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COMMUTATION FORMULAS FOR δ-DIFFERENTIATION IN A GENERALIZED FINSLER SPACE

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Abstract. In this work we define a generalized Finsler space $\mathbb{GF}_N$ as N- dimensional differentiable manifold on which a non-symmetric basic tensor $g_{ij}(x,\dot{x})$ is defined by virtue (1.3). Some basic properties of $\mathbb{GF}_N$ are given (§1).

Based on non-symmetry of basic tensor, we define (§2) two kinds of covariant derivative of a tensor in the Rund's sense and obtain 10 Ricci type identities. In these identities appear 3 curvature tensors and 15 magnitudes, which we call "curvature pseudotensors" in $\mathbb{GF}_N$. All mentioned curvature tensors and pseudotensors reduce to known curvature tensor in usual Finsler space $\mathbb{F}_N$. The cited Ricci type identities are proved by total induction method in a general case.

1. Preliminaries

The generalized Finsler space ($\mathbb{GF}_N$) is a differentiable manifold with non-symmetric basic tensor $g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N) \equiv g_{ij}(x,\dot{x})$, where

$$g_{ij}(x,\dot{x}) \neq g_{ji}(x,\dot{x}), \quad (g = det(g_{ij}) \neq 0,\ \dot{x} = dx/dt). \tag{1.1}$$

Based on (1.1), one can defined the symmetric respectively anti-symmetric part of g_{ij}

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}), \qquad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}), \tag{1.2}$$

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where, following [11], is

$$(1.3)\qquad a)\ g_{\underline{ij}}(x,\dot{x})=\frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j},\qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}(x,\dot{x})}{\partial\dot{x}^k}=0,$$

where $F(x,\dot{x})$ is a metric function in $\mathbb{GF}_N$, having the properties known from the theory of usual Finsler space ($\mathbb{F}_N$) (see e.g. [10]).

The lowering and the raising of indices one defines by the tensors $g_{\underline{ij}}$ and h^{ij} respectively, where h^{ij} is defined as follows

$$(1.4)\qquad g_{\underline{ij}}\,h^{jk}=\delta_i^k,\quad (\underline{g}=det(g_{\underline{ij}})\neq 0).$$

We can define **generalized Cristoffel symbols** of the 1^{st} and the 2^{nd} kind:

$$(1.5)\qquad \gamma_{i.jk}=\frac{1}{2}(g_{ji,k}-g_{jk,i}+g_{ik,j})\neq\gamma_{i.kj},$$

$$(1.6)\qquad \gamma^i_{jk}=h^{ip}\gamma_{p.jk}=\frac{1}{2}h^{ip}(g_{jp,k}-g_{jk,p}+g_{pk,j})\neq\gamma^i_{kj},$$

where, e.g., $g_{ji,k}=\partial g_{ji}/\partial x^k$.

Then we have

$$(1.7)\qquad \gamma^p_{jk}g_{\underline{ip}}=\gamma_{s.jk}h^{ps}g_{\underline{ip}}=\gamma_{s.jk}\delta^s_i=\gamma_{i.jk}.$$

Theorem 1.1. *In $\mathbb{GF}_N$ the following relations are valid*

$$(1.8)\qquad \gamma_{i.jk}+\gamma_{j.ik}=g_{\underline{ij},k},\quad \gamma_{i.jk}+\gamma_{k.ji}=g_{\underline{ik},j},$$

$$(1.9)\qquad \gamma_{i.jk}\dot{x}^i\dot{x}^j=\frac{1}{2}g_{\underline{ij},k}\dot{x}^i\dot{x}^j=\frac{1}{2}g_{ij,k}\dot{x}^i\dot{x}^j=\frac{1}{4}F^2_{\dot{x}^i\dot{x}^jx^k}\dot{x}^i\dot{x}^j,$$

$$(1.10)\qquad \gamma_{i.jk}\dot{x}^i\dot{x}^k=\frac{1}{2}g_{\underline{ik},j}\dot{x}^i\dot{x}^k=\frac{1}{2}g_{ik,j}\dot{x}^i\dot{x}^k=\frac{1}{4}F^2_{\dot{x}^ix^j\dot{x}^k}\dot{x}^i\dot{x}^k,$$

$$(1.11)\qquad \gamma_{i.jk}\dot{x}^j\dot{x}^k=(g_{\underline{ij},k}-\frac{1}{2}g_{\underline{jk},i})\dot{x}^j\dot{x}^k,$$

where, for example, $F^2_{\dot{x}^i\dot{x}^jx^k}=\dfrac{\partial^3F^2}{\partial\dot{x}^i\partial\dot{x}^j\partial x^k}$.

Proof. The equations (1.8) follow from (1.5), (1.2) and the equations (1.9)-(1.11) from (1.5), (1.2), (1.3). □

Introducing a tensor C_{ijk} like as at $\mathbb{F}_N$, we have

$$(1.12)\qquad C_{ijk}(x,\dot{x})\overset{def}{=}\frac{1}{2}g_{ij,\dot{x}^k}\underset{(1.3b)}{=}\frac{1}{2}g_{\underline{ij},\dot{x}^k}=\frac{1}{4}F^2_{\dot{x}^i\dot{x}^j\dot{x}^k},$$

where " $\underset{(1.3b)}{=}$ " signifies "equal based on (1.3b)". We see that C_{ijk} is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip}C_{pjk} \underset{(1.12)}{=} h^{ip}C_{jpk} = h^{ip}C_{jkp}. \tag{1.13}$$

With help of coefficients

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^s \neq P^i_{kj} \tag{1.14}$$

one obtains coefficients of non-symmetric affine connections in the Rund's sence [10, 13]:

$$P^{*i}_{jk} = \gamma^i_{jk} - h^{il}(C_{jlp}P^p_{ks} + C_{klp}P^p_{js} - C_{jkp}P^p_{ls})\dot{x}^s \underset{(1.6)}{\neq} P^{*i}_{kj}, \tag{1.15}$$

$$P^*_{i.jk} = P^{*r}_{jk}g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^s \neq P^*_{i.kj}. \tag{1.16}$$

Based on non-symmetry of the connection, it is possible to define in $\mathbb{GF}_N$ **two kinds of covariant derivative** of a tensor. For example, for a tensor $a^{r_1\ldots r_u}_{t_1\ldots t_v}(x,\xi)$ we define **covariant derivative of the 1^{st} kind**

$$\begin{aligned} &a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{1}{|}m}(x,\xi) \\ &=a^{\cdots}_{\ldots,m} + a^{r_1\ldots r_u}_{t_1\ldots t_v,\dot{p}}\xi^p_{,m} + \sum_{\alpha=1}^{n} P^{*r_\alpha}_{pm}a^{r_1\ldots,r_{\alpha-1}pr_{\alpha+1}\ldots r_u}_{t_1\ldots t_v} - \sum_{\beta=1}^{v} P^{*p}_{t_\beta m}a^{r_1\ldots r_u}_{r_1\ldots r_{\beta-1}pr_{\beta+1}\ldots r_v} \end{aligned} \tag{1.17}$$

and **covariant derivative of the 2^{nd} kind**

$$\begin{aligned} &a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{2}{|}m}(x,\xi) \\ &=a^{\cdots}_{\ldots,m} + a^{r_1\ldots r_u}_{t_1\ldots t_v,\dot{p}}\xi^p_{,m} + \sum_{\alpha=1}^{n} P^{*r_\alpha}_{mp}a^{r_1\ldots,r_{\alpha-1}pr_{\alpha+1}\ldots r_u}_{t_1\ldots t_v} - \sum_{\beta=1}^{v} P^{*p}_{mt_\beta}a^{r_1\ldots r_u}_{r_1\ldots r_{\beta-1}pr_{\beta+1}\ldots r_v}, \end{aligned} \tag{1.18}$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_N(x)$, and $a^{\cdots}_{\ldots,\dot{p}} = \frac{\partial a^{\cdots}_{\ldots}}{\partial \dot{x}^p}$.

By the procedure that is similar to that one in a Finsler space, it can be proved that covariant derivative (1.17), (1.18) of a tensor also is a tensor.

Theorem 1.2. *For the tensor $g_{\underline{ij}}(x,\dot{x})$ based on both kinds of derivative* (1.17), (1.18) *is valid*

$$g_{\underline{ij}\underset{\theta}{|}m}(x,\xi) = 2C_{ijp}(\xi^p_{,m} + P^{*p}_{ms}\dot{x}^s), \quad \theta = 1,2, \tag{1.19}$$

$$g_{\underline{ij}\underset{\theta}{|}m}(x,\dot{x}) = 2C_{ijp}\dot{x}^p_{\underset{2}{|}m}, \quad \theta = 1,2. \tag{1.20}$$

Proof. Starting from (1.17), we get

$$g_{\underline{ij}\underset{1}{|}m}(x,\xi) = g_{\underline{ij},m} + g_{\underline{ij},\dot{x}^p}\xi^p_{,m} - P^{*p}_{im}g_{\underline{pj}} - P^{*p}_{jm}g_{\underline{ip}} \underset{(1.12)}{=} g_{\underline{ij},m} + 2C_{ijp}\xi^p_{,m} - (P^*_{i.jm} + P^*_{j.im}). \tag{1.21}$$

Based on (1.16) and using (1.8) one obtains

$$P^*_{i.jm} + P^*_{j.im} = g_{\underline{ij},m} - 2C_{ijp}P^p_{ms}\dot{x}^s. \tag{1.22}$$

By substituting (1.22) into (1.21) we get

$$g_{\underline{ij}\underset{1}{|}m}(x,\xi) = 2C_{ijp}(\xi^p_{,m} + P^p_{ms}\dot{x}^s).$$

As

$$P^p_{ms}\dot{x}^s = P^{*p}_{ms}\dot{x}^s \tag{1.23}$$

we have

$$g_{\underline{ij}\underset{1}{|}m}(x,\xi) = 2C_{ijp}(\xi^p_{,m} + P^{*p}_{ms}\dot{x}^s).$$

The same result one obtains for $g_{ij\underset{2}{|}m}$, and we have proved (1.19).

For $\xi = \dot{x}$ one obtains

$$g_{\underline{ij}\underset{1}{|}m}(x,\dot{x}) = 2C_{ijp}(\dot{x}^p_{,m} + P^{*p}_{ms}\dot{x}^s), \quad \theta = 1,2.$$

Because $\dot{x}^p$ does not depend on ξ, the previous equation becomes (1.20). □

Theorem 1.3. *For the Chronecker symbol is in force*

$$\delta^i_{j\underset{1}{|}m} = 0, \qquad \delta^i_{j\underset{2}{|}m} = 0. \tag{1.24}$$

The proof is obtained by using the definition of δ-symbols and covariant derivative (1.17), (1.18).

Theorem 1.4. *For h^{ij} is force*

$$h^{ij}_{\underset{\theta}{|}m} = -h^{ip}h^{jq}g_{\underline{pq}\underset{\theta}{|}m}, \quad \theta = 1,2. \tag{1.25}$$

Proof. From $h^{ip}g_{\underline{pq}} = \delta^i_q$ one obtains

$$h^{ip}_{\underset{\theta}{|}m}g_{\underline{pq}} + h^{ip}g_{\underline{pq}\underset{\theta}{|}m} = 0.$$

Composing with h^{jq}, from here is

$$h^{ip}_{\underset{\theta}{|}m}\delta^j_p + h^{ip}h^{jq}g_{\underline{pq}\underset{\theta}{|}m} = 0, \quad \theta = 1,2 \Rightarrow (1.25).$$

□

2. Ricci type identities in $\mathbb{GF}_N$

It is known that in Finsler space one Ricci identity exists for δ-differentiation, corresponding to alternated covariant derivative of the 2^{nd} order. In the case of non-symmetric affine connection there exist 10 possibilities to form the difference

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{\lambda}{|}m\underset{\mu}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\nu}{|}m\underset{\omega}{|}n} \quad (\lambda,\mu,\nu,\omega = 1,2), \tag{2.1}$$

where $\underset{1}{|}, \underset{2}{|}$, denotes two kinds of covariant derivative based on (1.17), (1.18), and from that one obtains 10 Ricci type identities [3] and three curvature tensors. Here we prove corresponding identities in $\mathbb{GF}_N$ by total induction method. The mentioned possibilities are obtained for

$$\begin{aligned}(\lambda,\mu;\nu,\omega) \in \Big\{&(1,1;1,1),(2,2;2,2),(1,2;1,2),(2,1;2,1),(1,1;2,2),\\ &(1,1;1,2),(1,1;2,1),(2,2;1,2),(2,2;2,1),(1,2;2,1)\Big\}.\end{aligned} \tag{2.2}$$

To examine the general cases based on (2.1), we observe firstly the case of a tensor $a^{hij}_{kl}(x,\xi)$. By virtue of (1.17), (1.18) we obtain firstly for the 1^{st} and 2^{nd} kind of derivative:

$$\begin{aligned}&a^{hij}_{kl\underset{\theta}{|}mn}(x,\xi) - a^{hij}_{kl\underset{\theta}{|}nm}(x,\xi)\\ =&\underset{\theta}{\widetilde{K}}{}^{h}{}_{pmn}a^{pij}_{kl} + \underset{\theta}{\widetilde{K}}{}^{i}{}_{pmn}a^{ipj}_{kl} + \underset{\theta}{\widetilde{K}}{}^{j}{}_{pmn}a^{ijp}_{kl}\\ &-\underset{\theta}{\widetilde{K}}{}^{p}{}_{kmn}a^{hij}_{pl} - \underset{\theta}{\widetilde{K}}{}^{p}{}_{lmn}a^{hij}_{kp} + (-1)^{\theta}T^{*p}_{mn}a^{hij}_{kl\underset{\theta}{|}p}, \quad \theta = 1,2,\end{aligned} \tag{2.3}$$

where

$$\underset{1}{\widetilde{K}}{}^{i}{}_{jmn}(x,\xi) = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \tag{2.4}$$

$$\underset{2}{\widetilde{K}}{}^{i}{}_{jmn}(x,\xi) = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \tag{2.5}$$

are Rund's type **curvature tensors of the** 1^{st} **and the** 2^{nd} **kind** respectively, and

$$T^{*i}_{jk}(x,\dot{x}) = P^{*i}_{[jk]} = P^{*i}_{jk} - P^{*i}_{kj} \tag{2.6}$$

is the **torsion tensor** of the connection P^*. We can also denote

$$P^{*i}_{(jk)} = P^{*i}_{jk} + P^{*i}_{kj}. \tag{2.6'}$$

Further, by direct calculation one obtains

$$
\begin{aligned}
&a^{hij}_{kl\underset{1}{|}m\underset{2}{|}n}(x,\xi) - a^{hij}_{kl\underset{1}{|}n\underset{2}{|}m}(x,\xi)\\
&= \underset{1}{\widetilde{A}}{}^{h}_{pmn} a^{pij}_{kl} + \underset{1}{\widetilde{A}}{}^{i}_{pmn} a^{hpj}_{kl} + \underset{1}{\widetilde{A}}{}^{j}_{pmn} a^{hip}_{kl}\\
&- \underset{2}{\widetilde{A}}{}^{p}_{kmn} a^{hij}_{pl} - \underset{2}{\widetilde{A}}{}^{p}_{lmn} a^{hij}_{kp} + a^{hij}_{kl<[mn]>} + a^{hij}_{kl\leqslant[mn]\geqslant} + T^{*p}_{mn} a^{hij}_{kl\underset{1}{|}p},
\end{aligned}
\tag{2.7}
$$

where we denoted

$$
\begin{aligned}
a^{hij}_{kl<mn>} &= T^{*h}_{pm}(a^{pij}_{kl,n} + a^{pij}_{kl,\dot{s}}\xi^s_{,n}) + T^{*i}_{pm}(a^{hpj}_{kl,n} + a^{hpj}_{kl,\dot{s}}\xi^s_{,n}) + T^{*j}_{pm}(a^{hip}_{kl,n} + a^{hip}_{kl,\dot{s}}\xi^s_{,n})\\
&\quad - T^{*p}_{km}(a^{hij}_{pl,n} + a^{hij}_{pl,\dot{s}}\xi^s_{,n}) - T^{*p}_{lm}(a^{hij}_{kp,n} + a^{hij}_{kp,\dot{s}}\xi^s_{,n}),
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
a^{hij}_{kl\leqslant mn\geqslant} &= a^{psj}_{kl} P^{*h}_{[\breve{p}\breve{m}} P^{*i}_{\breve{n}\breve{s}]} + a^{pis}_{kl} P^{*h}_{[\breve{p}\breve{m}} P^{*j}_{\breve{n}\breve{s}]} + a^{hps}_{kl} P^{*i}_{[\breve{p}\breve{m}} P^{*j}_{\breve{n}\breve{s}]}\\
&\quad - a^{pij}_{sl} P^{*h}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{k}]} - a^{pij}_{ks} P^{*h}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{l}]} - a^{hpj}_{sl} P^{*i}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{k}]} - a^{hpj}_{ks} P^{*i}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{l}]}\\
&\quad - a^{hip}_{sl} P^{*j}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{k}]} - a^{hip}_{ks} P^{*j}_{[\breve{p}\breve{m}} P^{*s}_{\breve{n}\breve{l}]} + a^{hij}_{ps} P^{*p}_{[\breve{k}\breve{m}} P^{*s}_{\breve{n}\breve{l}]},
\end{aligned}
\tag{2.9}
$$

$$
P^{*h}_{[\breve{p}\breve{m}} P^{*i}_{\breve{n}\breve{s}]} = P^{*h}_{pm} P^{*i}_{ns} - P^{*h}_{mp} P^{*i}_{sn},
\tag{2.10}
$$

$$
\underset{1}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{jn} P^{*i}_{mp} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m},
\tag{2.11}
$$

$$
\underset{2}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{mj} P^{*i}_{pn} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}.
\tag{2.12}
$$

The following theorems are related to general 10 cases obtained by virtue of (2.1).

Theorem 2.1. *In the generalized Finsler space $\mathbb{GF}_N$ the next two Ricci type identity are valid*

$$
\begin{aligned}
&a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}nm}\\
&= \sum_{\alpha=1}^{u} \underset{\theta}{\widetilde{K}}{}^{r_\alpha}_{pmn} \binom{p}{r_\alpha} a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{\theta}{\widetilde{K}}{}^{p}_{t_\beta mn} \binom{t_\beta}{p} a^{\cdots}_{\cdots} + (-1)^\theta T^{*p}_{mn} a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}p},
\end{aligned}
\tag{2.13}
$$

where $\underset{\theta}{\widetilde{K}}$, $\theta = 1, 2$ *are given in* (2.4), (2.5), *and*

$$
\binom{p}{r_\alpha} a^{\cdots}_{\cdots} = a^{r_1\dots r_{\alpha-1} p r_{\alpha+1}\dots r_u}_{t_1\dots t_v}, \qquad \binom{t_\beta}{p} a^{\cdots}_{\cdots} = a^{r_1\dots r_u}_{t_1\dots t_{\beta-1} p r_{\beta+1}\dots t_v}.
\tag{2.14}
$$

Theorem 2.2. *In $\mathbb{GF}_N$ the 3^{rd} Ricci type identity is in force:*

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m} = \sum_{\alpha=1}^{u} \underset{1}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{2}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} + a^{r_1\dots r_u}_{t_1\dots t_v<[mn]>} + a^{r_1\dots r_u}_{t_1\dots t_v\leqslant[mn]\geqslant} + T^{*p}_{mn}a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}p}\,, \tag{2.15}$$

where $\underset{1}{\widetilde{A}}{}^i_{jmn}, \underset{2}{\widetilde{A}}{}^i_{jmn}$ are given at (2.11), (2.12) *and*

$$a^{r_1\dots r_u}_{t_1\dots t_v<mn>} = \sum_{\alpha=1}^{u} T^{*r_\alpha}_{pm}\binom{p}{r_\alpha}(a^{\cdots}_{\cdots,n} + a^{\cdots}_{\cdots,\dot{s}}\xi^s_{,n}) - \sum_{\beta=1}^{u} T^{*p}_{t_\beta m}\binom{t_\beta}{p}(a^{\cdots}_{\cdots,n} + a^{\cdots}_{\cdots,\dot{s}}\xi^s_{,n})\,, \tag{2.16}$$

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\leqslant mn\geqslant} = &\sum_{\substack{\alpha=1 \\ (\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*r_\beta}_{n\underset{\smile}{s}]}\binom{p}{r_\alpha}\binom{s}{r_\beta}a^{\cdots}_{\cdots} \\ &- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*s}_{n\underset{\smile}{t_\beta}]}\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ &+ \sum_{\substack{\alpha=1 \\ (\alpha<\beta)}}^{v-1}\sum_{\beta=1}^{v} P^{*p}_{[\underset{\smile}{t_\alpha}m}P^{*s}_{n\underset{\smile}{t_\beta}]}\binom{t_\alpha}{p}\binom{t_\beta}{s}a^{\cdots}_{\cdots}\,, \end{aligned} \tag{2.17}$$

$$\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} = a^{r_1\dots r_{\alpha-1}pr_{\alpha+1}\dots r_u}_{t_1\dots t_{\beta-1}sr_{\beta+1}\dots t_v}\,.$$

Proof. For a tensor $a^i(x,\xi)$ from (2.17) one obtains zero and the identity (2.15) becomes

$$a^i_{\underset{1}{|}m\underset{2}{|}n} - a^i_{\underset{1}{|}n\underset{2}{|}m} = \underset{1}{\widetilde{A}}{}^i_{pmn}a^p + T^{*i}_{pm}(a^p_{,n} + a^p_{,\dot{s}}\xi^s_{,n}) - T^{*i}_{pn}(a^p_{,m} + a^p_{,\dot{s}}\xi^s_{,m}) + T^{*p}_{mn}a^i_{\underset{1}{|}p}. \tag{2.18}$$

Also, from (2.15) is

$$a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} = \underset{1}{\widetilde{A}}{}^i_{pmn}a^p_j - \underset{2}{\widetilde{A}}{}^p_{jmn}a^i_p + a^i_{j<[mn]>} + a^i_{j\leqslant[mn]\geqslant} + T^{*p}_{mn}a^i_{j\underset{1}{|}p}\,, \tag{2.19}$$

where

$$a^i_{j<mn>} = T^{*i}_{pm}(a^p_{j,n} + a^p_{j,\dot{s}}\xi^s_{,n}) - T^{*p}_{jm}(a^i_{p,n} + a^i_{p,\dot{s}}\xi^s_{,n}), \tag{2.20}$$

$$a^i_{j\leqslant mn\geqslant} = (P^{*i}_{mp}P^{*s}_{jn} - P^{*i}_{pm}P^{*s}_{nj})a^p_s. \tag{2.21}$$

It can be proved by direct calculation that (2.18) and (2.19) are valid. To prove the general identity (2.15) by total induction method, we suppose that (2.15) is valid, and prove that for arbitrary tensor $b^{r_1\dots r_u r_{u+1}}_{t_1\dots t_v t_{v+1}}$ is also valid.

Applying (2.15) to the tensor $a^{r_1\dots r_u}_{t_1\dots t_v} = b^{r_1\dots r_u r_{u+1}}_{t_1\dots t_v t_{v+1}} c^{t_{v+1}}_{r_{u+1}}$ we get

$$\begin{aligned}
&a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m}\\
&= \sum_{\alpha=1}^{u} \underset{1}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha} b^{\cdots}_{\cdots} c^{t_{v+1}}_{r_{u+1}} - \sum_{\beta=1}^{v} \underset{2}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p} b^{\cdots}_{\cdots} c^{t_{v+1}}_{r_{u+1}}\\
&+(b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{<[mn]>} + (b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{\leqslant[mn]\geqslant} + T^{*p}_{mn}(b^{\cdots}_{\cdots\underset{1}{|}p}c^{\cdots}_{\cdots} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots\underset{1}{|}p}) \ .
\end{aligned} \tag{2.22}$$

From (2.17), (2.19) one obtains

$$\begin{aligned}
(b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{< mn >} &= \sum_{\alpha=1}^{u} P^{*r_\alpha}_{[mn]}\binom{p}{r_\alpha}(b^{\cdots}_{\cdots,n}c^{\cdots}_{\cdots} + b^{\cdots}_{\cdots,\dot{s}}c^{\cdots}_{\cdots}\xi^s_{,n} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots,n} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots,\dot{s}}\xi^s_{,n})\\
&- \sum_{\beta=1}^{u} P^{*p}_{[t_\beta n]}\binom{t_\beta}{p}(b^{\cdots}_{\cdots,n}c^{\cdots}_{\cdots} + b^{\cdots}_{\cdots,\dot{s}}c^{\cdots}_{\cdots}\xi^s_{,n} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots,n} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots,\dot{s}}\xi^s_{,n}) \ ,
\end{aligned}$$

$$\begin{aligned}
(b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{\leqslant mn\geqslant} &= c^{\cdots}_{\cdots}\Big[\sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} P^{*r_\alpha}_{[\breve{p}m}P^{*r_\beta}_{\breve{n}s]}\binom{p}{r_\alpha}\binom{s}{t_\beta} b^{\cdots}_{\cdots}\\
&- \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} P^{*r_\alpha}_{[\breve{p}m}P^{*s}_{\breve{n}t_\beta]}\binom{p}{r_\alpha}\binom{t_\beta}{s} b^{\cdots}_{\cdots}\\
&+ \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=1}^{v} P^{*p}_{[t_\beta \breve{m}}P^{*s}_{\breve{n}t_\beta]}\binom{t_\alpha}{p}\binom{t_\beta}{s} b^{\cdots}_{\cdots}\Big] \ .
\end{aligned} \tag{2.23}$$

On the other hand, based on (2.22) and (1.17), (1.18), we have

$$\begin{aligned}
a^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n} - a^{\cdots}_{\cdots\underset{1}{|}n\underset{2}{|}m} &= (b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{\underset{1}{|}m\underset{2}{|}n} - (b^{\cdots}_{\cdots}c^{\cdots}_{\cdots})_{\underset{1}{|}n\underset{2}{|}m}\\
&= (b^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n}c^{\cdots}_{\cdots} + b^{\cdots}_{\cdots}c^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n} + b^{\cdots}_{\cdots\underset{1}{|}m}c^{\cdots}_{\cdots\underset{2}{|}n} + b^{\cdots}_{\cdots\underset{2}{|}m}c^{\cdots}_{\cdots\underset{1}{|}n})_{[mn]}\\
&= (b^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n} - b^{\cdots}_{\cdots\underset{1}{|}n\underset{2}{|}m})c^{\cdots}_{\cdots} + (c^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n} - c^{\cdots}_{\cdots\underset{1}{|}n\underset{2}{|}m})b^{\cdots}_{\cdots}\\
&+ \Big\{[b^{\cdots}_{\cdots,m} + \sum_{\alpha=1}^{u+1} P^{*r_\alpha}_{pm}\binom{p}{r_\alpha}b^{\cdots}_{\cdots} - \sum_{\beta=1}^{v+1} P^{*p}_{t_\beta m}\binom{t_\beta}{p}b^{\cdots}_{\cdots} + b^{\cdots}_{\cdots,\dot{s}}\xi^s_{,m}]\\
&[c^{\cdots}_{\cdots,n} + P^{*t_{v+1}}_{ns}c^s_{r_{u+1}} - P^{*s}_{nr_{u+1}}c^{t_{v+1}}_s + c^{\cdots}_{\cdots,\dot{s}}\xi^s_{,n}]\\
&+ [c^{\cdots}_{\cdots,m} + P^{*t_{v+1}}_{sm}c^s_{r_{u+1}} - P^{*s}_{r_{u+1}m}c^{t_{v+1}}_s + c^{\cdots}_{\cdots,\dot{s}}\xi^s_{,m}]\\
&[b^{\cdots}_{\cdots,n} + \sum_{\alpha=1}^{u+1} P^{*r_\alpha}_{np}\binom{p}{r_\alpha}b^{\cdots}_{\cdots} - \sum_{\beta=1}^{v+1} P^{*p}_{nt_\beta}\binom{t_\beta}{p}b^{\cdots}_{\cdots} + b^{\cdots}_{\cdots,\dot{s}}\xi^s_{,m}]\Big\}_{[mn]}.
\end{aligned} \tag{2.24}$$

Substituting the expression in the second brackets of the right side of previous equation by virtue of (2.21), substituting the expressions (2.23) in (2.22) and then equaling the right sides of equations (2.22), (2.24), after a longer arranging one obtains

$$\begin{aligned}
&c^{t_{v+1}}_{r_{u+1}}\big(b^{r_1\dots r_{u+1}}_{t_1\dots t_{v+1}\underset{1}{|}m\underset{2}{|}n} - b^{r_1\dots r_{u+1}}_{t_1\dots t_{v+1}\underset{1}{|}n\underset{2}{|}m}\big) = c^{t_{v+1}}_{r_{u+1}}\Big\{\sum_{\alpha=1}^{u}\underset{1}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}b^{\cdots}_{\cdots}\\
&+\underset{1}{\widetilde{A}}{}^{r_{u+1}}_{pmn}\binom{p}{r_{u+1}}b^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}\underset{2}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}b^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}\underset{2}{\widetilde{A}}{}^{p}_{t_{v+1}mn}\binom{t_{v+1}}{p}b^{\cdots}_{\cdots}\\
&+\Big[\sum_{\alpha=1}^{u}P^{*r_\alpha}_{[pm]}\binom{p}{r_\alpha}(b^{\cdots}_{\dots,n} + b^{\cdots}_{\dots,\dot{s}}\xi^{s}_{,n}) + P^{*r_{u+1}}_{[pm]}\binom{p}{r_{u+1}}(b^{\cdots}_{\dots,n} + b^{\cdots}_{\dots,\dot{s}}\xi^{s}_{,n})\\
&-\sum_{\beta=1}^{v}P^{*p}_{[t_\beta m]}\binom{t_\beta}{p}(b^{\cdots}_{\dots,n} + b^{\cdots}_{\dots,\dot{s}}\xi^{s}_{,n}) - P^{*p}_{[t_{v+1}]m}\binom{t_{v+1}}{p}(b^{\cdots}_{\dots,n} + b^{\cdots}_{\dots,\dot{s}}\xi^{s}_{,n})\Big]_{[mn]} \qquad (2.25)\\
&+\Big[\sum_{\alpha=1}^{u-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{u}P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*r_\beta}_{\underset{\smile}{n}s]}\binom{p}{r_\alpha}\binom{s}{t_\beta}b^{\cdots}_{\cdots} + \sum_{\alpha=1}^{u}P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*r_{u+1}}_{\underset{\smile}{n}s]}\binom{p}{r_\alpha}\binom{s}{r_{u+1}}b^{\cdots}_{\cdots}\\
&-\sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*s}_{\underset{\smile}{n}t_\beta]}\binom{p}{r_\alpha}\binom{t_\beta}{s}b^{\cdots}_{\cdots} - \sum_{\alpha=1}^{u+1}P^{*r_\alpha}_{[\underset{\smile}{p}m}P^{*s}_{\underset{\smile}{n}t_{v+1}]}\binom{p}{r_\alpha}\binom{t_{v+1}}{s}b^{\cdots}_{\cdots}\\
&-\sum_{\beta=1}^{v}P^{*r_{u+1}}_{[\underset{\smile}{p}m}P^{*s}_{\underset{\smile}{n}t_\beta]}\binom{p}{r_{u+1}}\binom{t_\beta}{s}b^{\cdots}_{\cdots} + \sum_{\alpha=1}^{v-1}\sum_{\substack{\beta=2\\(\alpha<\beta)}}^{v}P^{*p}_{[t_\alpha \underset{\smile}{m}}P^{*s}_{\underset{\smile}{n}t_\beta]}\binom{t_\alpha}{p}\binom{t_\beta}{s}b^{\cdots}_{\cdots}\\
&+\sum_{\alpha=1}^{v}P^{*p}_{[t_\alpha \underset{\smile}{m}}P^{*s}_{\underset{\smile}{n}t_{v+1}]}\binom{t_\alpha}{p}\binom{t_{v+1}}{s}b^{\cdots}_{\cdots}\Big]_{[mn]} + T^{*p}_{mn}b^{\cdots}_{\dots\underset{1}{|}m}\Big\}.
\end{aligned}$$

Since $c^{t_{v+1}}_{r_{u+1}}$ is an arbitrary tensor, the previous equation, based on (2.16), (2.17) becomes

$$\begin{aligned}
b^{r_1\dots r_{u+1}}_{t_1\dots t_{v+1}\underset{1}{|}m\underset{2}{|}n} - b^{r_1\dots r_{u+1}}_{t_1\dots t_{v+1}\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u+1}\underset{1}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}b^{\cdots}_{\cdots} - \sum_{\beta=1}^{v+1}\underset{2}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}b^{\cdots}_{\cdots}\\
&\quad + b^{\cdots}_{\dots<[mn]>} + b^{\cdots}_{\dots\leqslant[mn]\geqslant} + T^{*p}_{mn}b^{\cdots}_{\dots\underset{1}{|}p}\,.
\end{aligned}$$

that is (2.15) is in force for a tensor $b^{r_1\dots r_{u+1}}_{t_1\dots t_{v+1}}$ and the theorem is proved. □

Analogously can be proved the following theorems (2.3)–(2.9).

Theorem 2.3. *Applying two kinds of covariant derivative in an inversed order than in preceding case, we obtain the* 4^{th} *Ricci type identity in* $\mathbb{GF}_N$

$$\begin{aligned}
a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}m\underset{1}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u}\underset{3}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v}\underset{4}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots}\\
&\quad - a^{\cdots}_{\dots<[mn]>} - a^{\cdots}_{\dots\leqslant[mn]\geqslant} - T^{*p}_{mn}a^{\cdots}_{\dots\underset{2}{|}p}, \qquad (2.26)
\end{aligned}$$

where

$$\underset{3}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{pn} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{mj,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}, \tag{2.27}$$

$$\underset{4}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}. \tag{2.28}$$

Theorem 2.4. *In* $\mathbb{GF}_N$ *is in force the* 5^{th} *Ricci type identity*

$$\begin{aligned} a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{1}{|}mn} - a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{2}{|}nm} &= \sum_{\alpha=1}^{u} \underset{5}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{6}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<(mn)>} + a^{\cdots}_{\cdots\leqslant(mn)\geqslant} - P^{*p}_{mn}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}), \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} a^{r_1\ldots r_u}_{t_1\ldots t_v\leqslant mn\geqslant} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} P^{*r_\alpha}_{[\underset{\smile}{pm}}P^{*r_\beta}_{\underset{\smile}{sn}]}\binom{p}{r_\alpha}\binom{s}{t_\beta}a^{\cdots}_{\cdots} \\ &\quad - \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} P^{*r_\alpha}_{[\underset{\smile}{pm}}P^{*s}_{\underset{\smile}{t_\beta n}]}\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ &\quad + \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} P^{*p}_{[\underset{\smile}{t_\alpha m}}P^{*s}_{\underset{\smile}{t_\beta n}]}\binom{t_\alpha}{p}\binom{t_\beta}{s}a^{\cdots}_{\cdots}, \end{aligned} \tag{2.30}$$

$$\underset{5}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{jm,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}, \tag{2.31}$$

$$\underset{6}{\widetilde{A}}{}^{i}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^{s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{s}_{,m}. \tag{2.32}$$

Theorem 2.5. *In* $\mathbb{GF}_N$ *is in force the* 6^{th} *Ricci type identity*

$$\begin{aligned} a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{1}{|}mn} - a^{r_1\ldots r_u}_{t_1\ldots t_v\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u} \underset{7}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{8}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<mn>} + a^{\cdots}_{\cdots\leqslant mn\geqslant}, \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} a^{r_1\ldots r_u}_{t_1\ldots t_v\leqslant mn\geqslant} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u} (P^{*r_\alpha}_{[pm]}P^{*r_\beta}_{sn} + P^{*r_\alpha}_{pn}P^{*r_\beta}_{[sm]})\binom{p}{r_\alpha}\binom{s}{r_\beta}a^{\cdots}_{\cdots} \\ &\quad - \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} (P^{*r_\alpha}_{[pm]}P^{*s}_{t_\beta n} + P^{*r_\alpha}_{pn}P^{*s}_{[t_\beta n]})\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ &\quad + \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=2}^{v} (P^{*p}_{[t_\alpha m]}P^{*s}_{t_\beta n} + P^{*p}_{t_\alpha n}P^{*s}_{[t_\beta n]})\binom{t_\alpha}{p}\binom{t_\beta}{s}a^{\cdots}_{\cdots}, \end{aligned} \tag{2.34}$$

$$\underset{7}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{mp} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \tag{2.35}$$

$$\underset{8}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{mj}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}. \tag{2.36}$$

Theorem 2.6. *In $\mathbb{GF}_N$ the 7^{th} Ricci type identity is valid*

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{9}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{10}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad + a^{\cdots}_{\cdots<nm>} + a^{\cdots}_{\cdots\leqslant nm\geqslant} - (P^{*p}_{mn}a^{\cdots}_{\cdots\underset{1}{|}p} - P^{*p}_{nm}a^{\cdots}_{\cdots\underset{2}{|}p}), \end{aligned} \tag{2.37}$$

where

$$\underset{9}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \tag{2.38}$$

$$\underset{10}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}. \tag{2.39}$$

Theorem 2.7. *In $\mathbb{GF}_N$ the 8^{th} Ricci type identity is valid*

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}n\underset{2}{|}m} &= \sum_{\alpha=1}^{u} \underset{11}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{12}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad - a^{\cdots}_{\cdots<nm>} + a^{\cdots}_{\cdots\leqslant mn\geqslant} + P^{*p}_{mn}a^{\cdots}_{\cdots\underset{1}{|}p} - P^{*p}_{nm}a^{\cdots}_{\cdots\underset{2}{|}p}, \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\leqslant mn\geqslant} &= \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{u-1}\sum_{\beta=2}^{u}(P^{*r_\alpha}_{mp}P^{*r_\beta}_{[ns]} + P^{*r_\alpha}_{[np]}P^{*r_\beta}_{ms})\binom{p}{r_\alpha}\binom{s}{r_\beta}a^{\cdots}_{\cdots} \\ &\quad - \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v}(P^{*r_\alpha}_{mp}P^{*s}_{[nt_\beta]} + P^{*r_\alpha}_{[np]}P^{*s}_{mt_\beta})\binom{p}{r_\alpha}\binom{t_\beta}{s}a^{\cdots}_{\cdots} \\ &\quad + \sum_{\substack{\alpha=1\\(\alpha<\beta)}}^{v-1}\sum_{\beta=1}^{v}(P^{*p}_{mt_\alpha}P^{*s}_{[nt_\beta]} + P^{*p}_{[nt_\alpha]}P^{*s}_{mt_\beta})\binom{t_\alpha}{p}\binom{t_\beta}{s}a^{\cdots}_{\cdots}, \end{aligned} \tag{2.41}$$

$$\underset{11}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{mj,n} - P^{*i}_{jn,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \tag{2.42}$$

$$\underset{12}{\widetilde{A}}{}^i_{jmn} = P^{*i}_{mj,n} - P^{*i}_{jn,m} + P^{*p}_{mj}P^{*i}_{pn} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}. \tag{2.43}$$

Theorem 2.8. *In $\mathbb{GF}_N$ the 9^{th} Ricci type identity is valid*

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}mn} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} &= \sum_{\alpha=1}^{u} \underset{13}{\widetilde{A}}{}^{r_\alpha}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{14}{\widetilde{A}}{}^{p}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ &\quad - a^{\cdots}_{\cdots<mn>} + a^{\cdots}_{\cdots\leqslant nm\geqslant}, \end{aligned} \tag{2.44}$$

where

$$\underset{13}{\widetilde{A}}{}^i{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \tag{2.45}$$

$$\underset{14}{\widetilde{A}}{}^i{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}. \tag{2.46}$$

Theorem 2.9. *In $\mathbb{GF}_N$ the 10^{th} Ricci type identity is valid*

$$\begin{aligned} a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} = \sum_{\alpha=1}^{u} \underset{15}{\widetilde{A}}{}^{r_\alpha}{}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{15}{\widetilde{A}}{}^{p}{}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} \\ - P^{*p}_{nm}(a^{\cdots}_{\cdots\underset{1}{|}p} - a^{\cdots}_{\cdots\underset{2}{|}p}), \end{aligned} \tag{2.47}$$

where

$$\underset{15}{\widetilde{A}}{}^i{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}. \tag{2.48}$$

The identity (2.47) can be written in another form. Namely, counting the difference in the last brackets in (2.47) , one obtains

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} = \sum_{\alpha=1}^{u} \underset{3}{\widetilde{K}}{}^{r_\alpha}{}_{pmn}\binom{p}{r_\alpha}a^{\cdots}_{\cdots} - \sum_{\beta=1}^{v} \underset{3}{\widetilde{K}}{}^{p}{}_{t_\beta mn}\binom{t_\beta}{p}a^{\cdots}_{\cdots} ,$$

where

$$\begin{aligned} &\underset{3}{\widetilde{K}}{}^i{}_{jmn} \\ &= \underset{15}{\widetilde{A}}{}^i{}_{jmn} + P^{*p}_{nm}T^{*i}_{pj} \\ &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{nm}(P^{*i}_{pj} - P^{*i}_{jp}) + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}. \end{aligned} \tag{2.49}$$

Remark 2.1. The magnitudes $\underset{\theta}{\widetilde{K}}$, $\theta = 1,2,3$ are tensors and magnitudes $\underset{t}{\widetilde{A}}$, $t = \overline{1,15}$, have the form and the role of the curvature tensor, but they are not tensors. For example, from (2.33) one obtains

$$a^i_{\underset{1}{|}mn} - a^i_{\underset{1}{|}n\underset{2}{|}m} = \underset{7}{\widetilde{A}}{}^i{}_{pmn}a^p + T^{*i}_{pm}(a^p_{,n} + a^p_{,\dot{s}}\xi^s_{,n}),$$

from where we see that $\underset{7}{\widetilde{A}}$ is not a tensor, because the expression in the bracket is not a tensor. These magnitudes $\underset{t}{\widetilde{A}}$, $t = \overline{1,15}$ we call **curvature pseudotensors** of the $1^{st}, \dots, 15^{th}$ kind in $\mathbb{GF}_N$.

Remark 2.2. For $g_{ij}(x,\dot{x}) = g_{ji}(x,\dot{x})$ we obtain usual Finsler space $\mathbb{F}_N$. If $g_{ij}(x) \neq g_{ji}(x)$ one obtains a generalized Riemannian space $\mathbb{GR}_N$ [2]. For $g_{ij}(x) = g_{ji}(x)$, $\mathbb{GF}_N$ reduces to the Riemannian space $\mathbb{R}_N$.

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