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DERIVATIONAL EQUATIONS OF SUBMANIFOLDS IN AN ASYMMETRIC AFFINE CONNECTION SPACE

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Abstract. In a space L_N of asymmetric affine connection one observes a submanifold, defined in local coordinates. Because of the asymmetry of the connection in the space, the connection of submanifolds is generally asymmetric. Based on this, it follows that 4 kinds of covariant derivatives and 4 kinds of derivational equations are possible. In the present paper is proved that by applying the 3^{rd}, or the 4^{th} kind of covariant derivative, it follows that the induced connection is symmetric (Theorem 1.2.).

In the pseudonormal submanifold are defined 2 connections (2.4) and 4 kinds of covariant derivative. It is proved that by applying the 3^{rd} or the 4^{th} kind of derivative one concludes that the induced connections in this case is unique (Theorem 2.2).

In §3 are examined some properties of coefficients of derivational equations and induced connection in pseudonormal subspace.

1. Introduction

Let L_N be a space of asymmetric affine connection with a torsion tensor (in local coordinates x^i)

$$T^i_{jk} = L^i_{jk} - L^i_{kj}. \tag{1.1}$$

A submanifold $X_M \subset L_N$ is defined by equations

$$x^i = x^i(u^1, \ldots, u^M) = x^i(u^\alpha), \quad i = 1, \ldots, N. \tag{1.2}$$

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The partial derivatives

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad (\mathrm{rank}(B^i_\alpha) = M) \tag{1.3}$$

define tangent vectors on X_M.

Consider $N - M$ contravariant vectors C^i_A, $(A, B, \ldots \in \{M+1, \ldots, N\})$ defined on X_M and linearly independent, and let the matrix $\binom{B^\alpha_i}{C^A_i}$ be inverse for the matrix (B^i_α, C^i_A) provided that the following conditions are satisfied [4]:

$$\begin{aligned} &a)\ B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad b)\ B^i_\alpha C^A_i = 0, \quad c)\ B^\alpha_i C^i_A = 0, \quad d)\ C^i_A C^B_i = \delta^B_A, \\ &e)\ B^i_\alpha B^\alpha_j + C^i_A C^A_j = \delta^i_j. \end{aligned} \tag{1.4}$$

The magnitudes B^i_α, B^α_i are **projection factors (tangent vectors)**, and the magnitudes C^i_A, C^A_i are **afine pseudonormals** [4]-[7] of the submanifold X_M. If a vector v^i is defined in the points of X_M, we present it as linear combination of tangents and pseudonormals

$$v^i = B^i_\alpha v^\alpha + C^i_A v^A, \tag{1.5}$$

wherefrom, composing with B^β_i, and then with C^B_i we obtain

$$a)\ v^\beta = v^i B^\beta_i, \quad b)\ v^B = v^i C^B_i. \tag{1.6}$$

2. Induced connection on $X_M \subset L_N$ and derivational equations

2.1. In order to define an affine connection on X_M, consider a vector $v^i = B^i_\beta v^\beta$ from L_N, tangent to X_M. Its covariant differential along X_M, because of asymmetry of the connection L^i_{jk}, is possible to define in two manners

$$\underset{\substack{1\\2}}{\delta} v^i = dv^i + L^i_{\underset{kj}{jk}} v^j dx^k = d(B^i_\beta v^\beta) + L^i_{\underset{kj}{jk}} (B^i_\beta v^\beta) B^k_\gamma du^\gamma,$$

from where, transvecting with B^α_i, it follows that

$$\underset{\substack{1\\2}}{\delta} v^\alpha = B^\alpha_i \underset{\substack{1\\2}}{\delta} v^i = dv^\alpha + \widetilde{L}^\alpha_{\underset{\gamma\beta}{\beta\gamma}} v^\beta du^\gamma, \tag{2.1}$$

where

$$\widetilde{L}^\alpha_{\beta\gamma} = B^\alpha_i (B^i_{\beta,\gamma} + L^i_{jk} B^j_\beta B^k_\gamma) \tag{2.2}$$

are coefficients of **induced connection for** X_M and

$$B^i_{\beta,\gamma} = \frac{\partial}{\partial u^\gamma} B^i_\beta = \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma}.$$

Because in general it is $\widetilde{L}^\alpha_{\beta\gamma} \neq \widetilde{L}^\alpha_{\gamma\beta}$, the **induced torsion tensor** for X_M is

$$\widetilde{T}^\alpha_{\beta\gamma} = \widetilde{L}^\alpha_{\beta\gamma} - \widetilde{L}^\alpha_{\gamma\beta} = B^\alpha_i B^j_\beta B^k_\gamma \widetilde{T}^i_{jk}. \tag{2.3}$$

By virtue of the connection $\widetilde{L}^\alpha_{\beta\gamma}$, X_M becomes a subspace $L_M \subset L_N$.

2.2. Supposing that the both connections L and $\widetilde{L}$ are asymmetric, one can define four kinds of covariant derivative for a tensor defined at points of X_M. For example:

$$\underset{\substack{1\\2\\3\\4}}{\nabla}_\mu t^{i\alpha}_{j\beta} \equiv t^{i\alpha}_{j\beta\underset{\substack{1\\2\\3\\4}}{|}\mu} = t^{i\alpha}_{j\beta,\mu} + L^i_{\substack{pm\\mp\\pm\\mp}} t^{p\alpha}_{j\beta} B^m_\mu - L^p_{\substack{jm\\mj\\mj\\jm}} t^{i\alpha}_{p\beta} B^m_\mu + \widetilde{L}^\alpha_{\substack{\pi\mu\\\mu\pi\\\pi\mu\\\mu\pi}} t^{i\pi}_{j\beta} - \widetilde{L}^\pi_{\substack{\beta\mu\\\mu\beta\\\mu\beta\\\beta\mu}} t^{i\alpha}_{j\pi}, \tag{2.4}$$

and in this way four connections $\underset{\theta}{\nabla}$, $\theta \in \{1, \dots, 4\}$ are defined on the submanifold $X_M \subset L_N$. The X_M becomes L_M, and obtained structure we shall note $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{1, \dots, 4\})$.

2.3. We shall investigate the presentation of covariant derivatives of the projection factors (tangent vectors), i.e. the objects $B^i_{\alpha\underset{\theta}{|}\mu}$, with the help of B^i_α and C^i_A. In that way we obtain **derivational equations for tangents**.

Putting

$$B^i_{\alpha\underset{\theta}{|}\mu} = \underset{\theta}{\Phi}{}^\pi_{\alpha\mu} B^i_\pi + \underset{\theta}{\Omega}{}^P_{\alpha\mu} C^i_P, \tag{2.5}$$

in relation to (1.4), one gets

$$a)\ \underset{\theta}{\Phi}{}^\pi_{\alpha\mu} = B^\pi_i B^i_{\alpha\underset{\theta}{|}\mu}, \quad b)\ \underset{\theta}{\Omega}{}^P_{\alpha\mu} = C^P_i B^i_{\alpha\underset{\theta}{|}\mu}. \tag{2.6}$$

A) Let us examine, firstly, $\underset{\theta}{\Phi}$. For $\theta = 1$, based on (2.4) and (1.4a), it follows that

$$\begin{aligned} \underset{1}{\Phi}{}^\pi_{\alpha\mu} &= B^\pi_i (B^i_{\alpha,\mu} + L^i_{pm} B^p_\alpha B^m_\mu - \widetilde{L}^\rho_{\alpha\mu} B^i_\rho) \\ &= B^\pi_i (B^i_{\alpha,\mu} + L^i_{pm} B^p_\alpha B^m_\mu) - \widetilde{L}^\pi_{\alpha\mu} \underset{(2.2)}{=} 0 \end{aligned} \tag{2.7}$$

where $\underset{(2.2)}{=}$ designates „= based on (2.2)”. In the same manner

$$\underset{2}{\Phi}{}^\pi_{\alpha\mu} = 0. \tag{2.7$'$}$$

Further we have

$$\underset{3}{\Phi}{}^\pi_{\alpha\mu} \underset{(1.4a)}{=} B^\pi_i (B^i_{\alpha,\mu} + L^i_{pm} B^p_\alpha B^m_\mu) - \widetilde{L}^\pi_{\mu\alpha} \underset{(2.2)}{=} \widetilde{L}^\pi_{\alpha\mu} - \widetilde{L}^\pi_{\mu\alpha} = \widetilde{T}^\pi_{\alpha\mu}, \tag{2.8}$$

and in the same way

$$\underset{4}{\Phi}{}^\pi_{\alpha\mu} = -\widetilde{T}^\pi_{\alpha\mu} = -\underset{3}{\Phi}{}^\pi_{\alpha\mu}. \tag{2.8$'$}$$

Presenting derivatives $B^{\alpha}_{i\underset{\theta}{|}\mu}$ by virtue of B^{π}_i and C^P_i we shall obtain **another form of derivational equations for tangents**, as follows.

Putting

$$B^{\alpha}_{i\underset{\theta}{|}\mu} = \underset{\theta}{\widehat{\Phi}}{}^{\alpha}_{\pi\mu} B^{\pi}_i + \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu} C^P_i, \tag{2.9}$$

we get

$$a)\ \underset{\theta}{\widehat{\Phi}}{}^{\alpha}_{\pi\mu} = B^i_{\pi} B^{\alpha}_{i\underset{\theta}{|}\mu}, \quad b)\ \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu} = C^i_P B^{\alpha}_{i\underset{\theta}{|}\mu}. \tag{2.10}$$

Now, we have to find $\underset{\theta}{\widehat{\Phi}}$ and $\underset{\theta}{\widehat{\Omega}}$ $(\theta = 1, \dots, 4)$.

Starting from $(1.4a)$ and using $(2.6a)$, $(2.10a)$, we have

$$B^i_{\alpha\underset{\frac{1}{2}}{|}\mu} B^{\beta}_i + B^i_{\alpha} B^{\beta}_{i\underset{\frac{1}{2}}{|}\mu} = \delta^{\beta}_{\alpha\underset{\frac{1}{2}}{|}\mu} = 0,$$

i.e.

$$\underset{\frac{1}{2}}{\widehat{\Phi}}{}^{\beta}_{\alpha\mu} = -\underset{\frac{1}{2}}{\Phi}{}^{\beta}_{\alpha\mu} \underset{(2.7,7')}{=} 0. \tag{2.11}$$

Using the 3^{rd} kind of derivative, we obtain

$$B^i_{\alpha\underset{3}{|}\mu} B^{\beta}_i + B^i_{\alpha} B^{\beta}_{i\underset{3}{|}\mu} = \delta^{\beta}_{\alpha\underset{3}{|}\mu} = \delta^{\beta}_{\alpha,\mu} + \widetilde{L}^{\beta}_{\pi\mu}\delta^{\pi}_{\alpha} - \widetilde{L}^{\pi}_{\mu\alpha}\delta^{\beta}_{\pi} = \widetilde{L}^{\beta}_{\alpha\mu} - \widetilde{L}^{\beta}_{\mu\alpha} = \widetilde{T}^{\beta}_{\alpha\mu},$$

i.e., on the base of $(2.6a)$, $(2.10a)$:

$$\underset{3}{\widehat{\Phi}}{}^{\beta}_{\alpha\mu} + \underset{3}{\Phi}{}^{\beta}_{\alpha\mu} = \widetilde{T}^{\beta}_{\alpha\mu} \underset{(2.8)}{\Rightarrow} \underset{3}{\widehat{\Phi}}{}^{\beta}_{\alpha\mu} = 0.$$

By the same manner:

$$\underset{3}{\widehat{\Phi}}{}^{\beta}_{\alpha\mu} = \underset{4}{\widehat{\Phi}}{}^{\beta}_{\alpha\mu} = 0. \tag{2.12}$$

Further, we have

$$\begin{aligned}\underset{3}{\widehat{\Phi}}{}^{\alpha}_{\pi\mu} &\underset{(2.10)}{=} B^i_{\pi}(B^{\alpha}_{i,\mu} + L^p_{mi} B^{\alpha}_p B^m_{\mu}) + \widetilde{L}^{\alpha}_{\pi\mu} \\ &\underset{(2.2)}{=} B^i_{\pi}(B^{\alpha}_{i,\mu} + L^p_{mi} B^{\alpha}_p B^m_{\mu}) + B^{\alpha}_i(B^i_{\pi,\mu} + L^i_{jk} B^j_{\pi} B^k_{\mu}) \\ &= (B^{\alpha}_i B^i_{\pi})_{,\mu} + T^p_{im} B^{\alpha}_p B^i_{\pi} B^m_{\mu} \underset{(1.4a)}{=} \delta^{\alpha}_{\pi,\mu} + \widetilde{T}^{\alpha}_{\pi\mu} = \widetilde{T}^{\alpha}_{\pi\mu},\end{aligned}$$

i.e. wrp to (2.8)

$$\underset{3}{\widehat{\Phi}}{}^{\alpha}_{\pi\mu} = \underset{3}{\Phi}{}^{\alpha}_{\pi\mu} = \widetilde{T}^{\alpha}_{\pi\mu}. \tag{2.13}$$

Now, from $(2.8')$, (2.12), (2.13) it follows that

$$\underset{\theta}{\Phi}{}^{\alpha}_{\pi\mu} = \underset{\theta}{\widehat{\Phi}}{}^{\alpha}_{\pi\mu} = \widetilde{T}^{\alpha}_{\pi\mu} = 0, \quad \theta = 3, 4. \tag{2.14}$$

B) Let us find $\underset{\theta}{\Omega}$, $\underset{\theta}{\widehat{\Omega}}$ in (2.5), respectively (2.9). On the base of (2.6b), (2.4) and (1.4b) it follows that

$$\underset{\substack{1\\2}}{\Omega}{}^P_{\alpha\mu} = \underset{\substack{3\\4}}{\Omega}{}^P_{\alpha\mu} = C^P_i(B^i_{\alpha,\mu} + \underset{mp}{L^i_{pm}} B^p_\alpha B^m_\mu) \tag{2.15}$$

and by virtue of (2.10b), (2.4), (1.4c)

$$\underset{\substack{1\\2}}{\widehat{\Omega}}{}^\alpha_{P\mu} = \underset{\substack{4\\3}}{\widehat{\Omega}}{}^\alpha_{P\mu} = C^i_P(B^\alpha_{i,\mu} - \underset{mi}{L^p_{im}} B^\alpha_p B^m_\mu). \tag{2.16}$$

From exposed, we can state next two theorems.

Theorem 2.1. *Derivational equations for tangents of a submanifold $X_M \subset L_N$ with a structure $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$ are*

$$B^i_{\underset{\theta}{\alpha|\mu}} = \underset{\theta}{\Omega}{}^P_{\alpha\mu} C^i_P, \quad \theta \in \{1,2\}, \tag{2.17}$$

$$B^\alpha_{\underset{\theta}{i|\mu}} = \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu} C^P_i, \quad \theta \in \{1,2\}, \tag{2.17$'$}$$

where $\underset{\theta}{\Omega}$, $\underset{\theta}{\widehat{\Omega}}$ are given in (2.15) *and* (2.16) *respectively, and C^i_P are pseudonormals of subspace L_M of a space L_N with asymmetric affine connection.*

Theorem 2.2. *For $\theta \in \{3,4\}$, that is in the structure $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{3,4\})$, derivational equations for tangents are also of the form* (2.17), (2.17$'$), *but in this case the induced connection is symmetric* ($\widetilde{T} = 0$).

2.4. Supposing that the indices $A, B, \ldots \in \{M+1, \ldots, N\}$ have not a tensor character [2,3], with respect to (2.4) for covariant derivative of the pseudonormal C^i_A in the structure $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{1,2,3,4\})$, we have

$$C^i_{\underset{\substack{1\\2}}{A|\mu}} = C^i_{\underset{\substack{4\\3}}{A|\mu}} = C^i_{A,\mu} + \underset{mp}{L^i_{pm}} C^p_A B^m_\mu. \tag{2.18}$$

Starting from **derivational equations for pseudonormals**

$$C^i_{\underset{\theta}{A|\mu}} = \underset{\theta}{\Lambda}{}^\pi_{A\mu} B^i_\pi + \underset{\theta}{\Psi}{}^P_{A\mu} C^i_P, \quad \theta \in \{1,2\}, \tag{2.19}$$

composing with B^ρ_i and C^Q_i and using (1.4) we obtain

$$C^i_{\underset{\theta}{A|\mu}} B^\rho_i = \underset{\theta}{\Lambda}{}^\pi_{A\mu} \delta^\rho_\pi = \underset{\theta}{\Lambda}{}^\rho_{A\mu},$$

$$C^i_{\underset{\theta}{A|\mu}} C^Q_i = \underset{\theta}{\Psi}{}^P_{A\mu} \delta^Q_P = \underset{\theta}{\Psi}{}^Q_{A\mu},$$

from where, using (2.4):

$$\Lambda_{\frac{1}{2}}{}^{\pi}_{A\mu} = B^{\pi}_{i} C^{i}_{A|\mu \atop \frac{1}{2}} = B^{\pi}_{i}(C^{i}_{A,\mu} + L^{i}_{pm \atop mp} C^{p}_{A} B^{m}_{\mu}), \tag{2.20}$$

$$\Psi_{\frac{1}{2}}{}^{P}_{A\mu} = C^{P}_{i} C^{i}_{A|\mu \atop \frac{1}{2}} = C^{P}_{i}(C^{i}_{A,\mu} + L^{i}_{pm \atop mp} C^{p}_{A} B^{m}_{\mu}). \tag{2.21}$$

On the other hand, differentiating (1.4c), for $\alpha = \pi$ one gets

$$B^{\pi}_{i|\mu \atop \theta} C^{i}_{A} + B^{\pi}_{i} C^{i}_{A|\mu \atop \theta} = 0 \underset{(2.10b),(2.20)}{\Rightarrow} \Lambda_{\theta}{}^{\pi}_{A\mu} = -\widehat{\Omega}_{\theta}{}^{\pi}_{A\mu},$$

and, in place of (2.20), we have

$$\Lambda_{\frac{1}{2}}{}^{\pi}_{A\mu} = -\widehat{\Omega}_{\frac{1}{2}}{}^{\pi}_{A\mu} \underset{(2.16)}{=} -C^{i}_{A}(B^{\pi}_{i,\mu} + L^{p}_{im \atop mi} B^{\pi}_{p} B^{m}_{\mu}). \tag{2.20$'$}$$

It is easy to prove that the values (2.20) and (2.20$'$) are equal. So, we have

Theorem 2.3. *Derivational equations for pseudonormals of a submanifolds $X_M \subset L_N$ in the structure $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$ are*

$$C^{i}_{A|\mu \atop \theta} = -\widehat{\Omega}_{\theta}{}^{\pi}_{A\mu} B^{i}_{\pi} + \Psi_{\theta}{}^{P}_{A\mu} C^{i}_{P}, \quad \theta \in \{1,2\} \tag{2.22}$$

where $\underset{\theta}{\widehat{\Omega}}$ are given in (2.16)*, and $\underset{\theta}{\Psi}$ in* (2.21)*.*

If we explain $C^{A}_{i|\mu \atop \theta}$ with help of tangents and pseudonormals we shall obtain **the** 2^{nd} **form of derivational equations for pseudonormals**. Accordingly:

$$C^{A}_{i|\mu \atop \frac{1}{2}} = C^{A}_{i|\mu \atop \frac{4}{3}} = C^{A}_{i,\mu} - L^{p}_{im \atop mi} C^{A}_{p} B^{m}_{\mu}. \tag{2.23}$$

Now we can write

$$C^{A}_{i|\mu \atop \theta} = \widehat{\Omega}_{\theta}{}^{A}_{\pi\mu} B^{\pi}_{i} + \widehat{\Psi}_{\theta}{}^{A}_{P\mu} C^{P}_{i}, \quad \theta \in \{1,2\}, \tag{2.24}$$

from where

$$\widehat{\Lambda}_{\frac{1}{2}}{}^{A}_{\pi\mu} = B^{i}_{\pi} C^{A}_{i|\mu \atop \frac{1}{2}} = B^{i}_{\pi}(C^{A}_{i,\mu} - L^{p}_{im \atop mi} C^{A}_{p} B^{m}_{\mu}), \tag{2.25}$$

$$\widehat{\Psi}_{\frac{1}{2}}{}^{A}_{P\mu} = C^{i}_{P} C^{A}_{i|\mu \atop \frac{1}{2}} = C^{i}_{P}(C^{A}_{i,\mu} - L^{p}_{im \atop mi} C^{A}_{p} B^{m}_{\mu}). \tag{2.26}$$

Starting of (1.4b) for $\alpha = \pi$ one obtains

$$\widehat{\Lambda}_{\theta}{}^{A}_{\pi\mu} = -\Omega_{\theta}{}^{A}_{\pi\mu}, \quad \theta = 1, 2, \tag{2.27}$$

and using $(1.4d)$:

$$\underset{\theta}{\widehat{\Psi}}{}^{A}_{P\mu} = -\underset{\theta}{\Psi}{}^{A}_{P\mu}, \tag{2.28}$$

and (2.24) becomes

$$C^{A}_{i\underset{\theta}{|}\mu} = -\underset{\theta}{\Omega}{}^{A}_{\pi\mu} B^{\pi}_{i} - \underset{\theta}{\Psi}{}^{A}_{P\mu} C^{P}_{i}, \quad \theta \in \{1,2\}. \tag{2.29}$$

So, we have proved:

Theorem 2.4. *Derivational equations for pseudonormals C^A_i of submanifold $X_M \subset L_N$ in the structure $(X_M \subset L_N, \widetilde{L}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$ are given by* (2.29)*, where $\underset{\theta}{\Omega}$ is given wrp to* (2.6*b*)*, and $\underset{\theta}{\Psi}$ wrp to* (2.21)*.*

3. Induced connection on X^N_{N-M}

3.1. Denote by X^N_{N-M} submanifold of L_N containing pseudonormals of X_M, and whose vectors are linear combinations of the pseudonormals. Let v^i be such a vector, defined at points of X_M. Then we have [4]-[6]:

$$v^i = C^i_Q v^Q. \tag{3.1}$$

Consider absolute differential δv^i along X_M, that can be defined in two manners

$$\underset{\substack{1\\2}}{\delta} v^i = dv^i + L^i_{\substack{jk\\kj}} v^j dx^k \underset{(3.1)}{=} d(C^i_Q v^Q) + L^i_{\substack{jk\\kj}} C^j_Q v^Q B^k_\mu du^\mu,$$

i.e.

$$\underset{\substack{1\\2}}{\delta} v^i = C^i_Q dv^Q + (C^i_{Q,\mu} + L^i_{\substack{jk\\kj}} C^j_Q B^k_\mu) v^Q du^\mu. \tag{3.2}$$

Let us consider the projection of $\underset{\theta}{\delta} v^i$ on X^N_{N-M}. Transvecting the equation (3.2) with C^A_i, one obtains

$$\underset{\substack{1\\2}}{\delta} v^A = C^A_i \underset{\substack{1\\2}}{\delta} v^i \underset{(1.4d)}{=} \delta^A_Q dv^Q + C^A_i (C^i_{Q,\mu} + L^i_{\substack{jk\\kj}} C^j_Q B^k_\mu) v^Q du^\mu,$$

i.e.

$$\underset{\substack{1\\2}}{\delta} v^A = dv^A + \underset{\substack{1\\2}}{\overline{L}}{}^A_{Q\mu} v^Q du^\mu, \tag{3.3}$$

where

$$\underset{\substack{1\\2}}{\overline{L}}{}^A_{Q\mu} = C^A_i (C^i_{Q,\mu} + L^i_{\substack{jk\\kj}} C^j_Q B^k_\mu) \tag{3.4}$$

are coefficients of **induced connections in pseudonormal subspace**.

For a tensor on X_M, whose some indices are related to L_N, and other ones to L^N_{N-M}, one can consider 4 kinds of covariant derivative. For example [1], [4],

$$\overline{\nabla}_{\substack{1\\2\\3\\4}}{}_\mu t^{iA}_{jB} = t^{iA}_{jB\underset{\substack{1\\2\\3\\4}}{\perp}\mu} = t^{iA}_{jB,\mu} + L^i_{\substack{pm\\mp\\pm\\mp}} t^{pA}_{jB} B^m_\mu - L^p_{\substack{jm\\mj\\mj\\jm}} t^{iA}_{pB} B^m_\mu + \overline{L}^A_{\substack{1\\2\\1\\2}}{}_{P\mu} t^{iP}_{jB} - \overline{L}^P_{\substack{1\\2\\2\\1}}{}_{B\mu} t^{iA}_{jP}. \tag{3.5}$$

In this manner 4 connections $\underset{\theta}{\overline{\nabla}}$, $\theta \in \{1, \dots, 4\}$ on the submanifold $X^N_{N-M} \subset L_N$ are defined. We note the obtained structures $(X^N_{N-M} \subset L_N, \overline{L}, \underset{\theta}{\overline{\nabla}}, \theta \in \{1, \dots, 4\})$.

3.2. Further, consider a presentation of the objects $C^i_{A\underset{\theta}{\perp}\mu}$ by virtue of B^i_α and C^i_A in the mentioned structure. Putting

$$C^i_{A\underset{\theta}{\perp}\mu} = \underset{\theta}{\overline{\Lambda}}{}^\pi_{A\mu} B^i_\pi + \underset{\theta}{\overline{\Psi}}{}^P_{A\mu} C^i_P, \quad \theta \in \{1, 2, 3, 4\}, \tag{3.6}$$

we obtain

$$a)\ C^i_{A\underset{\theta}{\perp}\mu} B^\pi_i = \underset{\theta}{\overline{\Lambda}}{}^\pi_{A\mu}, \quad b)\ C^i_{A\underset{\theta}{\perp}\mu} C^P_i = \underset{\theta}{\overline{\Psi}}{}^P_{A\mu}. \tag{3.7}$$

A) For $\underset{\theta}{\overline{\Lambda}}$, wrp of (3.7$a$), (1.4$c$) one obtains the same values as for $\underset{\theta}{\Lambda}$ in (2.20).

The other form of derivational equations for normals in the cited structure one gets by presenting of derivative $C^A_{i\underset{\theta}{\perp}\mu}$ by virtue of B^π_i and C^P_i:

$$C^A_{i\underset{\theta}{\perp}\mu} = \underset{\theta}{\widehat{\Lambda}}{}^A_{\pi\mu} B^\pi_i + \underset{\theta}{\widehat{\Psi}}{}^A_{P\mu} C^P_i, \quad \theta \in \{1, 2, 3, 4\}. \tag{3.8}$$

Therefore, we obtain

$$a)\ \underset{\theta}{\widehat{\Lambda}}{}^A_{\pi\mu} = B^i_\pi C^A_{i\underset{\theta}{\perp}\mu}, \quad b)\ \underset{\theta}{\widehat{\Psi}}{}^A_{P\mu} = C^i_P C^A_{i\underset{\theta}{\perp}\mu}. \tag{3.9}$$

From (3.9a), (3.5) is

$$\underset{\substack{1\\2}}{\widehat{\Lambda}}{}^A_{\pi\mu} = \underset{\substack{4\\3}}{\widehat{\Lambda}}{}^A_{\pi\mu} = B^i_\pi (C^A_{i,\mu} - L^p_{\substack{im\\mi}} C^A_p B^m_\mu). \tag{3.10}$$

On the other hand, differentiating the equation $B^i_\pi C^A_i = 0$ and using (3.9a), (2.6b), it follows that

$$\underset{\theta}{\widehat{\Lambda}}{}^A_{\pi\mu} = -\underset{\theta}{\Omega}{}^A_{\pi\mu}, \tag{3.11}$$

where $\underset{\theta}{\Omega}$ are given in (2.6b).

B) Examine, further, the coefficients $\underset{\theta}{\overline{\Psi}}$, $\underset{\theta}{\widehat{\Psi}}$ in (3.6) and (3.8). In relation to (3.7b), (3.5), (1.4d), (3.4), we get

$$\underset{1}{\overline{\Psi}}{}^P_{A\mu} = \underset{2}{\overline{\Psi}}{}^P_{A\mu} = 0, \tag{3.12}$$

$$\overline{\underset{3}{\Psi}}{}^{P}_{A\mu} = -\overline{\underset{4}{\Psi}}{}^{P}_{A\mu} = T^{i}_{pm} C^{P}_{i} C^{p}_{A} B^{m}_{\mu}, \tag{3.13}$$

and from (3.9b), (3.5)

$$\widehat{\underset{1}{\Psi}}{}^{A}_{P\mu} = \widehat{\underset{2}{\Psi}}{}^{A}_{P\mu} = 0, \tag{3.14}$$

$$\widehat{\underset{3}{\Psi}}{}^{A}_{P\mu} = -\widehat{\underset{4}{\Psi}}{}^{A}_{P\mu} = T^{p}_{im} C^{i}_{P} C^{A}_{p} B^{m}_{\mu} = \underset{3}{\Psi}{}^{A}_{P\mu}. \tag{3.15}$$

On the other hand, differentiating the equation $C^{i}_{P} C^{A}_{i} = \delta^{A}_{P}$ one obtains

$$\widehat{\underset{\theta}{\Psi}}{}^{A}_{P\mu} = -\overline{\underset{\theta}{\Psi}}{}^{A}_{P\mu}. \tag{3.16}$$

Comparing the equation (3.12)-(3.16) it follows that

$$\widehat{\underset{\theta}{\Psi}}{}^{P}_{A\mu} = \overline{\underset{\theta}{\Psi}}{}^{P}_{A\mu} = 0, \quad \theta \in \{1,2,3,4\}. \tag{3.17}$$

Further based on (3.4), (3.13), (3.17), we have

$$\overline{\underset{1}{L}}{}^{A}_{Q\mu} - \overline{\underset{2}{L}}{}^{A}_{Q\mu} = T^{i}_{jk} C^{A}_{i} C^{j}_{Q} B^{k}_{\mu} = \overline{\underset{3}{\Psi}}{}^{A}_{Q\mu} = 0, \tag{3.18}$$

i.e.

$$\overline{\underset{1}{L}}{}^{A}_{Q\mu} = \overline{\underset{2}{L}}{}^{A}_{Q\mu} \equiv \overline{L}^{A}_{Q\mu}. \tag{3.18$'$}$$

Thus, we state theorems:

Theorem 3.1. *Derivational equations for pseudonormals of submanifold $X_M \subset L_N$ considered in a structure $(X^{N}_{N-M}, \overline{L}, \underset{\theta}{\nabla}, \theta \in \{1,2\})$ are*

$$C^{i}_{A \underset{\theta}{\perp} \mu} = -\underset{\theta}{\widehat{\Omega}}{}^{\pi}_{A\mu} B^{i}_{\pi}, \tag{3.19}$$

$$C^{A}_{i \underset{\theta}{\perp} \mu} = -\underset{\theta}{\Omega}{}^{A}_{\pi\mu} B^{\pi}_{i}, \quad \theta \in \{1,2,3,4\}. \tag{3.19$'$}$$

where the coefficients $\underset{\theta}{\widehat{\Omega}}$, $\underset{\theta}{\Omega}$ are given at (2.16), (2.15), and B^{i}_{π}, B^{π}_{i} are projection factors (tangent vectors) of the submanifold $X_M \subset L_N$.

Theorem 3.2. *In the structure $(X^{N}_{N-M}, \overline{L}, \underset{\theta}{\nabla}, \theta \in \{3,4\})$ derivational equations for pseudonormals of submanifold $X_M \subset L_N$ are of the form (3.19), (3.19$'$) and then there exists an unique connection (3.18$'$) $(\overline{\underset{1}{L}} = \overline{\underset{2}{L}} = \overline{L})$ in the submanifold X_M.*

4. Some properties of derivational equations coefficients and connection coefficients

4.1. With respect to exposed above, we see that in derivational equations there appear the quantities $\underset{\theta}{\Omega}{}^P_{\alpha\mu}, \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu}, \underset{\theta}{\Psi}{}^P_{A\mu}$, and we shall investigate their properties.

Noting the symmetrization and antisymmetrization wrp to indices j, k, with $\underline{jk}$, and $\underset{\vee}{jk}$, respectively, we have

$$L^i_{\underline{jk}} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}(L^i_{jk} - L^i_{kj}) = \frac{1}{2}T^i_{jk}, \quad L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}, \tag{4.1}$$

and analogously in other cases. By virtue of (2.15) is:

$$\underset{1}{\Omega}{}^P_{\alpha\mu} = \underset{3}{\Omega}{}^P_{\alpha\mu} = C^P_i(B^i_{\alpha,\mu} + L^i_{\underline{pm}}B^p_\alpha B^m_\mu) + L^i_{\underset{\vee}{pm}}C^P_i B^p_\alpha B^m_\mu = \Omega^P_{\alpha\mu} + \underset{1}{\Omega}{}^P_{\underset{\vee}{\alpha\mu}} \tag{4.2}$$

where

$$\Omega^P_{\alpha\mu} = \underset{1}{\Omega}{}^P_{\underline{\alpha\mu}}. \tag{4.2'}$$

In the same manner, we conclude

$$\underset{2}{\Omega}{}^P_{\alpha\mu} = \underset{4}{\Omega}{}^P_{\alpha\mu} = \Omega^P_{\alpha\mu} - \underset{1}{\Omega}{}^P_{\underset{\vee}{\alpha\mu}}.$$

So, we have

$$\underset{\theta}{\Omega}{}^P_{\alpha\mu} = \Omega^P_{\alpha\mu} + (-1)^{\theta+1}\underset{1}{\Omega}{}^P_{\underset{\vee}{\alpha\mu}}, \quad \theta \in \{1,2,3,4\}. \tag{4.3}$$

Also, using (2.16), we prove

$$\begin{aligned} a)\ \underset{1}{\widehat{\Omega}}{}^\alpha_{P\mu} &= \underset{4}{\widehat{\Omega}}{}^\alpha_{P\mu} = \widehat{\Omega}^\alpha_{P\mu} - L^p_{\underset{\vee}{im}}C^i_P B^\alpha_p B^m_\mu, \\ b)\ \underset{2}{\widehat{\Omega}}{}^\alpha_{P\mu} &= \underset{3}{\widehat{\Omega}}{}^\alpha_{P\mu} = \widehat{\Omega}^\alpha_{P\mu} + L^p_{\underset{\vee}{im}}C^i_P B^\alpha_p B^m_\mu. \end{aligned} \tag{4.4}$$

Substituting at (2.21) L^i_{pm} by virtue of (4.1), one gets

$$\underset{\theta}{\Psi}{}^P_{A\mu} = \Psi^P_{A\mu} + (-1)^{\theta+1}L^i_{\underset{\vee}{pm}}C^P_i C^p_A B^m_\mu, \quad \theta \in \{1,2\}, \tag{4.5}$$

where $\Psi^P_{A\mu}$ is obtained from (2.21) substituting L^i_{pm} with $L^i_{\underline{pm}}$.

On the base of exposed, we have:

Theorem 4.1. *For the coefficients $\underset{\theta}{\Omega}, \underset{\theta}{\widehat{\Omega}}, \underset{\theta}{\Psi}$, at derivational equations of submanifold of a space L_N with asymmetric affine connection are valid the equations* (4.3)-(4.5), *where $\Omega, \widehat{\Omega}, \Psi$, are values of $\underset{1}{\Omega}, \underset{1}{\widehat{\Omega}}, \underset{1}{\Psi}$, respectively, substituting at* (2.15), (2.16), (2.21) *connection coefficients L^i_{jk} with the symmetric part $L^i_{\underline{jk}}$.*

4.2. Consider, at the end, some properties of coefficients $\overline{L}_1, \overline{L}_2$, of the induced connection in pseudonormal submanifold X^N_{N-M} of L_N. First of all, Kronecker symbols being constants, we have

$$\delta^A_{B\underset{\theta}{\perp}\mu} = \delta_{AB\underset{\theta}{\perp}\mu} = \delta^{AB}_{\underset{\theta}{\perp}\mu} = 0, \quad \theta = \{1,2,3,4\}. \tag{4.6}$$

On the other hand, from (3.5) one gets

$$\delta^{AB}_{\underset{1}{\perp}\mu} = 0 + \underset{1}{\overline{L}}{}^A_{P\mu}\delta^{PB} + \underset{1}{\overline{L}}{}^B_{P\mu}\delta^{AP} = \underset{1}{\overline{L}}{}^A_{B\mu} + \underset{1}{\overline{L}}{}^B_{A\mu} \underset{(4.6)}{=} 0.$$

The analog equation is valid for $\overline{L}_2$, and we conclude

$$\underset{\omega}{\overline{L}}{}^A_{B\mu} = -\underset{\omega}{\overline{L}}{}^B_{A\mu}, \quad \omega \in \{1,2\}, \text{ for } \underset{\theta}{\overline{\nabla}}, \quad \theta \in \{1,2\}. \tag{4.7}$$

Further, we have

$$\delta^A_{B\underset{\substack{3\\4}}{\perp}\mu} \underset{(3.5)}{=} \underset{\substack{1\\2}}{\overline{L}}{}^A_{P\mu}\delta^P_B - \underset{\substack{2\\1}}{\overline{L}}{}^P_{B\mu}\delta^A_P \underset{(4.6)}{=} 0,$$

i.e.

$$\underset{1}{\overline{L}}{}^A_{B\mu} = \underset{2}{\overline{L}}{}^A_{B\mu} \text{ for } \underset{\theta}{\overline{\nabla}}, \quad \theta \in \{3,4\}. \tag{4.8}$$

From (4.7), (4.8) it follows that applying $\underset{1}{\overline{\nabla}}$ and $\underset{3}{\overline{\nabla}}$ or $\underset{1}{\overline{\nabla}}$ and $\underset{4}{\overline{\nabla}}$ or $\underset{2}{\overline{\nabla}}$ and $\underset{3}{\overline{\nabla}}$ or $\underset{2}{\overline{\nabla}}$ and $\underset{4}{\overline{\nabla}}$ one obtains

$$\underset{1}{\overline{L}}{}^A_{B\mu} = -\underset{2}{\overline{L}}{}^A_{B\mu}. \tag{4.9}$$

So, we stated:

Theorem 4.2. *The coefficients* $\overline{L}_1, \overline{L}_2$ (3.4) *of induced connection in the pseudonormal submanifold* X^N_{N-M} *of* L_N *possess the properties:*

(a) *the property* (4.7) *in the structures* $(X^N_{N-M}, \underset{\substack{1\\2}}{\overline{L}}, \underset{\theta}{\overline{\nabla}}, \theta \in \{1,2\})$

(b) *the property* (4.8) *in the structures* $(X^N_{N-M}, \underset{\substack{1\\2}}{\overline{L}}, \underset{\theta}{\overline{\nabla}}, \theta \in \{3,4\})$

(c) *the property* (4.9) *in the structures*

$$\Big(X^N_{N-M}, \underset{\substack{1\\2}}{\overline{L}}, \underset{\theta}{\overline{\nabla}}, \underset{\omega}{\overline{\nabla}}, (\theta,\omega) \in \{(1,2),(1,4),(2,3),(2,4)\}\Big).$$

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