 = \cdots = d_n = d$, we conclude that f takes the form

$$f = \gamma \left(a_1 x_1^d + a_2 x_2^d + \cdots + a_n x_n^d\right)^{\frac{1}{d}} + c. \tag{5.10}$$

Therefore, up to constants, f is a generalized CES production function. This gives case (iv).

Conversely, it is straightforward to verify that each one of cases (i)-(iv) implies that f has vanishing Hessian determinant. Therefore, in each case the Hessian matrix $H(f)$ is a singular matrix. □

Theorem 5.1 implies immediately the following simple characterization of generalized CES (or ACMS) production functions with homogeneity degree one.

Theorem 5.2. *Let $F(u)$ be a twice differentiable function with $F'(u) \neq 0$ and let f be the composite function given by*

$$f = F(a_1 x_1^{d_1} + a_2 x_2^{d_2} + \cdots + a_n x_n^{d_n}) \tag{5.11}$$

with $a_1, \ldots, a_n \neq 0$ and $d_1, \ldots, d_n \neq 0, 1$. Then the Hessian matrix $H(f)$ of f is singular if and only if, up to a suitable constant, f is a generalized CES production function with the degree of homogeneity equal to one.

Two further immediate consequences of Theorem 5.1 are the following.

Corollary 5.1. *The Hessian matrix of the generalized CES production function given by (2.4) is invertible if and only if the degree h of homogeneity satisfies $h \neq 1$.*

Corollary 5.2. *The Hessian matrix of the utility function given by (2.7) is always singular.*

Another simple characterization of the generalized CES production functions with homogeneity degree one is the following.

Theorem 5.3. *Let $F(u) = u^r$ be a power function with $r \neq 0, 1$ and let f be the composite function given by*

$$f = F(h_1(x_1) + \cdots + h_n(x_n)),$$

where $h_1(x_1), \ldots, h_n(x_n)$ are twice differentiable functions with $h_i'' \neq 0$, $i = 1, \ldots, n$. Then the Hessian matrix $H(f)$ is a singular matrix if and only if, up to suitable translations of $f, x_1, \ldots, x_n$, the composite function f is a generalized CES production function with homogeneity degree one.

Proof. Under the hypothesis of the theorem, we have

$$F'(u) = ru^{r-1}, \quad F''(u) = r(r-1)u^{r-2}.$$

Thus, by applying the Hessian Determinant Formula, we find

$$\begin{aligned} &\det(H(f)) = \\ &r^n u^{nr-n-1} \Big\{ u h_1'' \cdots h_n'' + (r-1) \sum_{j=1}^{n} h_1'' \cdots h_{j-1}'' h_j'^2 h_{j+1}'' \cdots h_n'' \Big\}. \end{aligned} \tag{5.12}$$

with $u = h_1(x_1) + \cdots + h_n(x_n)$. Consequently, $H(f)$ is a singular matrix if and only if

$$\sum_{j=1}^{n} \left(\frac{(r-1) h_j'(x_j)^2}{h_j''(x_j)} + h_j(x_j) \right) = 0 \tag{5.13}$$

holds.

By taking the partial derivative of (5.13) with respect to x_i for a given $i \in \{1, \dots, n\}$, we obtain

$$(2r-1)h_i''(x_i)^2 = (r-1)h_i'(x_j)h_i'''(x_i), \quad i = 1, \dots, n. \tag{5.14}$$

Therefore, after solving (5.14), we get

$$h_i(x_i) = a_i + b_i(x + c_i)^{\frac{1}{r}}, \quad i = 1, \dots, n, \tag{5.15}$$

for some constants a_i, b_i, c_i. Consequently, after applying some suitable translations of $f, x_1, \dots, x_n$, f is a generalized CES production function with one as its degree of homogeneity.

The converse can be verified directly. □

6. Characterization of CD production functions

Next, we provide the following characterization of the generalized CD production function of homogeneity degree one via the Hessian Determinant Formula.

Theorem 6.1. *Let $h_1(x_1), \dots, h_n(x_n)$ be twice differentiable functions with $h_i'' \neq 0$ for $i = 1, \dots, n$ and let*

$$f = \exp(h_1(x_1) + \cdots + h_n(x_n)). \tag{6.1}$$

Then the Hessian matrix $H(f)$ is singular if and only if, up to suitable translations of $x_1, \dots, x_n$, f is a generalized CD production function with degree of homogeneity equal to one.

Proof. Under the hypothesis of the theorem, we have $F' = F'' = F$. Thus, after applying the Hessian Determinant Formula, we find

$$\det(H(f)) = \left\{ h_1'' \cdots h_n'' + \sum_{j=1}^{n} h_1'' \cdots h_{j-1}'' h_j'^2 h_{j+1}'' \cdots h_n'' \right\} f^n. \tag{6.2}$$

Hence the Hessian matrix $H(f)$ is singular if and only if we have

$$0 = 1 + \sum_{j=1}^{n} \frac{h_j'^2(x_j)}{h_j''(x_j)}. \tag{6.3}$$

From (6.3) we find

$$h_j'^2(x_j) = -\alpha_j h_j''(x_j), \quad j = 1, \dots, n, \tag{6.4}$$

for some nonzero constants α_j satisfying $\alpha_1 + \cdots + \alpha_n = 1$. After solving (6.4) we obtain

$$h_j(x_j) = \ln\left(a_j(x_j + b_j)^{\alpha_j}\right), \quad j = 1, \dots, n, \tag{6.5}$$

for some constants a_j, b_j with $a_j \neq 0$.

By combining (6.5) with (6.1) we derive that

$$f = \gamma(x_1 + b_1)^{\alpha_1} \cdots (x_n + b_n)^{\alpha_n}, \tag{6.6}$$

with $\gamma = a_1 \cdots a_n$. Therefore, after applying some suitable translations of $x_1, \ldots, x_n$, we obtain the generalized CD production function (2.2) with $\sum_{i=1}^n \alpha_i = 1$.

The converse is easy to verify. □

7. A further application

Finally, we provide the following result via the Hessian Determinant Formula.

Theorem 7.1. *Let $h_1(x_1), \ldots, h_n(x_n)$ be twice differentiable functions with $h_i'' \neq 0$ for $i = 1, \ldots, n$ and let*

$$f = \ln(h_1(x_1) + \cdots + h_n(x_n)). \tag{7.1}$$

Then the Hessian matrix $H(f)$ is singular if and only if, up to suitable translations of $x_1, \ldots, x_n$, f is of the form:

$$f = \ln\left(\sum_{i=1}^n a_i e^{b_i x_i}\right) \tag{7.2}$$

for some nonzero constants $a_i, b_i, i = 1, \ldots, n$.

Proof. Under the hypothesis of the theorem, we have

$$F(u) = \ln u, \quad F'(u) = \frac{1}{u}, \quad F''(u) = -\frac{1}{u^2}.$$

Thus, after applying the Hessian Determinant Formula, we find

$$\det(H(f)) = \frac{1}{u^{n+1}}\left\{uh_1'' \cdots h_n'' - \sum_{j=1}^n h_1'' \cdots h_{j-1}'' h_j'^2 h_{j+1}'' \cdots h_n''\right\}. \tag{7.3}$$

with $u = h_1(x_1) + \cdots + h_n(x_n)$. Consequently, $H(f)$ is singular if and only if

$$\sum_{j=1}^n \left(\frac{h_j'^2(x_j)}{h_j''(x_j)} - h_j(x_j)\right) = 0 \tag{7.4}$$

holds. From (7.4) we obtain

$$(\alpha_j + h_j(x_j))h_j''(x_j) = h_j'^2(x_j), \quad j = 1, \ldots, n, \tag{7.5}$$

for some constants α_j with $\sum_{i=1}^n \alpha_i = 0$.

After solving the differential equation (7.5) we get

$$h_j(x_j) = b_j e^{a_j x_j} - \alpha_j, \quad j = 1, \ldots, n, \tag{7.6}$$

for some nonzero constants a_j, b_j. Thus, by applying some suitable translations of x_j, we obtain

$$h_j(x_j) = b_j e^{a_j x_j}, \quad j = 1, \ldots, n, \tag{7.7}$$

Now, by combining (7.7) with (7.1) we obtain (7.2).

The converse can be verified directly. □

By combining Lemma 4.1 and Theorem 7.1 we obtain the following.

Corollary 7.1. *Let $h_1(x_1), \ldots, h_n(x_n)$ be twice differentiable functions with $h_i'' \neq 0$ for $i = 1, \ldots, n$ and let $f = \ln(h_1(x_1) + \cdots + h_n(x_n))$. Then the graph of f has null Gauss-Kronecker curvature if and only if, up to translations, $h_i(x_i) = a_i e^{b_i x_i}$ for some nonzero constants a_i, b_i, $i = 1, \ldots, n$.*

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