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ON GROWTH AND FORM AND GEOMETRY. I

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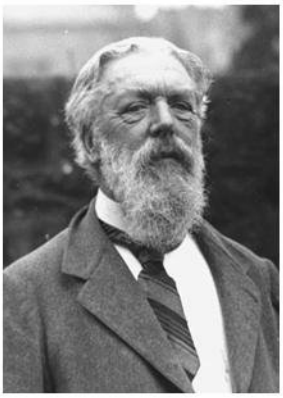

D'Arcy Thompson

Bang–Yen Chen

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1. "*The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any science which interprets experience on a higher than purely descriptive level.*", as von Neumann stated in his article entitled "The Mathematician". And, here follows a quote in this respect from the foreword of Weber's edition of Riemann's lectures on "Die Partiellen Differentialgleichungen der Mathematischen Physik": "*Alles in Allem haben wir es hier mit einer Anshauung zu thun, die auch den zu erfreuen geeignet ist, der in der physikalischen Theorien mehr sught, als die blosse Darstellung oder Beschreibung der Erscheinungen.*".

In this sense, in *the biological theory of growth and form* one may consider the hereafter recalled *geometrical model for planar natural growth of D'Arcy Thompson* and its thereafter -as far as we know- firstly formulated *natural nD generalisation* and the geometrical forms which these models do characterise: *the logarithmic spirals or the equiangular curves of Descartes* amongst the curves in Euclidean planes $\mathbb{E}^2$ and *the constant ratio or equiangular submanifolds of Bang-Yen Chen* amongst the nD submanifolds M^n in Euclidean $(n+m)$D spaces $\mathbb{E}^{n+m}$.

The most immediately relevant biological extension of the original $\mathbb{E}^2$ model, namely the case of surfaces M^2 in Euclidean spaces $\mathbb{E}^3$, in particular, applies to *shells*, in agreement with the following citation from D'Arcy Thompson's "On Growth and Form": "*The surface of any shell, whether discoid or turbinate, may be imagined to be generated by the revolution about a fixed axis of a closed curve, which, remaining always geometrically similar to itself, increases its dimension continually; and, since the scale of the figure increases in geometrical progression while the angle of rotation increases in arithmetical, and the centre of similitude remains fixed, the curve traced in space by corresponding points in the generating curve is, in all such cases, an equiangular spiral. In discoid shells, the generating figure revolves in a plane perpendicular to the axis, as in the Nautilus, the Argonaut and the Ammonite. In turbinate shells, it follows a skew path with respect to the axis of revolution, and the curve in space generated by any given point makes a constant angle to the axis of the enveloping cone, and partakes, therefore, of the character of a helix, as well as of a logarithmic spiral; it may be strictly entitled a helico-spiral. When the envelope of the shell is a right cone -and it is seldom far from being so- then our helico-spiral is a loxodromic curve. Such turbinate or helico-spiral shells include the snail, the periwinkle and all the common typical gastropodes.*". Our extended *basic principle of growth* however is the same for *all dimensions n* and for *all co–dimensions m*, and, besides in biology, may be applied in *other fields of science* too, and will be discussed further in following parts of our paper, also for *pseudo-Euclidean ambient spaces*.

2. From the 1942 edition of D'Arcy Thompson's book "On Growth and Form" [17] (of which book the first version appeared in 1917), we recall the following passage of the section "*The equiangular spiral in its dynamical aspect*".

"In mechanical structures, *curvature* is essentially a mechanical phenomenon. It is found in flexible structures as a result of *bending*, or it may be introduced into construction for the purpose of resisting such a bending-moment. But neither shell nor tooth nor claw are flexible structures; they have not been *bent* into their peculiar curvature, they have *grown* into it.

We may for a moment, however, regard the equiangular or logarithmic spiral of our shell from the dynamic point of view, by looking at *growth* itself as the force concerned. In the growing structure, let growth at a point P be resolved into a force F acting along the line joining P to a pole O, and a force T acting in a direction perpendicular to OP; and let the magnitude of these forces (or of these rates of growth) remain constant. It follows that the resultant of the forces F and T (as PQ) makes a constant angle with the radius vector. But a constant angle between tangent and radius vector is a fundamental property of the "equiangular" spiral: the very property with which Descartes started his investigation, and that which gives its alternative name to the curve.

In such a spiral, radial growth and growth in the direction of the curve bear a constant ratio to one another. For, if we consider a consecutive radius vector OP', whose increment as compared with OP is dr, while ds is the small arc PP', then $dr/ds = \cos\alpha = \text{constant}$.

In the growth of a shell, we can conceive no simpler law than this, that it shall widen and lengthen in the same unvarying proportions: and this simplest of laws is that which Nature tends to follow. The shell, like the creature within it, grows in size but *does not change its shape*; and the existence of this constant relativity of growth, or constant similarity of form, is of the essence, and may be made the basis of a definition, of the equiangular spiral.

Such a definition, though not commonly used by mathematicians, has been occasionally employed; and it is one from which the other properties of the curve can be deduced with great ease and simplicity. In mathematical terms it would run as follows: "Any [plane] curve proceeding from a fixed point (which is called the pole), and such that the arc intercepted between any two radii at a given angle to one another is always similar to itself, is called an equiangular, or logarithmic, spiral."

In this definition, we have the most fundamental and "intrinsic" property of the curve, namely the property of the continual similarity, and the very property by reason of which it is associated with organic growth in such structures as the horn or the shell. For it is peculiarly characteristic of the spiral shell, for instance, that it does not alter as it grows; each increment is similar to its predecessor, and the whole, after every spurt of growth, is just like it was before. We feel no surprise when the animal which secretes the shell, or any other animal whatsoever, grows by such symmetrical expansion as to preserve its form unchanged; though even there, as we have already seen, the unchanging form denotes a nice balance between the rates of growths in various directions, which is but seldom accurately maintained for long. But the shell retains its unchanging form in spite of its *asymmetrical* growth; it grows at one end only, and so does the horn. And this remarkable property of increasing by *terminal* growth, but nevertheless retaining unchanged the form of the entire figure, is characteristic of the equiangular spiral, and of no other mathematical curve. It well deserves the name, by which James Bernoulli was wont to call it, of *spira mirabilis*.

We may at once illustrate this curious phenomenon by drawing the outline of a little *Nautilus* shell within a big one. We know, or we may see at once, that they are of precisely the same shape; so that, if we look at the little shell through a magnifying glass, it becomes identical with the big one. But we know, on the other hand, that the little *Nautilus* shell grows into the big one, not by growth or magnification in all parts and directions, as when a boy grows into a man, but by growing *at one end only.*"

At the end of this paper, one can find a collection of pictures, photos and drawings which may be well to look at from time to time while reading the present text. For example, for Section 2, there one can find the two figures which were printed in D'Arcy Thompson's book within the above quoted passage.

3. At this stage, concerning the citation of Section 2, we restrict to the following few comments.

(i) When considering a [planar] form as a result of growth, *the constant ratio between the resolvents of the force of growth "in length and in width"*, or, in other words, between the radial component of growth and the component of growth in the direction perpendicular to the radial one, as proposed by D'Arcy Thompson, indeed may well be considered as *one of the most natural laws of natural growth.* And, similarly, Bang-Yen Chen's *constant ratio nD submanifolds M^n in $(n+m)D$ Euclidean spaces* $\mathbb{E}^{n+m}$ [4, 5, 6, 7, 8] are defined by *one of the most natural conditions by which pure geometry may determine the shape of submanifolds M^n in* $\mathbb{E}^{n+m}$, namely, by imposing *a constant ratio between the normal and the tangential components of the position vector* of such submanifolds of Euclidean spaces (cfr. [19]). For dimension $n = 1$ and for codimension $m = 1$, that is for curves M^1 in a Euclidean plane $\mathbb{E}^2$, *the constant ratio between the biological "growths in length and in width"*, or, still, *the self-similarity in form, is equivalent, geometrically, to the constant ratio between the magnitudes of the position vector's normal and tangential components along these Euclidean planar curves.* By the way, some of us can not let pass the chance, here and now, to refer to Simon Stevin's logo and motto as these were shown on the cover of his 1586 book "The Beghinselen der Weeghconst", and, moreover, to hereby draw attention to the fundamental connections between the psychology of our kind's natural perceptions and our experience of the gravitational cross, in particular, and the workings of our minds trying "to understand" forms or shapes and our developments of the so-called Euclidean and of other natural geometries [2, 14, 20].

(ii) *The biological growing at one end after in one or other way having started off to grow anyway at some pole*, is crucial in the above formulated growing process of "things", and, equivalently, *the description of submanifolds in Euclidean spaces by means of their position vector* is crucial in geometry [9, 3].

(iii) In the above citation of D'Arcy Thompson, the term *curvature* seems to serve as an alternative way to speak of *form or shape*, what, indeed, also in the present terminology, at least for curves, in some sense is perfectly all right, thinking of *the congruence theorem of Euclidean planar curves*. Some of the following treatment concerning the geometry of planar curves will be presented at times in a rather rough way; to compensate for this, we are so free to strongly recommend [12] as likely the most serious, precise and delicate presentation of this geometry.

(iv) -Off the record-: hopefully D'Arcy Thompson (1860-1948) was spared to see the abridged editions of his book "On Growth and Form", since in this respect the comparison pops up with, say, a deaf-born person reworking, of course "for the benefit of the people", the 9th symphony of Beethoven, making several changes here and there, and, for sure, as crown of his job, dryly, deleting the choral part.

4. In an analytical study of curves Γ in a Euclidean plane $\mathbb{E}^2$, when hereby having some planar biological growth in mind, the choice of *polar co-ordinates* (ρ, θ) may be considered to be not so unnatural. Namely, such co-ordinates might lead one to speculate that one or other kind of thing, somewhile after, in one or other way, having been originating about a certain point, say, *a pole* O, depending on *directions as described by polar angles* θ, then starts to evolve within a plane, to proper *polar distances* $\rho(\theta)$ from O, thus yielding *polar parametrisations* $X(\theta) = \rho(\theta).(\cos\theta, \sin\theta)$, [(1)], for curves Γ, which, somewhat like this, might be thought of as being traced out by such thing in the course of its growth in a plane. *The position vector field* of a curve Γ in $\mathbb{E}^2$ will be denoted by OX or by X alone, for short, and derivatives with respect to θ will also be denoted by accents, for short, such that, respectively, $X' = \mathrm{d}X/\mathrm{d}\theta$ and $X'' = \mathrm{d}^2X/\mathrm{d}\theta^2$ are *the angular velocity vector field* and *the angular acceleration or angular curvature vector field* of Γ.

As is well known, *the equiangular spirals*, i.e. the curves Γ in $\mathbb{E}^2$ for which *the angle* $\alpha = \angle(X, X') \in [0, \pi/2]$ *between the tangent direction of the curve and the direction of its position vector is constant*, are *the logarithmic spirals* $\rho(\theta) = k.\mathrm{e}^{a\,\theta}$, [(2)], whereby $a = \mathrm{cotg}\,\alpha$ and k is any real constant. And, conversely, these spirals geometrically define the natural logarithmic and exponential functions; in some sense, most directly, the constant $(\pi/4)$-angled spiral with pole O and passing through the point $E = (1, 0)$ defines the natural exponential and logarithmic functions in a geometrical way: in particular, *this* $(\pi/4)$*-spiral* defines the value of the natural exponential function by its distances $\rho(\theta)$ from their pole O for all angles $\theta \in \mathbb{R}$.

Let $T = X'/\|X'\|$ be *the unit tangent vector field* of a curve Γ in $\mathbb{E}^2$ and let $N = T^{\perp}$ be *the unit normal vector field* such that $\{T, N\}$ is a positively oriented orthonormal frame field along Γ in $\mathbb{E}^2$. Then, of course, for such a curve Γ *to have constant angle* $\alpha = \angle(T, X)$ *is equivalent to have constant angle* $\alpha^{\perp} = \angle(N, X) = (\pi/2) - \alpha$, and, so, the polar equation (2) of the equiangular spirals with a constant angle α or equivalently with a constant angle $\alpha^{\perp}$ can be rewritten as $\rho(\theta) = k.\mathrm{e}^{a\,\theta}$, [(3)], whereby $a = \mathrm{tg}\,\alpha^{\perp}$.

Since, for a general curve $\rho(\theta)$ in $\mathbb{E}^2$, $X'(\theta) = \rho'(\theta).(\cos\theta, \sin\theta) + \rho(\theta).(-\sin\theta, \cos\theta)$, [(4)], it follows that the *arclength parameter* s *based at* $\theta = 0$, is given by $s(\theta) = \int_0^{\theta} \|X'(\theta)\| \mathrm{d}\theta = \int_0^{\theta} \left[\rho'(\theta)^2 + \rho(\theta)^2\right]^{1/2} \mathrm{d}\theta$, [(5)], such that $\mathrm{d}s(\theta)/\mathrm{d}\theta = \left(\rho'^2 + \rho^2\right)^{1/2}(\theta)$, [(6)]. Hence $\mathrm{d}\rho(\theta)/\mathrm{d}s = [\mathrm{d}\rho(\theta)/\mathrm{d}\theta]\,.\,[\mathrm{d}\theta(s)/\mathrm{d}s] = \left[\rho'/\,(\rho'^2+\rho^2)^{1/2}\right](\theta)$, [(7)], which shows that $\operatorname{grad}\rho = \mathrm{d}\rho/\mathrm{d}s = \text{constant}$ *is a characterisation of the equiangular spirals in the Euclidean plane*, (cfr. also D'Arcy Thompson's above quotation).

According to the so-called *Frenet theory of Euclidean curves*: $\mathrm{d}X/\mathrm{d}s = T$, [(8)], $\mathrm{d}^2X/\mathrm{d}s^2 = \mathrm{d}T/\mathrm{d}s = \kappa N$, [(9)], whereby κ denotes the *curvature function* of $\Gamma \subset \mathbb{E}^2$. And, as asserted by *the congruence theorem of Euclidean planar curves*, the magnitude of the second derivative with respect to an arclength parameter s of the position vector field X of a curve Γ in $\mathbb{E}^2$, i.e. *the curvature* κ *of* Γ *as a function of* s, completely determines the curve -apart of its actual location in the plane-, i.e. *completely determines the form or the shape of the curve* Γ *in the Euclidean plane* $\mathbb{E}^2$. As such, *the equiangular spirals are characterised by the property that their radius of curvature* $R = 1/\kappa$ *is a first degree function of their arclength* s*:* $R = A\,s + B$ *for some real constants* A *and* B, (hereby, strictly speaking or maybe by showing some good will, including the *straight lines* and the *circles* as the particular cases of *0-spirals* and $(\pi/2)$-*spirals*, respectively).

The *Laplace operator* of any curve Γ in $\mathbb{E}^2$ being given by $\Delta = -\mathrm{d}^2/\mathrm{d}s^2$, by (9), *the curvature vector field* or *tension field* of Γ in $\mathbb{E}^2$ satisfies $\mathrm{d}^2X/\mathrm{d}s^2 = \kappa\,N = -\Delta X$. So, the direction of the Laplacian ΔX is automatically fixed by the tangent direction $T = \mathrm{d}X/\mathrm{d}s$ of the curve by the Euclidean structure in the plane, -T and N being mutually orthogonal-, and, so, *this direction as such does not contain more geometrical information about the curve than already given by the tangent direction.* Yet, it may be used to rephrase a previous statement as follows: *the equiangular spirals are characterised by the property to have a constant angle* $\alpha^\perp = \angle(X, \Delta X)$.

5. Returning to parametrisations in *polar co–ordinates* (ρ,θ) of arbitrary curves Γ with position vector X in a Euclidean plane $\mathbb{E}^2$, the first derivative $X' = \mathrm{d}X/\mathrm{d}\theta$ (of course, still) determines the tangent direction of Γ, but, now, in general, *the direction of the second derivative* $X'' = d^2X/d\theta^2$ *does offer additional geometrical information about* Γ. The *angular curvature vector* or *angular tension vector* X'', when thinking of such curve Γ as associated with some process of planar growth, may be interpreted as *a measure of the tension created in this curve by its form or shape as these evolve in terms of the polar angle.* From (4.4) it follows that $X''(\theta) = [\rho''(\theta) - \rho(\theta)]\,.(\cos\theta, \sin\theta) + 2\rho'(\theta).(-\sin\theta, \cos\theta)$, [(1)]. Thus $\|X''\| = [(\rho''-\rho)^2 + 4\rho'^2]^{1/2}$, or rather its converse, gives a *numerical measure for this angular tension.*

Next, let us express that *the thing* Γ *grows in such a way that always its position vector* $X(\theta)$ *makes a constant angle* φ *with the direction of its angular tension vector* $X''(\theta)$, i.e. that $\varphi = \angle(X, X'')$ *is constant*, or, equivalently, that $\cos\varphi = (X\cdot X'')/(\|X\|.\|X''\|)$ *is constant.* Generically, which here means *excluding at this stage the straight lines through the pole* O *and the circles centered at the pole* O *from the immediately following scene*, this amounts to the condition $\rho'' - 2C\rho' - \rho = 0$, [(2)], whereby $C = \operatorname{cotg}\varphi$, of which the solutions are given by $\rho(\theta) =$

$e^{C\theta}.\left(C_1.e^{\theta/\sin\varphi}+C_2.e^{-\theta/\sin\varphi}\right)$, [(3)], whereby C_1 and C_2 are real constants of integration. *From a practical point of view*, or, put otherwise, *by considering the curves (3) in approximation*, whereby thinking in particular at increasing values of the polar angle θ, *the planar forms which result from the basic principle of growth according to which the angle φ between the position X and the angular tension X'' be constant all along the growing process are the equiangular spirals $\rho(\theta) = K.e^{[(1+\cos\varphi)/\sin\varphi]\theta}$*, [(4)], whereby K is a real constant.

The particular cases of *"growing in full accordance with the experienced angular tension"*, (i.e.: the case $\varphi = 0$ or $X \parallel X''$), and of "*growing as much as possible opposing the experienced angular tension*", (i.e.: the case $\varphi = \pi/2$ or $X \perp X''$), are determined by the conditions $\rho' = 0$, [(5)], and $\rho'' - \rho = 0$, [(6)], respectively, and, so, these growing processes, respectively, do characterise *the circles of radius R which are centered at O, $\rho(\theta) = R$*, [(7)], and, in the above sense of approximation, *the natural logarithmic spirals $\rho(\theta) = K.e^{\theta}$*, [(8)], or, still, *the natural spirae mirabilis.* And, similarly as noted above in relation to the equiangular $(\pi/4)$–spiral, but now likely in an even more natural way, at least from the Euclidean geometrical point of view, having in mind the connection between the gravitational cross and the Theorem of Pythagoras (cfr. [2, 20]), *one may consider the curves originating about a pole and growing such that their position permanently remains perpendicular to the direction of their angular tension and which pass through the point E*, (the angles θ being measured with respect to the axis OE), *as to geometrically define the natural exponential function*, in that, for each angle θ, the radial distance $\rho = d(O, X)$ determines the value of e^{θ}.

6. Before presenting the announced extension of *D'Arcy Thompson's basic principle of planar growth* to, in particular, *a basic principle of growth of surfaces in space*, from [13, 15, 18], we would prefer first to recall some related observations which essentially concern the fundamental characteristic of *Euclidean isotropy.*

A Euclidean plane $\mathbb{E}^2$, that is the Riemannian 2D space which is determined on the *standard plane* $\mathbb{R}^2$, charted with co–ordinates (x, y), by the *Euclidean metric*, $ds^2 = dx^2+dy^2$, (cfr. *the Theorem of Pythagoras*), besides possessing other exemplary qualities, is *homogeneous* and *isotropic*: "*at every point a Euclidean plane looks the same in all directions and these looks are the same for all points*". However, in many natural processes and for many natural phenomena, as we may observe them, these qualities are not at all "realistic". Above already was commented on the connection between *the pole O of growth as utmost important natural point* in a plane, on the one hand, and *the geometry of the position vector X in the Euclidean plane*, on the other. And, as was shown before, in some sense, *amongst the most primitive planar curves which do have a non-trivial tension due to their form*, from our most primitive geometrical point of view, that is, giving preference to what is most special in Euclidean geometry as far as directions are concerned, *are (parts of) the circles and (parts of) the natural spirae mirabilis*, (cfr. (5.7) and (5.8)), since these curves are characterised by an angular growth which permanently is either *parallel with* *or perpendicular to* the direction of their angular tension or curvature ($X \parallel X''$ or $X \perp X''$); and, of course, *the straight lines are the curves of which the form yields no tension at all. The circles and the natural logarithmic spirals may be considered as the most distinguished specimen in the class of all the equiangular curves*, i.e. of the curves for which the angle α between the direction of their position X and their tangent direction T, or, equivalently, the angle $\alpha^{\perp}$ between the direction of their position X and their normal direction N, is constant. And, the equiangular curves which thus certainly figure amongst the basic Euclidean geometrical forms, so beautifully model some of nature's very privileged planar forms, in accordance with D'Arcy Thompson's law of planar growth, (cfr. Section 2).

An effective approach geometrically to deal with some of the, say, *global anisotropies* in many forms that do occur in nature, and, moreover, with certain kinds of, say, *repeated local deviations from Euclidean perfection* in such forms, consists in applying so-called *Gielis transformations* to the basic forms that do naturally show up in Euclidean geometry. What this essentially is all about will next, very briefly, be illustrated for *the planar case, by applying such transformations to the equiangular curves*, in (i)-(iv) hereafter. Yet, in complete analogy, one may equally well carry out the same programme in the higher dimensional cases too, and, this, in the definite and also in the indefinite setting for that matter, (cfr. [13, 15, 18] for more information and for more examples).

(i) The *unit circle* $x^2 + y^2 = 1$, [(1)], may be considered as "the ground figure" of the Euclidean planar geometry, in that it determines the Euclidean distance from the origin O to all its points to be equal to 1, hereby expressing the isotropy in the Euclidean plane, in accordance with the Theorem of Pythagoras, [of which the infinitesimal version amounts to endowing $\mathbb{R}^2$ with the Euclidean line element $\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}y^2)^{1/2}$].

(ii) The *generalised Lamé curves* $|(x/A)|^p + |(y/B)|^p = 1$, [(2)], whereby $A, B, p \in \mathbb{R}_0^+$ are some constants, (of which $2 \neq p \in \mathbb{N}$ for the original Lamé curves), similarly may be considered as *ground figures for a specific class of non-Euclidean planar geometries*, in the way that they determine their distance from 0 to all their points to be equal to 1, [namely the *Minkowski Finsler geometries* obtained by endowing $\mathbb{R}^2$ with the corresponding line elements $\mathrm{d}s = (|\mathrm{d}x/A|^p + |\mathrm{d}y/B|^p)^{1/p}$], which -whenever "(1)$\neq$(2)"- manifestly is in conflict with our natural sense of measuring lengths. In the words of Chern [10], these are (amongst the simplest -the authors-) *Riemannian geometries "without the quadratic restriction"*, and, in particular, *these above geometries' anisotropies are related to some 4-fold symmetries* in the plane. Some examples of such Lamé curves, for $A = B = R$ and also for $A \neq B$ and for various values of p (smaller than 1, equal to 1 and greater than 1 and much greater than 1) are shown below.

(iii) Turning now to polar co-ordinates (ρ, θ) and introducing hereby *a coefficient* $m/4$ to the polar angle θ, ($m \in \mathbb{N}_0$ or $m \in \mathbb{R}_0$ for that matter), which allows, whenever wanted or needed, "to escape" from the quadrants which are inherently present in a rectangular (x, y) Cartesian co–ordinate system and which introduces *anisotropies related to m–fold symmetries in the plane*, and, moreover, in some sense giving *"individual freedom" to the exponents in the equations*, (though for many natural applications, the upcoming exponents n_1 and n_2 "remain" equal), from (2) further follows *the extension from the unit circle of the Euclidean planar geometry to the curves with parametric polar equations* $\rho(\theta) = \{|[\cos(m\theta/4)]/A|^{n_1} + |[\sin(m\theta/4)]/B|^{n_2}\}^{-1/n_3}$, [(3)], whereby $A, B \in \mathbb{R}_0^+$ and $m, n_1, n_2, n_3 \in \mathbb{R}_0$, which too may be considered as ground figures of corresponding planar geometries. For example, up to scale, for $A = B$ and for (a) $[m = 4, n_1 = n_2 = 15, n_3 = 12]$, (b) $[m = 5, n_1 = n_2 = n_3 = 4]$ and (c) $[m = 7, n_1 = n_2 = 6, n_3 = 10]$, the resulting curves pretty accurately match the cross sections of the stems of (a) Scrophalaria nodosa, (b) Equisetum and (c) Raspberry, respectively, while, for $A = B$ and for (d) $[m = 5, n_1 = n_2 = 7, n_3 = 2]$ and (e) $[m = 5, n_1 = n_2 = 13, n_3 = 2]$, the resulting curves correspond to profiles of kinds of starfish, as suggested by the accompanying figures below.

(iv) In our present purpose, rather than viewing such curves as determining their own geometries, for any choice of values for A, B, m, n_1, n_2 and n_3, *we prefer to interpret the curves with equations (3) as being obtained from the unit circle centered at O*, i.e. from the curve with polar equation $\rho(\theta) = 1$, [(4)], *by the transformation given by the right hand side of (3)* for these values of A, B, m, n_1, n_2 and n_3. And similarly, instead of thus transforming the Euclidean unit circle, *any planar curve, say, determined by a polar equation* $\rho(\theta) = f(\theta)$, [(5)], *for any given function* $f : \mathbb{R} \to \mathbb{R}^+$, *may be transformed into the planar curve with polar equation* $\rho(\theta) = f(\theta).\{|[\cos(m\theta/4)]/A|^{n_1} + |[\sin(m\theta/4)]/B|^{n_2}\}^{-1/n_3}$, [(6)]. For example, up to scale, starting from *the logarithmic spiral* (a), $\rho(\theta) = f(\theta) = \mathrm{e}^{(0.2)\theta}$, by the transformations (6) with parameter values (b) $[A = B, m = 4, n_1 = n_2 = n_3 = 100]$ and (c) $[A = B, m = 10, n_1 = n_2 = n_3 = 5]$, one finds the corresponding curves shown further on.

We end this section by making a remark, on the side, which may emphasise the remarkable place indeed of "our" Euclidean planar geometry within the set of *all Minkowski Finsler (p)-geometries*, i.e. the geometries which are associated with the *ground figures* $|x|^p + |y|^p = 1$, $[(*p)]$, for all $p \in \mathbb{R}_0^+$, or, still, the geometries determined on $\mathbb{R}^2$ by the *line elements* $\mathrm{d}s_p = [|\mathrm{d}x|^p + |\mathrm{d}y|^p]^{1/p}$. Each such (p)-geometry has its own "pi", namely $\pi(p) :=$ *half the perimeter of its metric ground figure* $(*p)$, whereby this perimeter is measured making use of the very metric of this (p)-geometry itself. And, the property alluded to then is the following: the Euclidean geometry is characterised among all these (p)-geometries by the fact that $\forall p \neq 2 : \pi(p) > \pi(2) = \pi$, or, still: *the Euclidean geometry is the Minkowski Finsler geometry* $(\mathbb{R}^2, ds_p)$ *for which the metric ground figure* $(*p)$ *has the smallest perimeter*, (cfr. [15]).

7. D'Arcy Thompson's *basic principle or law of planar growth* concerns curves Γ with position vector field X starting from some pole O in a Euclidean plane $\mathbb{E}^2$ and states that *the magnitudes of the components Gp and $Gp^{\perp}$ of a growing force G*, which geometrically amounts to some tangential vector field along Γ, in the direction of the position X and in the direction perpendicular to the position X, respectively, *have a constant ratio*: $\|Gp\|/\|Gp^\perp\| = \operatorname{tg}\alpha^\perp$ is constant, whereby $\alpha^\perp = \angle(Gp^\perp, G)$, (cfr. the arced rectangular triangle in the drawing corresponding to this situation which follows later on), hereby maintaining the previous notations $\alpha = \angle(G, Gp) = \angle(T, X)$ and $\alpha^\perp = \angle(Gp^\perp, G) = \angle(X, N)$. Thus, *the curves* Γ *following this law of growth are precisely the equiangular spirals in* $\mathbb{E}^2$, (namely, it are the curves Γ which are characterised by the property that $\alpha^\perp$ *is constant*, or equivalently, that α *is constant*), as already explicited by D'Arcy Thompson right away, (cfr. Section 2). In passing, it could be remarked that, of course, equivalently, this law could have been formulated as $\|Gp^\perp\|/\|Gp\| = \operatorname{tg}\alpha$ is constant, (cfr. the dotted rectangular triangle in the drawing corresponding to this situation which follows later on), and that, as very special particular cases for which $\alpha^\perp = 0$ or $Gp = 0$ and $\alpha = 0$ or $Gp^\perp = 0$, the concerned curves are nothing but *the circles* and *the straight lines*.

Our extension of this principle of growth, say, firstly focussing on surfaces M^2 in $\mathbb{E}^3$, then goes as follows. Hereby, we refer to the related drawing shown further on, of which "*the plane of the paper*" is choosen as to be the plane which is determined by *the direction of the position vector* X of M^2 in $\mathbb{E}^3$ *and* by *the normal direction* $T_X^\perp M$ *of this surface* at X, in which direction is also shown a unit normal vector N. Denoting as before by ρ *the polar distance*, i.e. $\rho = \mathrm{d}(O, X) = \|X\|$, one can then readily observe that for a generic point X on a generic surface M^2 in $\mathbb{E}^3$, *the change of* ρ *when moving* X *along* M^2 *on some infinitesimal distance* (suggested by the radius of the sphere shown in the drawing) *is maximal in the tangent direction to* M^2 *at* X *in which the tangent plane* $T_X M$ *is cut by the normal plane* $[X, N]$ *spanned by* X *and* N*, or, still, that this tangent direction is nothing but the direction of the vector grad*ρ *at* X. Put otherwise, the plane $[X, N]$, or, for any non-minimal surface, equivalently, the plane $[X, \Delta X]$ whereby Δ is *the Laplace operator* of the surface M^2, (since by Beltrami's formula $\Delta X = -2HN$, whereby H is Germain's *mean curvature* of M^2 in $\mathbb{E}^3$), contains the vector $\operatorname{grad}\rho$, and so, this plane can also be determined as the plane $[X, \operatorname{grad}\rho]$; [unless $\operatorname{grad}\rho = 0$, but then M^2 is (part of) a sphere centered at the origin]. Now, consider *Euler's normal section* σ *of* M^2 *at* X *in the direction of grad*ρ *in this plane* $[X, \operatorname{grad}\rho] = \mathbb{E}^2$, with unit tangent vector T at X. For any generic surface M^2 in $\mathbb{E}^3$, speculated about as being one or other kind of thing, somewhile after in one or other way having been originating in a 3D space somewhere around a certain pole, taken to be the origin of the ambient space $\mathbb{E}^3$, further having been growing into some 2D form M^2 in $\mathbb{E}^3$, *certainly, for any point* X *of this surface* M^2*, this curve* σ *in this plane* $[X, grad, \rho]$ *is likely the curve through* X *which is most fundamentally associated with the growth of* M^2 *in* $\mathbb{E}^3$ *at* X *as such.* And, then, finally, we impose as *principle of growth for 2D forms in 3D spaces* that these surfaces M^2 in $\mathbb{E}^3$ should obey, at each of their points X, D'Arcy Thompson's above law of planar growth for this curve σ in this plane $[X, \operatorname{grad}\rho]$.

So, *for 2D surfaces* M^2 *in a Euclidean 3D space* $\mathbb{E}^3$ *with position vector field* X*, we propose the following generalised principle of natural growth out of a pole* O, the origin of $\mathbb{E}^3$: *the magnitudes of the components Gp and $Gp^{\perp}$ of a growing force G acting in the direction which is determined by the gradient of the radial distance function* $\rho = \mathrm{d}(O, X) = \|X\|$, (that is: the component Gp of G in the direction of the position X and the component $Gp^{\perp}$ in the direction perpendicular to the position X in the plane $[X, \mathrm{grad}\,\rho]$), *have a constant ratio.* And, geometrically, just like for the curves Γ in $\mathbb{E}^2$ before, this is *equivalent* to the fact that *the angle $\alpha^{\perp} = \angle(X, N)$ is constant*, or, still, to the fact that *the angle $\alpha = \angle(X, \mathrm{grad}\,\rho)$ is constant.*

In complete analogy, from the original law of growth of curves in planes, i.e. from 1D forms in 2D spaces, one comes to the following generalised D'Arcy Thompson *law of natural growth out of a pole O for nD submanifolds M^n with position vector field X in (n+m)D Euclidean ambient spaces: the magnitudes of the components Gp and $Gp^{\perp}$ of a growing force G acting in the direction which is determined by the gradient of the radial distance function $\rho = d(O, X) = \|X\|$*, (that is: the component Gp of G in the direction of the position X and the component $Gp^{\perp}$ in the direction perpendicular to the position X in the plane $[X, \mathrm{grad}\,\rho]$), *have a constant ratio.* And, as above, one sees that this is equivalent to the geometrical fact that *the angle $\alpha = \angle(X, T_X M)$ between the position vector X and the tangent space $T_X M$ of M^n at X is constant*, (X being projected onto $T_X M$ in the plane $[X, \mathrm{grad}\,\rho]$), or, still, to the geometrical fact that *the angle $\alpha^{\perp} = \angle(X, T_X^{\perp} M)$ between the position vector X and the normal space $T_X^{\perp} M$ of M^n in $\mathbb{E}^{n+m}$ at X is constant*, (X being projected onto $T_X^{\perp} M$ in the plane $[X, \mathrm{grad}\,\rho]$). And, hereby following [1], this may well justify to call such M^n in $\mathbb{E}^{n+m}$ *the equiangular submanifolds of the Euclidean spaces*, the equiangular spirals of Descartes thus being the equiangular 1D submanifolds in the Euclidean plane $\mathbb{E}^2$.

8. Thus, what may well be the *most straightforward extension of the principle of planar growth*, as this was originally given in terms of the constant ratio of the magnitudes of the orthogonal components of the growing forces with respect to the position vector field, from 1D to nD and from 1 co–D to m co–D, gives *a basic law of growth for submanifolds M^n of arbitrary dimension n and arbitrary co–dimension m in Euclidean spaces $\mathbb{E}^{n+m}$ which characterises these submanifolds M^n in $\mathbb{E}^{n+m}$ to be equiareal*, i.e. *which characterises these submanifolds by a basic property of their position vector field X, namely, that the angle $\alpha^{\perp} = \angle(X, T_X^{\perp} M)$ between X and $T_X^{\perp} M$, the normal space of M^n in $\mathbb{E}^{n+m}$ at X, is constant.*

It can not come as a big surprise, then, that this class of submanifolds had been studied already before, *from a purely geometrical point of view.* And, indeed, as part of his *general investigations of the geometry of the position vector of submanifolds M^n* in (pseudo) Euclidean spaces (p)$\mathbb{E}^{n+m}$, in [3, 4, 5, 6, 7, 8, 9], Bang–Yen Chen already before had *introduced these submanifolds, discovered several of their fundamental properties, obtained their full classification* (for all n and for all m) and, moreover, *put their theory in several wider geometrical contexts*, (some of which will also play some roles in following parts of our paper). Essentially, he introduced these submanifolds as follows [4]: *the position vector X of any submanifold M^n in (p)$\mathbb{E}^{n+m}$ naturally and canonically orthogonally decomposing into a tangential component $X^{\top}$*

and a normal component $X^\perp$, $X = X^\top + X^\perp$, he named a submanifold M^n in (p)$\mathbb{E}^{n+m}$ *a constant ratio submanifold* when *the ratio* $\|X^\top\|/\|X^\perp\| = tg\,\alpha^\perp$ *is constant*, (or, for that matter, when $X^\perp = 0$ or when the ratio $\|X^\top\|/\|X^\perp\|$ is a fixed real number), or, *equivalently*, when *the ratio* $\|X^\perp\|/\|X^\top\| = tg\,\alpha$ *is constant*, (or, for that matter, when $X^\top = 0$ or when the ratio $\|X^\perp\|/\|X^\top\|$ is a fixed real number), which, of course, amounts to, equivalently, when *the angle* $\alpha^\perp = \angle(X, T_X^\perp M)$ *between the position vector* X *and the normal space* $T_X^\perp M$ *of* M^n *in (p)*$\mathbb{E}^{n+m}$ *at* X *is constant*, or, still, equivalently, when *the angle* $\alpha = \angle(X, T_X M)$ *between the position vector* X *and the tangent space* $T_X M$ *of* M^n *in (p)*$\mathbb{E}^{n+m}$ *is constant.*

At this stage, from the above mentioned works of Bang–Yen Chen, only *two results* will be formulated in this part of our paper, and, herein, these results will even be *limited to the case of Euclidean ambient spaces.* Firstly, he generalised a classic characterisation of the equiangular spirals, which above also was recalled in a quote out of "On Growth and Form", as follows.

Theorem 1. [7]. *A submanifold* M^n *of a Euclidean space* $\mathbb{E}^{n+m}$ *is a constant ratio submanifold if and only if* $\|grad\,\rho\|$ *is constant.* □

In view of Theorem 1, we would like to call the extension of D'Arcy Thompson's basic law of planar growth which was given in the previous section, when formulated in the following *equivalent*, but, at least in our opinion, *more simple and more appealing version, the basic law of D'Arcy Thompson and Bang-Yen Chen for the natural growth out of a pole* of things of arbitrary dimensions and co–dimensions (and, so, in particular, for 2D things in 3D spaces): *the growth is such that the instantaneous maximal change of its polar distance is constant.*

And, secondly, he classified the forms which result from the growths according to this basic law as follows.

Theorem 2. [7]. *A submanifold* M^n *in a Euclidean space* $\mathbb{E}^{n+m}$ *is a submanifold of constant ratio if and only if: (i)* M^n *is a cone with vertex* O*, (ii)* M^n *is contained in a hypersphere of the ambient space, centered at* O*, or, (iii) there exist local co–ordinate systems* $(s, u_2, \ldots, u_n)$ *on* M^n *such that the immersion of* M^n *in* $\mathbb{E}^{n+m}$ *is given by* $X(s, u_2, \ldots, u_n) = b\,s.\,Y(s, u_2, \ldots, u_n)$, $[(*)]$, *for some number* $b \in]0, 1[$, *and whereby* $Y = Y(s, u_2, \ldots, u_n)$ *satisfies the following conditions: (1)* $\|Y\| = 1$*, (2)* Y_s *is perpendicular to* $Y_{u_2}, \ldots, Y_{u_n}$ *and (3)* $\|Y_s\| = (1 - b^2)^{1/2}/(b\,s)$. □

In particular, focussing in $(*)$ on the case of surfaces M^2 in $\mathbb{E}^3$, *the integral curves of the vector field grad* ρ *on equiangular surfaces*, in general, are twisted curves, which, however, certainly in small enough parts, *have "the looks" of equiangular spirals*, (cfr. their curvature as determined e.g. in [4]).

9. For equiangular surfaces M^2 in $\mathbb{E}^3$, an alternative presentation of their *classification* was given by M. Munteanu as follows.

Theorem. [16]. *A surface* M^2 *in the Euclidean space* $\mathbb{E}^3$ *is equiangular if and only if: (i)* M^2 *is a cone with vertex at* O*, (ii)* M^2 *is (part of) a sphere* $S_O^2(r)$ *of radius* r *centered at* O*, or, (iii)* M^2 *can be parametrised as* $X(u, v) = (\sin\alpha^\perp)u\,\{[\cos\phi(u)]\,.f(v) +$

$[\sin\phi(u)] \,.\, [f(v)\times f'(v)]\}$ $[(**)]$, *whereby* $\phi(u) = (cotg\,\alpha^{\perp})\,(ln\,u)$ *and* f *is a curve on* $S^2_O(1)$ *with arclength parameter* v. □

This parametrisation shows that *the non-trivial equiangular surfaces* M^2 *in* $\mathbb{E}^3$ *are generated by logarithmic* α*-spirals with poles at* O of which the planes *are normal to a curve lying on a sphere centered at* O, as described and illustrated in [16]. A later following figure shows parts of some constant ratio surfaces made according to the parametrisation $(**)$: in each case one of the spirals going out of the pole to a point on a spherical curve is emphasised, as is this spherical curve itself, hereafter being a part of (a) a *spherical loxodrome*, (b) a *sinusoidal curve around the equator*, and (c) a *small circle*, respectively.

10. When thinking of a surface M^2 in $\mathbb{E}^3$ as some 2D thing growing or having been growing out of some pole O in some 3D space, then the use of *polar* or *spherical co-ordinates* (ρ,λ,β) for the description of such concrete surfaces seems to be most natural indeed; hereby ρ are the *polar distances* and λ,β are the *geographical lengths and widths* or *geographical longitudes and latitudes*, respectively, such that $(\cos\beta.\cos\lambda, \cos\beta.\sin\lambda, \sin\beta)$ are the *rectangular Cartesian co-ordinates* of a point P with *geographical co-ordinates* (λ,β) on the *unit sphere* centered at O. And, in terms of such spherical co–ordinates, K. Boyadzhiev obtained the following *classification of the equiangular surfaces* M^2 *in* $\mathbb{E}^3$.

Theorem 1. [1]. *A surface* M^2 *in* $\mathbb{E}^3$ *is equiangular if and only if: (i)* M^2 *is a cone with vertex at* O*, (ii)* M^2 *is (part of)* $S^2_O(r)$ *for some* $r>0$*, or (iii) the polar equation of* M^2 *is of the form* $\rho(\lambda,\beta) = k.e^{a\lambda+h(\beta)}$, $[(***)]$, *whereby* $k\geq 0$ *and* a *are constants, and, when* $a=0$*:* $h(\beta) = (tg\,\alpha^{\perp}).\beta$, $[(1)]$ *and, when* $a\neq 0$*:* $h(\beta) = a\,\{h_1(\beta)-h_2(\beta)\}$, $[(2)]$, *whereby* $h_1(\beta) = (1+c^2)^{1/2}.arctg\left\{\left[(1+c^2)^{1/2}.tg\beta\right]/(c^2-tg^2\beta)^{1/2}\right\}$, $h_2(\beta) = arctg\left\{(tg\beta)/\left[(c^2-tg^2\beta)^{1/2}\right]\right\}$, $c = \left\{\left[(tg\,\alpha^{\perp})/a\right]^2-1\right\}^{1/2}$ *and* $-arctg\,c\leq\beta\leq arctg\,c$. □

In particular, $(***)$ shows that the *parameter lines* $\beta=$ constant, i.e. the intersections of these surfaces with the vertical *cones* with vertex at O and of which the rulers make an angle of β with the (xy)–plane, are *loxodromes on these cones*, (i.e., the *spirals on cones*, having the *circular helices* -the loxodromes on right circular cylinders- as limiting cases), which are also called *concho-spirals* or *helico-spirals*, while, since, for small $a\neq 0$, $h(\beta)\approx(\mathrm{tg}\,\alpha^{\perp})\,\beta$, $[(3)]$, the *parameter lines* $\lambda=$ constant, i.e. the intersections of these surfaces with the vertical planes through O -the meridian planes- are close to being *equiangular spirals with pole* O. In any case, in this parametrisation $(***)$ of the equiangular surfaces M^2 in $\mathbb{E}^3$ from the point of view of growth, cfr., in particular, D'Arcy Thompson and Bang-Yen Chen's basic law of growth out of a pole, as such also the *helico-spirals* naturally show up, corresponding to an earlier quote from "On Growth and From". We recall that the concho-spirals in $\mathbb{E}^3$ are characterised by the property that their first and second radii of curvature, R_1 and R_2, (i.e.: the inverses of their first and second curvatures κ_1 and κ_2, or, still, of their curvature $\kappa=\kappa_1$ and their torsion $\tau=\kappa_2$), are both first degree functions of their arclengths: $R_1 = A_1 s + B_1$ and $R_2 = A_2 s + B_2$, for some real constants A_1, A_2, B_1 and B_2.

For the *general equiangular surfaces* M^2 in $\mathbb{E}^3$, as may be plausible from the generalised D'Arcy Thompson law of natural growth in space, whereas *these surfaces always develop or evolve in a similar way in their* $\operatorname{grad}\rho$ *tangent direction as in their radial direction, in their tangent directions different from* $\operatorname{grad}\rho$ *they can not be expected also to grow in a similar way.* Thus, in general, *equiangular surfaces are not self-similar*, in contrast with the 1D situation where the equiangular curves, that is, *the equiangular spirals, are even to be characterised as thé self-similar curves*, (as their only tangent direction, of course, coincides with their $\operatorname{grad}\rho$ direction). In this respect, when following the *terminology of Boyadzhiev* in naming *self-similar* the surfaces M^2 in $\mathbb{E}^3$ for which the polar equation is of the form $\rho(\lambda, \beta) = k.\mathrm{e}^{a\lambda}.\mathrm{e}^{b\beta}$, [(4)], for some arbitrary constants a, b and for some constant $k \geq 0$, then, related to the remark made just before, we mention the following.

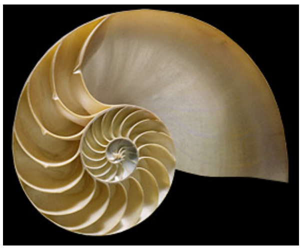
Theorem 2. [1]. *The only self-similar equiangular surfaces* M^2 *in* $\mathbb{E}^3$ *are given by the polar equations of the form* $\rho(\lambda, \beta) = k.\mathrm{e}^{(tg_{\alpha^\perp})\beta}$, [(5)], *for some constant* $k \geq 0$, *i.e. are the surfaces of revolution obtained by rotating an equiangular spiral with pole* O *around an axis through the pole.* □

In this very particular class (5) of equiangular surfaces, their *β-parameter lines*, i.e. the intersections of these surfaces with the planes through the z-axis, their meridians, are all *equiangular α-spirals*, while, their *λ–parameter lines*, i.e. the intersections of these surfaces with the circular cones with vertex at O and whose rulers make a fixed angle β with the (xy)–plane, of course, are nothing but the *parallel circles* of these surfaces of revolution, to which the former concho-spirals degenerate in this case.

In addition to the illustrations given in [1], in the final figure following below two further examples of constant ratio surfaces are shown, made according to $(***)$: first, it concerns the surface (a) for which $k = 1$ and $a = 0,1$ and $\alpha^\perp = \pi/6$, with ranges $\lambda \in [0, 3\pi]$ and $\beta \in [-\pi/2, \pi/2]$, and its part (b), for the range $\beta \in [-\pi/2, \pi/4]$, and, secondly, it concerns the surface of revolution (c) with $k = 1$ and $\alpha^\perp = \pi/3$, with ranges $\lambda \in [0, 2\pi]$ and $\beta \in [-\pi/2, \pi/2]$, and its part (d) for the range $\lambda \in [0, 3\pi/2]$.

And, finally, we may not miss to refer to "The Curves of Lives" [11], which book of 1914 very generously expresses T. A. Cook's general cultural and historical comments on some of the forms that we have been discussing in this paper.

1.



Nautilus popilius (Linnaeus) and Littorina scabra angulifera (Lamarck)

2.

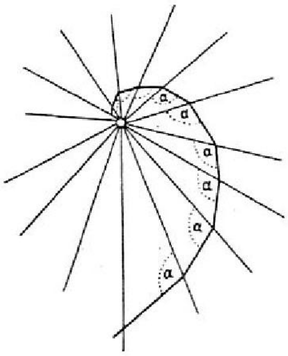


Fig. 351. Spiral path of an insect, as it draws towards a light. From Wigglesworth (after van Buddenbroek).

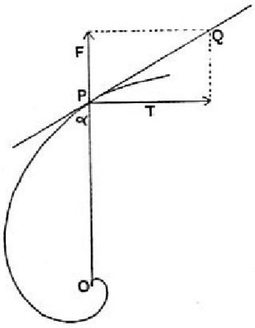
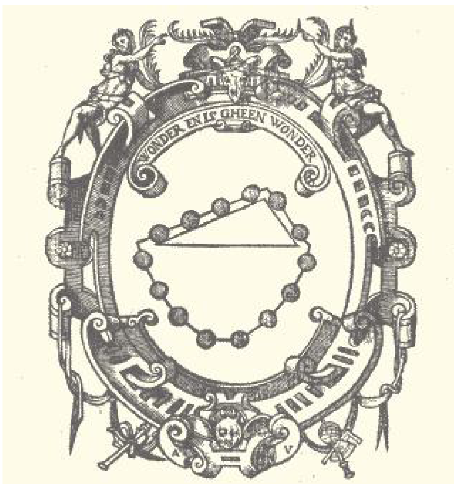


Fig. 352. Dynamical aspect of the equiangular spiral.

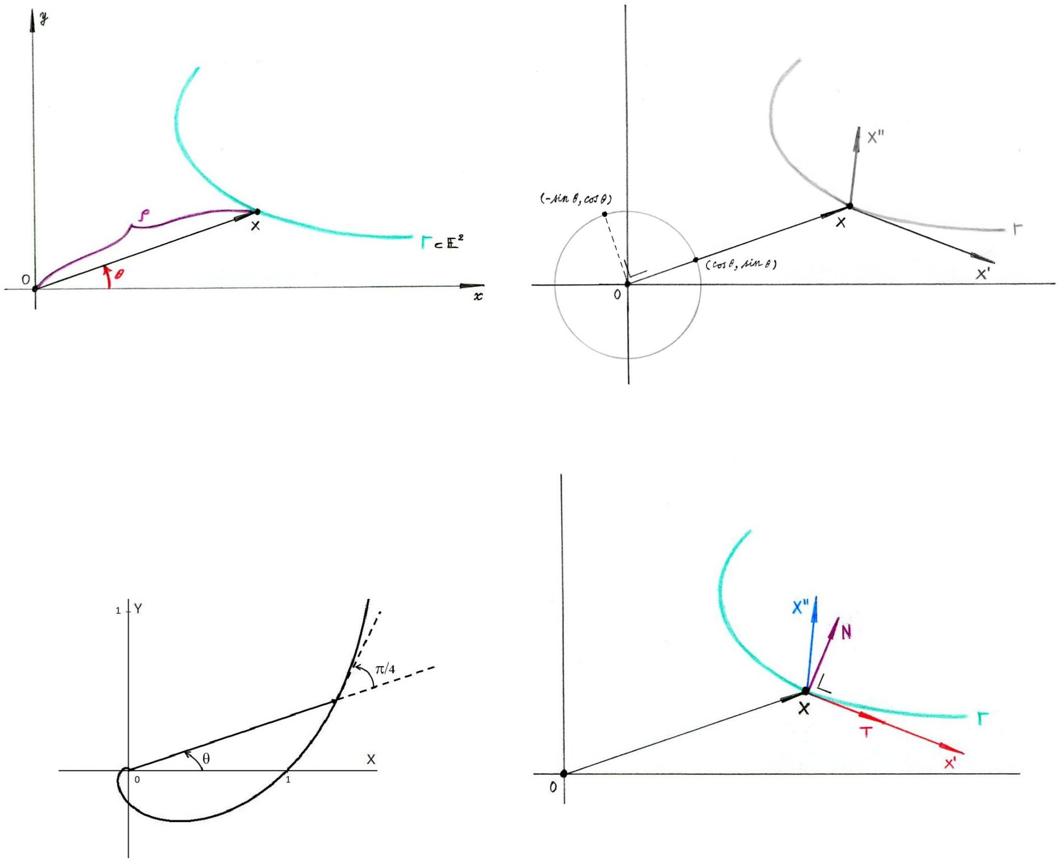
Figures from D'Arcy Thompson's book

3.



Simon Stevin's logo and motto

4.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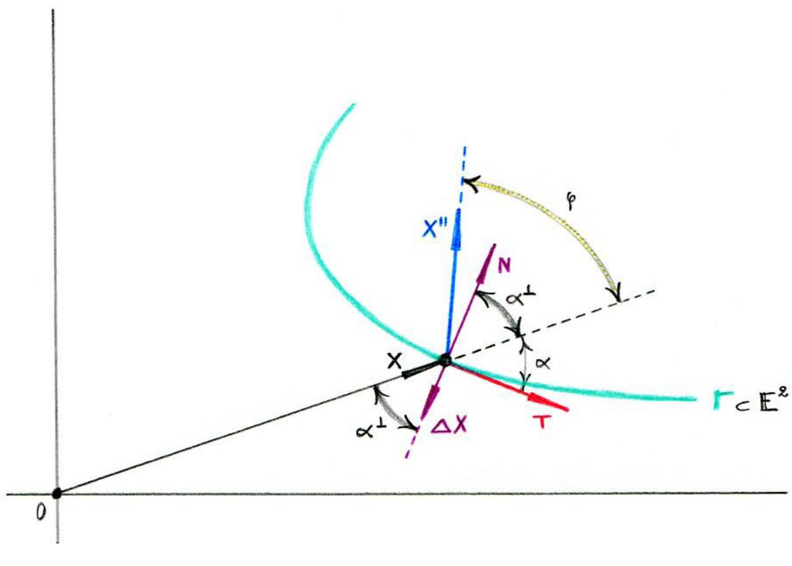
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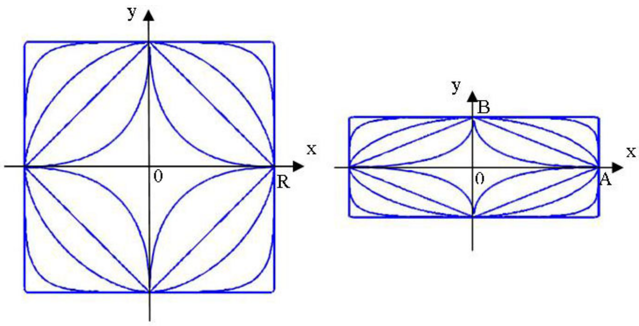

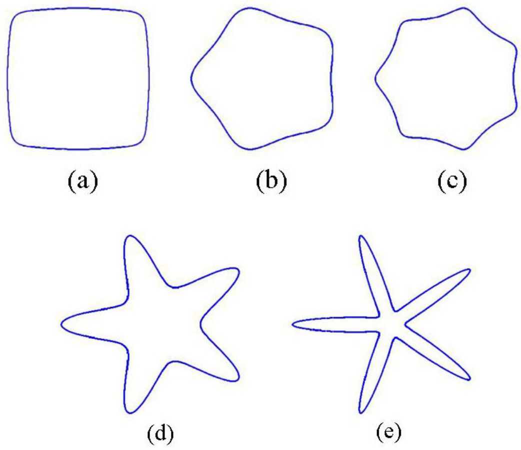

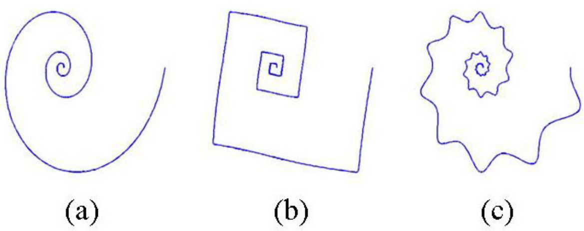

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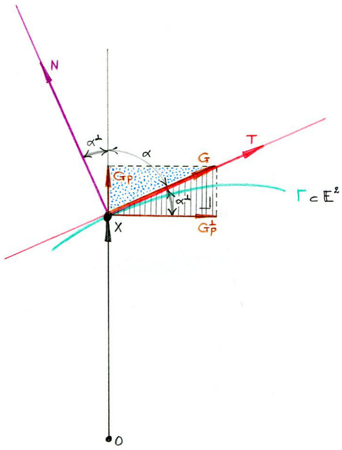

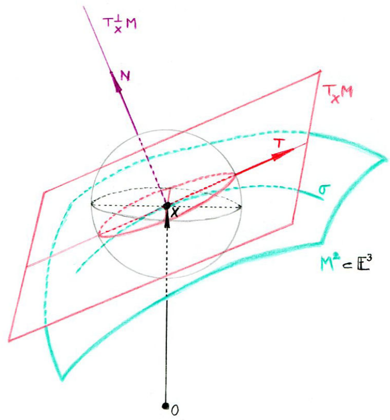

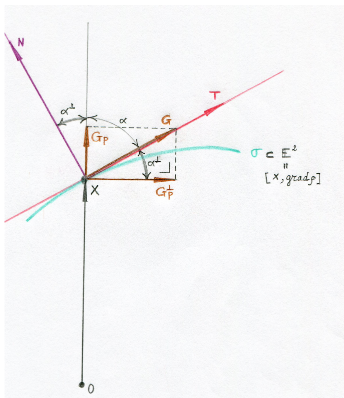

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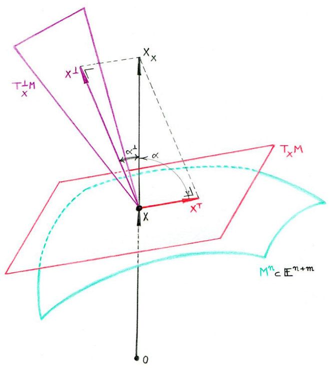

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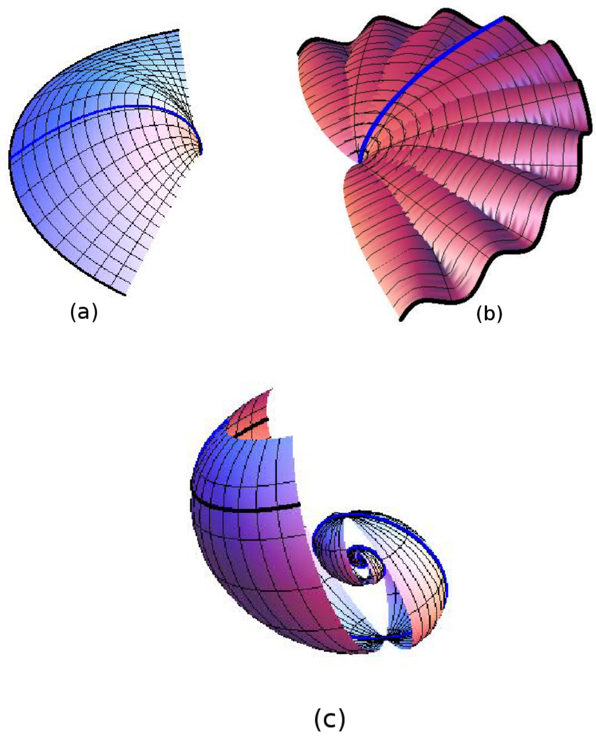

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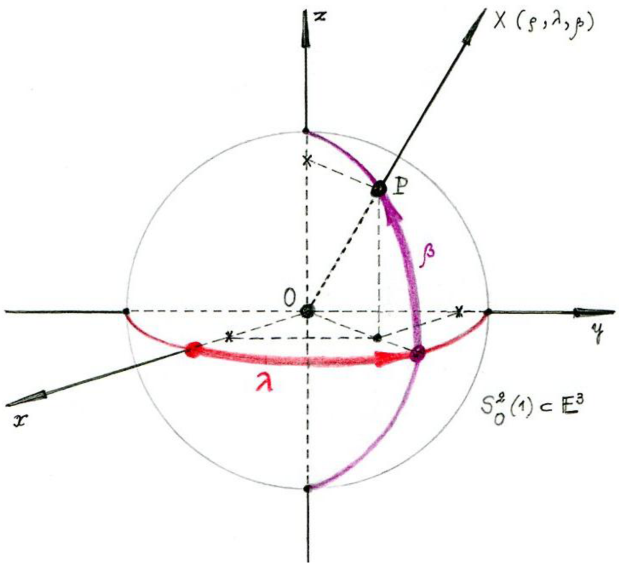

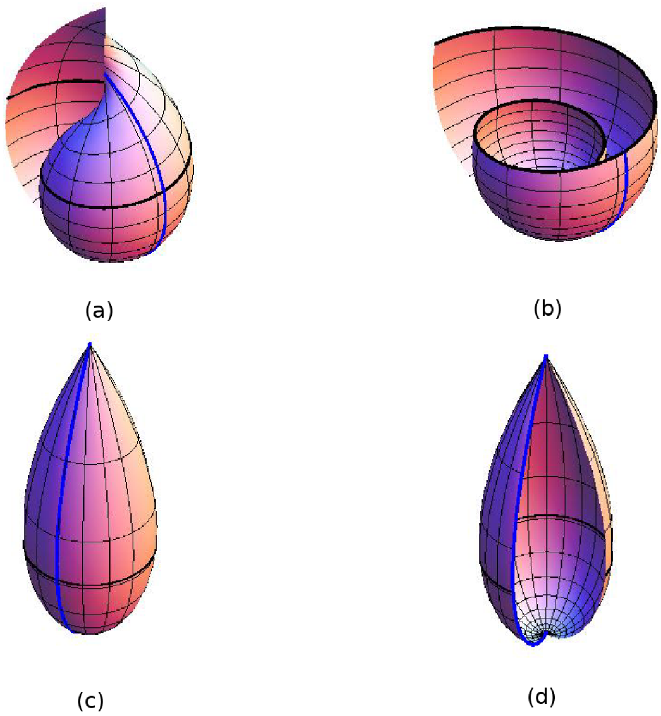

References

[1] K. N. Boyadzhiev, *Equiangular Surfaces, Self–Similar Surfaces, and the Geometry of Seashells*, The College Mathematics Journal, **38** (4) (2007), 265–271.

[2] J. Bronowski, *The Ascent of Man*, BBC Books, Reading, 2011.

[3] B. Y. Chen, *Riemannian submanifolds*, Chapter 3 in "*Handbook of Differential Geometry, Vol. I*", (eds. F. Dillen e.a.), Elsevier Publ. Co., Amsterdam, 2000.

[4] B. Y. Chen, *Constant Ratio Hypersurfaces*, Soochow J. Math., **27** (2001), 353–362.

[5] B. Y. Chen, *Geometry of position functions of Riemannian submanifolds in pseudo–Euclidean spaces*, J. Geom., **74** (2002), 61–77.

[6] B. Y. Chen, *Convolution of Riemannian manifolds and its applications*, Bull. Austral. Math. Soc., **66** (2002), 177–191.

[7] B. Y. Chen, *Constant-ratio space–like submanifolds in pseudo–Euclidean spaces*, Houston J. Math., **29** (2003), 281–294.

[8] B. Y. Chen, *More on convolutions of Riemannian manifolds*, Beiträge Algebra Geom., **44** (2003), 9–24.

[9] B. Y. Chen, *Pseudo-Riemannian Geometry, δ–invariants and Applications*, World. Sc. Publ. Co., Singapore, 2011.

[10] S. S . Chern, *Finsler Geometry is just Riemannian Geometry without the Quadratic Restriction*, Notices AMS (1996), 959–963.

[11] T. A. Cook, *The Curves of Life*, Dover Publ., New York, 1979.

[12] P. Dombrowski, *Wege in euklidischen Ebenen - Kinematik der Speziellen Relativitätstheorie*, Springer Verlag, Berlin, 1999.

[13] J. Gielis, S. Haesen and L. Verstraelen, *Universal Natural Shapes - From the supereggs of Piet Hein to the cosmic egg of Georges Lemaitre*, Kragujevac J. Math. **28** (2005), 57–68.
[14] J. Koenderink, *Color for the sciences*, MIT Press, Cambridge MA, 2010.
[15] K. Lenjou, *Krommen en oppervlakken van Lamé en Gielis*, Master Thesis K.U.Leuven, 2004-2005.
[16] M. I. Munteanu, *From golden spirals to constant slope surfaces*, J. Math. Phys. **51** (2010), 073507.
[17] D'Arcy Thompson, *On Growth and Form*, Cambridge University Press, 1942.
[18] L. Verstraelen, *Universal Natural Shapes*, Tangenta, Serbian Math. Soc., Kragujevac–Beograd (2005), 13–20.
[19] L. Verstraelen, *Laudatio for MSU Distinguished Professor Bang–Yen Chen at Riga 2008*, Bull. Transilv. Univ. Brasov Ser. III 1(50) (2008), 1–4.
[20] L. Verstraelen, *Foreword to B.Y. Chen's "Pseudo–Riemannian Geometry, δ–invariants and Applications"*, World Sc. Publ. Co., Singapore, 2011.

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