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DIFFERENTIAL SANDWICH THEOREMS OF p-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN FRACTIONAL DERIVATIVE OPERATOR

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Abstract. In the present paper we derive some subordination and superordination results for p-valent functions in the open unit disk by using certain fractional derivative operator. Some special cases are also considered.

1. Introduction and Preliminaries

Let $\mathcal{H}(\mathcal{U})$ denote the class of analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathcal{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C},\ p \in \mathbf{N}).$$

Also, let $\mathcal{A}(p)$ be the class of functions $f \in \mathcal{H}(\mathcal{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbf{N} \tag{1.1}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$.

Let $f, g \in \mathcal{H}(\mathcal{U})$. We say that the function f is subordinate to g, if there exist a Schwarz function w, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$.

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This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $p(z), h(z) \in \mathcal{H}(\mathcal{U})$, and let $\Phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\Phi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions, and if $p(z)$ satisfies the second-order superordination

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z) \tag{1.2}$$

then $p(z)$ is called to be a solution of the differential superordination (1.2). (If $f(z)$ is subordinatnate to $g(z)$, then $g(z)$ is called to be superordinate to $f(z)$). An analytic function $q(z)$ is called a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfies (1.2). An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.2) is said to be the best subordinant.

Recently, Miller amd Mocanu [5] obtained conditions on $h(z), q(z)$ and Φ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z) \quad \Longrightarrow \quad q(z) \prec p(z)$$

with the results of Miller amd Mocanu [5], Bulboacă [2] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1] used the results obtained by Bulboacă [3] and gave the sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by (see Srivastava and Karlsson [9])

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{when } n=0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & \text{when } n \in \mathbf{N}. \end{cases} \tag{1.4}$$

for $\lambda \neq 0, -1, -2, \ldots$.

We recall the following definitions of fractional derivative operators which were used by Owa [6], (see also [7]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi \tag{1.5}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.2. Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbf{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \tag{1.6}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon),\ z \longrightarrow 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.7}$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^\lambda f(z), \quad 0 \leq \lambda < 1. \tag{1.8}$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.9}$$

for $f(z) \in \mathcal{A}(p)$ and $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1$; $p \in \mathbf{N}$. Then it is observed that $M_{0,z}^{\lambda,\mu,\eta} f(z)$ maps $\mathcal{A}(p)$ onto itself as follows:

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n} \tag{1.10}$$

where

$$\delta_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}. \tag{1.11}$$

It is easily verified from (1.10) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)' = (p-\mu) M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) + \mu M_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.12}$$

Notice that

$$M_{0,z}^{0,0,\eta} f(z) = f(z)$$

and

$$M_{0,z}^{1,1,\eta} f(z) = \frac{z f'(z)}{p}.$$

The object of this paper is to derive several subordination and superordination results for p-valent functions involving certain fractional derivative operator.

In order to prove our results we mention the following known results which will be used in the sequel.

Lemma 1.1. [7] *Let $\lambda, \mu, \eta \in \mathbf{R}$, such that $\lambda \geq 0$ and $K > \max\{0, \mu - \eta\} - 1$. Then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu}. \tag{1.13}$$

Definition 1.4. [5] Denote by Q the set of all functions f that are analytic and injective in $\bar{\mathcal{U}} - E(f)$, where

$$E(f) = \{\xi \in \partial\mathcal{U} : \lim_{z\to\infty} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathcal{U} - E(f)$.

Lemma 1.2. [4] *Let the function q be univalent in the open unit disk $\mathcal{U}$, and θ and φ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(a) *Q is starlike univalent in $\mathcal{U}$, and*
(b) *$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathcal{U}$.*

If

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z))$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1.6, Shanmugam et al. [8] obtained the following lemma.

Lemma 1.3. [8] *Let* q *be univalent in the open unit disk* $\mathcal{U}$ *with* $q(0) = 1$ *and* $\alpha, \gamma \in \mathbb{C}$. *Further assume that*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\gamma}\right)\right\}.$$

If $p(z)$ *is analytic in* $\mathcal{U}$, *and*

$$\alpha p(z) + \gamma zp'(z) \prec \alpha q(z) + \gamma zq'(z)$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 1.4. [2] *Let the function* q *be univalent in the open unit disk* $\mathcal{U}$, *and* θ *and* φ *be analytic in a domain* D *containing* $q(\mathcal{U})$ *with* $\varphi(w) \neq 0$ *when* $w \in q(\mathcal{U})$. *Suppose that*

(a) $\mathrm{Re}\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0$ *for* $z \in \mathcal{U}$,
(b) $zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathcal{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$ *with* $p(\mathcal{U}) \subseteq D$, *and* $\theta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$, *and*

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z))$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1.8, Shanmugam et al. [8] obtained the following lemma.

Lemma 1.5. [8] *Let* q *be univalent in the open unit disk* $\mathcal{U}$ *with* $q(0) = 1$. *Let* $\alpha, \gamma \in \mathbb{C}$ *and* $\mathrm{Re}\left(\frac{\alpha}{\gamma}\right) > 0$. *If* $p(z) \in \mathcal{H}[q(0), 1] \cap Q, \alpha p(z) + \gamma zp'(z)$ *is univalent in* $\mathcal{U}$, *and*

$$\alpha q(z) + \gamma zq'(z) \prec \alpha p(z) + \gamma zp'(z)$$

then $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

2. Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 2.1. *Let q be univalent in $\mathcal{U}$ with $q(0) = 1$, and suppose that*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{1}{\gamma}\right)\right\}. \tag{2.1}$$

If $f(z) \in \mathcal{A}(p)$, and

$$\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) = \gamma\left[(p-\mu) - (p-\mu-1)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z) M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2}\right]$$
$$+ (1-\gamma)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \tag{2.2}$$

and if q satisfies the following subordination:

$$\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q(z) + \gamma z q'(z) \tag{2.3}$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}; \gamma \in \mathbb{C})$ *then*

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q(z) \tag{2.4}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

So, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{z(M_{0,z}^{\lambda,\mu,\eta} f(z))'}{M_{0,z}^{\lambda,\mu,\eta} f(z)} - \frac{z(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}. \tag{2.5}$$

Using the identity (1.12), a simple computation shows that

$$\gamma\left[(p-\mu) - (p-\mu-1)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z) M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2}\right]$$
$$+ (1-\gamma)\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} = p(z) + \gamma z p'(z).$$

The assertion (2.4) of Theorem 2.1 now follows by an application of Lemma 1.3, with $\alpha = 1$. □

Remark 2.1. For the choice $q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, in Theorem 2.1, we get the following Corollary.

Corollary 2.1. *Let* $-1 \leq B < A \leq 1$, *and suppose that*

$$\text{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\text{Re}\left(\frac{1}{\gamma}\right)\right\}. \tag{2.6}$$

If $f(z) \in \mathcal{A}(p)$ *and*

$$\Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2}$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}; \gamma \in \mathbb{C})$ *where* $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.2), *then*

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Next, by appealing to Lemma 1.5 of the preceding section, we prove the following.

Theorem 2.2. *Let* q *be convex in* $\mathcal{U}$ *and* $\gamma \in \mathbb{C}$ *with* $\text{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is univalent in* $\mathcal{U}$, *then*

$$q(z) + \gamma z q'(z) \prec \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \tag{2.7}$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N})$ *implies*

$$q(z) \prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \tag{2.8}$$

and q *is the best subordinant where* $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.2).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

Then from the assumption of Theorem 2.2, the function $p(z)$ is analytic in $\mathcal{U}$ and (2.5) holds. Hence, the subordination (2.7) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The assertion (2.8) of Theorem 2.2 now follows by an application of Lemma 1.5. $\square$

Combining Theorem 2.1 and Theorem 2.2, we get the following sandwich theorem.

Theorem 2.3. *Let* q_1 *and* q_2 *be convex functions in* $\mathcal{U}$ *with* $q_1(0) = q_2(0) = 1$. *Let* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$ *such that*

$$\frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is univalent in* $\mathcal{U}$, *then*

$$q_1(z) + \gamma z q_1'(z) \prec \Phi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda,\mu) - p - 1; p \in \mathbf{N})$ *implies*

$$q_1(z) \prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q_2(z) \tag{2.9}$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant where* $\Phi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.2).

Remark 2.2. For $\lambda = \mu = 0$ in Theorem 2.3, we get the following result.

Corollary 2.2. *Let* q_1 *and* q_2 *be convex functions in* $\mathcal{U}$ *with* $q_1(0) = q_2(0) = 1$. *Let* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$ *such that*

$$\frac{pf(z)}{zf'(z)} \in \mathcal{H}[1,1] \cap Q$$

and let

$$\Phi_1(\gamma, f)(z) = \gamma p \left[1 - \frac{f''(z)f(z)}{(f'(z))^2}\right] + p(1-\gamma)\frac{f(z)}{zf'(z)}, \quad p \in \mathbf{N}$$

is univalent in $\mathcal{U}$, *then*

$$q_1(z) + \gamma z q_1'(z) \prec \Phi_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{pf(z)}{zf'(z)} \prec q_2(z) \tag{2.10}$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

Theorem 2.4. *Let q be univalent in $\mathcal{U}$ with $q(0) = 1$, and assume that* (2.1) *holds. Let $f(z) \in \mathcal{A}(p)$, and*

$$\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) = [1 + \gamma(\mu - p - 1)] \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} + 2\gamma(p - \mu) \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}$$

$$- \gamma(p - \mu - 1) \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{z^p (M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2}. \tag{2.11}$$

If q satisfies the following subordination:

$$\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q(z) + \gamma z q'(z)$$

$(\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}; \gamma \in \mathbb{C})$ *then*

$$\frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q(z) \tag{2.12}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = (\frac{M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

So, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \frac{2z(M_{0,z}^{\lambda,\mu,\eta} f(z))'}{M_{0,z}^{\lambda,\mu,\eta} f(z)} - p - \frac{z(M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}. \tag{2.13}$$

Using the identity (1.12), a simple computation shows that

$$[1 + \gamma(\mu - p - 1)] \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} + 2\gamma(p - \mu) \frac{M_{0,z}^{\lambda,\mu,\eta} f(z)}{z^p}$$

$$- \gamma(p - \mu - 1) \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2 M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z)}{z^p (M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z))^2} = p(z) + \gamma z p'(z). \tag{2.14}$$

The assertion (2.12) of Theorem 2.4 now follows by an application of Lemma 1.3, with $\alpha = 1$. □

Remark 2.3. For the choice $q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, in Theorem 2.4, we get the following result.

Corollary 2.3. *Let $-1 \leq B < A \leq 1$, and assume that* (2.6) *holds. If $f(z) \in \mathcal{A}(p)$ and*

$$\Psi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec \frac{1 + Az}{1 + Bz} + \frac{\gamma(A - B)z}{(1 + Bz)^2}$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda,\mu) - p - 1; p \in \mathbf{N}; \gamma \in \mathbb{C})$ *where* $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.11), *then*

$$\frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Next, by appealing to Lemma 1.5 of the preceding section, we prove the following.

Theorem 2.5. *Let* q *be convex in* $\mathcal{U}$, *and* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$,

$$0 \neq \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is univalent in* $\mathcal{U}$, *then*

$$q(z) + \gamma z q'(z) \prec \Psi_{\lambda,\mu,\eta}(\gamma, f)(z) \tag{2.15}$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda,\mu) - p - 1; p \in \mathbf{N})$ *implies*

$$q(z) \prec \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \tag{2.16}$$

and q *is the best subordinant where* $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.11).

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}.$$

Then from the assumption of Theorem 2.5, the function $p(z)$ is analytic in $\mathcal{U}$ and (2.13) holds. Hence, the subordination (2.15) is equivalent to

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The assertion (2.16) of Theorem 2.5 now follows by an application of Lemma 1.5. □

Combining Theorem 2.4 and Theorem 2.5, we get the following sandwich theorem.

Theorem 2.6. *Let* q_1 *and* q_2 *be convex functions in* $\mathcal{U}$ *with* $q_1(0) = q_2(0) = 1$. *Let* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$ *such that*

$$\frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \in \mathcal{H}[1,1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is univalent in* $\mathcal{U}$, *then*

$$q_1(z) + \gamma z q_1'(z) \prec \Psi_{\lambda,\mu,\eta}(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N})$ *implies*

$$q_1(z) \prec \frac{(M_{0,z}^{\lambda,\mu,\eta} f(z))^2}{z^p M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant where* $\Psi_{\lambda,\mu,\eta}(\gamma, f)(z)$ *is as defined in* (2.11).

Remark 2.4. For $\lambda = \mu = 0$ in Theorem 2.6, we get the following result.

Theorem 2.7. *Let* q_1 *and* q_2 *be convex functions in* $\mathcal{U}$ *with* $q_1(0) = q_2(0) = 1$. *Let* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\gamma > 0$. *If* $f(z) \in \mathcal{A}(p)$ *such that*

$$\frac{p(f(z))^2}{z^{p+1} f'(z)} \in \mathcal{H}[1,1] \cap Q$$

and let

$$\Psi_1(\gamma, f)(z) = [1 - \gamma(p+1)] \frac{p(f(z))^2}{z^{p+1} f'(z)} + 2\gamma p \frac{f(z)}{z^p} - \gamma p \frac{f''(z)(f(z))^2}{z^p (f'(z))^2}, \quad p \in \mathbf{N}$$

is univalent in $\mathcal{U}$, *then*

$$q_1(z) + \gamma z q_1'(z) \prec \Psi_1(\gamma, f)(z) \prec q_2(z) + \gamma z q_2'(z)$$

implies

$$q_1(z) \prec \frac{p(f(z))^2}{z^{p+1} f'(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

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