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HOLOMORPHIC SOLUTION FOR CLASSES OF FORCED KORTEWEG-DE VRIES EQUATIONS OF FRACTIONAL ORDER

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Abstract. In this article, we consider some classes of forced Korteweg-de Vries equations of fractional order in complex domain. The existence and uniqueness of holomorphic solution are established. We illustrate our theoretical result by examples.

1. Introduction

Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. Various types play important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations require to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators, Erdlyi-Kober operators, Weyl-Riesz operators, Caputo operators and Grünwald-Letnikov operators, have appeared during the past three decades with its applications in other field [1-6]. Recently, the existence of analytic solutions for fractional differential equations in complex domain are posed [7-10].

Key words and phrases. Fractional calculus, fractional differential equation, holomorphic solution, unit disk, Riemann-Liouville operators, nonlinear, Korteweg-de Vries equation (KdV), Tsunami, forced (KdV).

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The study of nonlinear problems is of crucial importance in all areas of mathematics, mechanics and physics. Some of the most interesting features of physical systems are hidden in their nonlinear behavior, and can only be studied with appropriate methods designed to tackle and process nonlinear problems. One of the most important nonlinear problem is the Kortewegde Vries equation, it is used in many sections of nonlinear mechanics and physics. Recently, a numerical method is proposed for solving the KdVB equation. Zaki has used the collocation method with quintic B-spline finite element [11], Soliman has employed the collocation solution of the KdV equation using septic splines as element shape function [12], Kaya has implemented the Adomian decomposition method for solving the KdVB equation [13] and Jafari and Firoozjaee have developed for the numerical study of the Korteweg-de Vries (KdV) and the Korteweg-de Vries Burgers (KdVB) equations with initial conditions by a homotopy approach [14]. The Korteweg-de Vries equation with a forcing term is provided by recent studies as a mathematical model of describing the physics of a shallow layer of fluid subject to external forcing. Also, the basic hydrodynamic model of tsunami generation by the atmospheric disturbances is based on the well-known Korteweg-de Vries equation with a forcing term (see [15]).

It is well known that the solution to the Cauchy problem of the KdV equation with an analytic initial profile is analytic in the space variable for a fixed time. However, analyticity in the time variable fails (see [16]). The present paper deals with a nonlinear fractional differential equation, in sense of the Srivastava-Owa operators (see [17]).

Definition 1.1. The fractional derivative of order α is defined for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order α is defined, for a function $f(z)$ by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \qquad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Remark 1.1. From Definition 1.1 and Definition 1.2, we have

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1;\ 0<\alpha<1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1;\ \alpha > 0.$$

In the present work we consider the fractional differential equations

$$u_t + a(z) D_z^\alpha u + b(z) u u_z + c(z) u_{zzz} = \rho(z) D_z^\beta f + \sigma(z) f_z, \quad 0<\alpha,\beta<1 \tag{1.1}$$

subject to the initial condition $u(t,0)=0$, where $t\in J := [0,1]$, $z \in U := \{z \in \mathbb{C} : |z| < 1\}$, $u(t,z)$ is an unknown function, $a(z) := az^\alpha$, $b(z) := bz$, $c(z) := cz^3$, $\rho(z) := \rho z^\beta$, $\sigma(z) := \sigma z$ are complex valued functions in U, a,b,c,ρ,σ are constants and f is a prescribed forcing function on $J\times U$ which is holomorphic in t and z. Thus it has the following series:

$$f(t,z) := \sum_{i,j\geq 1} f_{i,j} t^i z^j, \quad t\in J,\ z\in U.$$

Using the majorant relations concept, we establish the existence and uniqueness of holomorphic solution for the problem (1.1):

Definition 1.3. If the majorant relations are described as: if $a(x) = \sum a_i x^i$ and $A(x) = \sum A_i x^i$, then we say that $a(x) \ll A(x)$ if and only if $|a_i| \leq A_i$ for each i.

2. Existence of unique solution

In this section, we pose the existence and uniqueness of holomorphic solution for problem (1.1). We have the following result:

Theorem 2.1. *Assume the problem* (1.1). *If*

$$a\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} + n[b+(n-1)(n-2)c] \neq 0, \quad n \in \mathbb{N}^* \tag{2.1}$$

then the equation (1.1) *has a unique holomorphic solution* $u(t,z)$ *near* $(0,0)\in J\times U$.

Proof. We realize that equation (1.1) has a formal solution

$$u(t,z)=\sum_{k=1}^{\infty}u_k(t)z^k,\quad (z\in U). \tag{2.2}$$

Without loss of generality, we assume that $u_0(t)=1$. Then, substituting the series (2.2) into the equation (1.1) and comparing the coefficients of z^k in two sides of the equation, we obtain

(2.3)

$$u_1'(t)+a\frac{\Gamma(2)}{\Gamma(2-\alpha)}u_1(t)+bu_1(t)=\sum_{i\geq 1}\left[\rho\frac{\Gamma(2)}{\Gamma(2-\beta)}+\sigma\right]f_{i,1}t^i$$
$$:=\phi_1(t)$$

$$u_2'(t)+a\frac{\Gamma(3)}{\Gamma(3-\alpha)}u_2(t)+2bu_2(t)=\sum_{i\geq 1}\left[\rho\frac{\Gamma(3)}{\Gamma(3-\beta)}+2\sigma\right]f_{i,2}t^i-cu_1^2(t)$$
$$:=\phi_2(t)$$

$$u_3'(t)+\left[a\frac{\Gamma(4)}{\Gamma(4-\alpha)}+3b+6c\right]u_3(t)=\sum_{i\geq 1}\left[\rho\frac{\Gamma(4)}{\Gamma(4-\beta)}+3\sigma\right]f_{i,3}t^i-3cu_1(t)u_2(t)$$
$$:=\phi_3(t)$$

$$\vdots$$

$$u_n'(t)+\left\{a\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}+n\left[b+(n-1)(n-2)c\right]\right\}u_n(t)=\phi_n(t).$$

Thus we obtain the following formula

$$u_n'(t)+\left\{a\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}+n\left[b+(n-1)(n-2)c\right]\right\}u_n(t)=\phi_n(t). \tag{2.4}$$

Then by the assumption (2.1), the equation (2.4) has a unique holomorphic solution $u_k(t)$ near $t=0$. Moreover, $u_k(t)$ is bounded for all $k\in\mathbb{N}$ such that

$$\|u_k\|\leq\frac{\|\phi\|}{|\delta_k|}, \tag{2.5}$$

where $\|\phi\|=\max_{t\in J}|\phi(\cdot)|$ and

$$\delta_k:=a\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}+k\left[b+(k-1)(k-2)c\right],\quad k\in\mathbb{N}^*.$$

Let us prove inequality (2.5). From equation (2.4) we have

$$\frac{d}{dt}\left[e^{\int_0^t\delta_k(s)ds}\times u_k(t)\right]=e^{\int_0^t\delta_k(s)ds}\times\phi(t).$$

Thus

$$\int_0^t \frac{d}{dy}\Big[e^{\int_0^y \beta_k(s)ds} \times u_k(y)\Big]dy = \int_0^t e^{\int_0^y \delta_k(s)ds} \times \phi(y)dy,$$

that is,

$$e^{\int_0^t \delta_k(s)ds} \times u_k(t) - u_k(0) = \int_0^t e^{\int_0^y \beta_k(s)ds} \times \phi(y)dy$$

which is equivalent to

$$u_k(t) = e^{-\int_0^t \delta_k(s)ds} \times \int_0^t e^{\int_0^y \delta_k(s)ds} \times \phi(y)dy.$$

Therefore,

$$\|u_k\| \le \max_{t\in J} \frac{\left| e^{-\int_0^t \delta_k(s)ds} \times \int_0^t e^{\int_0^y \delta_k(s)ds} \times \delta_k(y)dy \right|}{|\delta_k(t)|} \times \|\phi\|$$

$$\le \frac{\|\phi\|}{|\delta_k|}.$$

Now we proceed to prove that the formal series solution (2.2) is convergent near $(0,0) \in (J,U)$. We expand ϕ into Taylor series with respect to t, u, *i.e.*

$$\phi(t,u) = \sum_{m+p\ge 1}^{\infty} a_m(t) t^m u^p$$

such that

(i) $a_{m,p}(t)$ is holomorphic in J,
(ii) $|a_{m,p}(t)| \le A_{m,p}, \quad A_{m,p} > 0$ on J,
(iii) $\sum_{m,p\ge 2}^{\infty} A_{m,p} V^{m+p}$ converges in (t,V) where $V>0$ satisfies $|u| \le V$ and $|t| \le V$.

From the equation (2.4), we observe that

(2.6)
$$\begin{aligned} &\left[\frac{d}{dt} + \beta_1\right] u_1(t) = \phi_1, \\ &\vdots \\ &\left[\frac{d}{dt} + \beta_k\right] u_k(t) = -\sum_{m+p\ge 2}\Big[\sum_{\substack{k_1+\cdots+k_m \\ +l_1+\cdots+l_p = k}} a_{m,p} \times t_{k_1} \times \cdots \times t_{k_m} \times u_{l_1} \times \cdots \times u_{l_p}\Big]. \end{aligned}$$

Without loss of generality, we may assume that there exists a positive constant $K \ge 1$ such that

$$|u_1(t)| \le K, \quad \text{and} \quad t \le 1 \le K, \quad t \in J.$$

Denoting $C := \frac{1}{|\delta_k|}$, we pose the following formula:

$$V(z) = Kt + \frac{C}{1-r} \sum_{m+p\geq 2} \frac{A_{m,p}}{(1-r)^{m+p-2}} V^{m+p}, \tag{2.7}$$

where r is a parameter with $0 < r < 1$. Since the equation (2.7) is an analytic functional equation in V then, in view of the implicit function theorem, the equation (2.7) has a unique holomorphic solution $V(t)$ in a neighborhood of $t = 0$ with $V(0) = 0$. Expanding $V(t)$ into Taylor series in t we have

$$V(t) = \sum_{k\geq 1} V_k t^k \tag{2.8}$$

where

$$\begin{aligned} V_k = & \frac{C}{1-r} \sum_{m+p\geq 1} \Big[\sum_{\substack{k_1+\cdots+k_m \\ +l_1+\cdots+l_p = k}} \frac{A_{m,p}}{(1-r)^{m+p-2}} \times V_{k_1} \times \cdots \times V_{k_m} \times V_{l_1} \times \cdots \times V_{l_p} \Big] \\ := & \frac{C_k}{(1-r)^{k-1}}, \quad k \in \mathbb{N} \\ > & 0, \end{aligned} \tag{2.9}$$

with $C_1 = K$.

Next, our aim is to show that the series $\sum_{k\geq 1} V_k t^k$ is a majorant series for the formal series solution $\sum_{k\geq 1} u_k(t) t^k$ near $z = 0$. For this purpose we will show that

$$|u_k(t)| \leq V_k \quad \text{on} \quad J. \tag{2.10}$$

Since $(1-r) < 1$ implies

$$\frac{1}{(1-r)^{m+p-2}} \geq 1, \quad r < 1$$

then we have

$$\begin{aligned} |u_k(t)| &\leq C \sum_{m+p\geq 1} \Big[\sum_{\substack{k_1+\cdots+k_m \\ +l_1+\cdots+l_p = k}} A_{m,p} \times |u_{l_1}(t)| \times \cdots \times |u_{l_p}(t)| \Big] \\ &\leq C \sum_{m+p\geq 1} \Big[\sum_{\substack{k_1+\cdots+k_m \\ +l_1+\cdots+l_p = k}} A_{m,p} \times V_{k_1} \times \cdots \times V_{k_m} \times V_{l_1} \times \cdots \times V_{l_p} \Big] \\ &\leq C \sum_{m+p\geq 1} \Big[\sum_{\substack{k_1+\cdots+k_m \\ +l_1+\cdots+l_p = k}} \frac{A_{m,p}}{(1-r)^{m+p-2}} \times V_{k_1} \times \cdots \times V_{k_m} \times V_{l_1} \times \cdots \times V_{l_p} \Big] \\ &\leq \frac{C_k}{(1-r)^{k-2}} \leq \frac{C_k}{(1-r)^{k-1}} = V_k. \end{aligned}$$

Hence, we obtain the inequality (2.10). This completes the proof of Theorem 2.1. □

Example 2.1. Assume the following equation

$$(2.11)\qquad \begin{cases} u_t + \frac{z^{0.5}}{1.8} D_z^{0.5} u(t,z) + z u_z + z u u_z + z^3 u_{zzz} = z^6 sech(t), & z \in U \\ u(t,0) = 0, \quad \text{in an interval of} \quad t = 0 \end{cases}$$

where $u(t,z)$ is an unknown function. Put

$$u(t,z) = \mu(t) z^3 + v(t,z) \quad (v(t,z) = O(z^4))$$

as a formal solution. Therefore, $\mu(z)$ satisfies

$$3z^6 \mu(t)^2 + (1+z^3)\mu(t) + z^3 \mu'(t) - z^6 sech(t) = g(t,z).$$

Now, assuming

$$\mu(t) := q + \psi(t),$$

where q is a constant and $\psi(t) = O(t)$, we obtain that $q = \pm\frac{1}{\sqrt{3}}$. Hence, we impose the following equations:

(2.12)

$$\begin{cases} z^3\psi'(t) + 3z^6\Big(\frac{1}{3} - \frac{2}{\sqrt{3}}\psi(t) + \psi^2(t)\Big) + (1+z^3)(\psi(t) - \frac{1}{\sqrt{3}}) - z^6 sech(t) & = g(t,z), \\ & q = -\frac{1}{\sqrt{3}} \\ \psi(0) = 0, \end{cases}$$

(2.13)

$$\begin{cases} z^3\psi'(t) + 3z^6\Big(\frac{1}{3} + \frac{2}{\sqrt{3}}\psi(t) + \psi^2(t)\Big) + (1+z^3)(\psi(t) + \frac{1}{\sqrt{3}}) - z^6 sech(t) & = g(t,z), \\ & q = \frac{1}{\sqrt{3}} \\ \psi(0) = 0, \end{cases}$$

where the holomorphic solution $\psi(t)$ exists uniquely and converges in a neighborhood of the origin (Theorem 2.1).

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