Kragujevac Journal of Mathematics
Volume 35 Number 1 (2011), Pages 191–197.

DEGREE EQUITABLE DOMINATION ON GRAPHS

VENKATASUBRAMANIAN SWAMINATHAN [1] AND KUPPUSAMY MARKANDAN DHARMALINGAM [2]

Abstract. A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$, where $\deg(u)$ denotes the degree of vertex u and $\deg(v)$ denotes the degree of vertex v. The minimum cardinality of such a dominating set is denoted by γ^e and is called the equitable domination number of G. This Paper aims at the study of a new concept called degree equitable domination introduced by Prof. E. Sampathkumar. Minimal equitable dominating sets are characterized. The complexity of the new parameter namely equitable domination number is determined.

1. Introduction

New concepts of domination arise from practical considerations. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Equitability among citizens in terms of wealth, health, status etc is the goal of a democratic nation. In order to study this practical concept, a graph model is to be created. Prof. E. Sampathkumar is the first person to recognize the spirit and power of this concept and introduced various types of equitability in graphs like degree equitability, outward equitability, inward equitability, equitability in terms of number of equal degree neighbours, or in terms of number of strong degree neighbours etc. In general, if $G = (V, E)$ is a simple graph and $\phi : V(G) \rightarrow N$ is a function, we may define

Key words and phrases. Equitable Domination Number, Minimal Equitable Dominating set, Equitable isolate, Equitable independent set.
2010 *Mathematics Subject Classification.* Primary: 05C.
Received: September 15, 2010.
Revised: March 11, 2011.

equitability of vertices in terms of ϕ - values of the vertices. A wheel graph of order n is denoted by W_n. In this graph, one vertex lines at the centre of a circle (wheel) and $n-1$ vertical lies on the circumference. In this paper, a study is made of equitability defined by degree function.

2. Equitable Domination Number

Definition 2.1. A subset D of V is called an equitable dominating set if for every $v \in V-D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ^e and is called the equitable domination number of G.

In the following proposition the equitable domination number of some standard graphs are determined.

Proposition 2.1.

(i) *For the complete graph K_n on n vertices, $\gamma^e(K_n) = 1$.*

(ii) *For the paths P_n and the cycles C_n on n vertices, $\gamma^e(P_n) = \gamma^e(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

(iii) *If W_n denotes the wheel on n vertices, then*

$$\gamma^e(W_n) = \begin{cases} 1, & \text{if } n = 4, 5; \\ \left\lceil \frac{n-1}{3} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

(iv) *For the complete bipartite graph $K_{m,n}$, we have*

$$\gamma^e(K_{m,n}) = \begin{cases} 2, & \text{if } |m-n| \leq 1; \\ m+n, & \text{if } |m-n| \geq 2, \text{ where } m, n \geq 2. \end{cases}$$

Proof. (i) For the complete graph K_n, any single vertex forms an equitable dominating set of K_n. Hence $\gamma^e(K_n) = 1$.

(ii) Since the degree of any vertex of P_n is either 1 or 2 , any dominating set in P_n is clearly equitable. Hence $\gamma^e(P_n) = \gamma(P_n)$. But $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Therefore $\gamma^e(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Since C_n is regular, $\gamma^e(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

(iii) Let W_n be a wheel with $n-1$ vertices on the cycle and a single vertex at the centre. Let $V(W_n) = \{u, v_1, \ldots, v_{n-1}\}$ where u is the centre and v_i $(1 \leq i \leq n-1)$ is on the cycle. $\deg_{W_n}(v_i) = 3$ for all i, $1 \leq i \leq n-1$ and $\deg_{W_n}(u) = n-1$. Clearly $n \geq 4$.

Case 1: $n \leq 5$

Then $\deg_{W_n}(u) = n-1 \leq 4$. Since u is adjacent with v_i for all i, $1 \leq i \leq n-1$, $\{u\}$ is a dominating set of W_n. Also $\deg_{W_n}(u)$ is 3 if $n = 4$ and 4 if $n = 5$. In both cases, $\deg_{W_n}(v_i) = 3$, $1 \leq i \leq n-1$. Therefore $|\deg(u) - \deg(v_i)| \leq 1$, for all i, $1 \leq i \leq n-1$. Therefore $\{u\}$ is an equitable dominating set of W_n.

Case 2: $n \geq 6$

In this case $\deg_{W_n} u \geq 5$. While $\deg_{W_n} v_i = 3$, for all i, $1 \leq i \leq n-1$. Though $\{u\}$ is a dominating set it is not equitable. Let

$$D = \begin{cases} \{u, v_1, v_4, \ldots, v_{3k-2}\}, & \text{if } n-1 = 3k; \\ \{u, v_1, v_4, \ldots, v_{3k-2}, v_{3k-1}\}, & \text{if } n-1 = 3k+1; \\ \{u, v_1, v_4, \ldots, v_{3k-2}, v_{3k+1}\}, & \text{if } n-1 = 3k+2. \end{cases}$$

Then for any $v_i \in V - D$, there exists v_{i-1} or $v_{i+1} \in D$ such that $v_{i-1}v_i$ or $v_{i+1}v_i \in E(G)$ and $\deg(v_i) = \deg(v_{i-1}) = 3$ or $\deg(v_i) = \deg(v_{i+1}) = 3$. Therefore D is an equitable dominating set of W_n. Now,

$$|D| = \begin{cases} k+1, & \text{if } n = 3k; \\ k+2, & \text{if } n = 3k+1 \text{ or } 3k+2. \end{cases}$$

Also, when $n-1 = 3k$, $\left\lceil \frac{n-1}{3} \right\rceil = k$, when $n-1 = 3k+1$ or $3k+2$, $\left\lceil \frac{n-1}{3} \right\rceil = k+1$. Hence $|D| = \left\lceil \frac{n-1}{3} \right\rceil + 1$. If D' is an equitable dominating set of W_n then $u \in D'$ and $D' - \{u\}$ is an equitable dominating set of $C_n - 1$. Therefore $\left|D'\right| \geq 1 + \left\lceil \frac{n-1}{3} \right\rceil$. Therefore $\gamma^e(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.

(iv) Let $K_{m,n}$ be the complete bipartite graph with m vertices in one partition say V_1 and n vertices in another partition say V_2. Then

$$\deg_{K_{m,n}}(u) = \begin{cases} n, & \text{if } u \in V_1; \\ m, & \text{if } u \in V_2. \end{cases}$$

If $|m-n| \leq 1$, then any vertex say u from V_1 and any vertex say v from V_2 constitute a dominating set which is equitable. Therefore $\gamma^e(K_{m,n}) \leq 2$ if $|m-n| \leq 1$. Since $m, n \geq 2$, $K_{m,n}$ has no vertex which is adjacent with every other vertex. Therefore $\gamma^e(K_{m,n}) \geq 2$. Also $\gamma^e(K_{m,n}) = 2$ if $|m-n| \leq 1$. Suppose $|m-n| \geq 2$. Let D be a minimum equitable dominating set of $K_{m,n}$. Suppose $|D| < m+n$. Then there exists $u \in V - D$. Let $u \in V_1$. (Similar proof if $u \in V_2$. Therefore $\deg_{K_{m,n}}(u) = n$. Since D is an equitable dominating set of G, there exists $v \in D$ such that v is adjacent with u and $\left|\deg_{K_{m,n}}(v) - \deg_{K_{m,n}}(u)\right| \leq 1$ since $u \in V_1$ and since V_1 is independent, we get that $v \in V_2$. Therefore $\deg_{K_{m,n}}(v) = m$. Therefore

$\left|\deg_{K_{m,n}}(v) - \deg_{K_{m,n}}(u)\right| = |m - n| \geq 2$, which is a contradiction. Therefore $|D| = m + n$. Therefore $\gamma^e(K_{m,n}) = m + n$ if $|m - n| \geq 2$. □

Theorem 2.1. *If G is regular or $(k, k+1)$ bi-regular, for some k, then $\gamma^e(G) = \gamma(G)$.*

Proof. Suppose G is a regular graph. Then every vertex of G has the same degree say k. Let D be a minimum dominating set of G. Then $|D| = \gamma(G)$. Let $u \in V - D$. Then as D is a dominating set, there exists $v \in D$ and $uv \in E(G)$. Also $\deg(u) = \deg(v) = k$. Therefore $|\deg(u) - \deg(v)| = 0 \leq 1$. Therefore D is an equitable dominating set of G so that $\gamma^e(G) \leq |D| = \gamma(G)$. But $\gamma(G) \leq \gamma^e(G)$. Therefore $\gamma(G) = \gamma^e(G)$.

Now, suppose G is a bi-regular graph. Then every vertex of G has degree either k or $k + 1$ where, k is a positive integer. Let D be a minimum dominating set of G. Then $|D| = \gamma(G)$. Let $u \in V - D$. Then as D is a dominating set, there exists $v \in D$ such that $uv \in E(G)$. Also $\deg(u) = k$ or $k + 1$ and $\deg(v) = k$ or $k + 1$. Therefore $|\deg(u) - \deg(v)| = 1$. Therefore D is a equitable dominating set of G. Therefore $\gamma^e(G) \leq |D| = \gamma(G)$. But $\gamma(G) \leq \gamma^e(G)$. Therefore $\gamma(G) = \gamma^e(G)$. □

Remark 2.1. Since any equitable dominating set is also a dominating set, $\gamma(G) \leq \gamma^e(G)$ for any graph G.

3. Minimal Equitable Dominating Sets

Definition 3.1. A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$ if $|\deg(u) - \deg(v)| \leq 1$.

Remark 3.1. If D is an equitable dominating set then any super set of D is an equitable dominating set.

Definition 3.2. An equitable dominating set D is said to be a minimal equitable dominating set if no proper subset of D is an equitable dominating set.

Definition 3.3. A minimal equitable dominating set of maximum cardinality is called a Γ^e-set and its cardinality is denoted by Γ^e.

Definition 3.4. An equitable dominating set is said to be 1 - minimal if $D - v$ is not an equitable dominating set for all $v \in D$.

Remark 3.2. If a vertex $u \in V$ be such that $|\deg(u) - \deg(v)| \geq 2$ for all $v \in N(u)$ then u is in every equitable dominating set. Such points are called **equitable isolates**. Let I_e denote the set of all equitable isolates. Vacuously isolated points are equitable isolated points. Hence $I_s \subseteq I_e \subseteq D$ for every equitable dominating set D where I_s is the set of all isolated points of G.

Remark 3.3. An equitable dominating set D is minimal if and only if it is 1 - minimal.

Theorem 3.1. *An equitable dominating set D is minimal if and only if for every vertex $u \in D$ one of the following holds.*

(i) *Either $N(u) \cap D = \phi$ or $|\deg(v) - \deg(u)| \geq 2$ for all $N(u) \cap D$.*

(ii) *There exists a vertex $v \in V - D$ such that $N(v) \cap D = \{u\}$ and $|\deg(v) - \deg(u)| \leq 1$ (That is u has a private equitable neighbour in $V - D$).*

Proof. Assume that D is a minimal equitable dominating set. Suppose (i) and (ii) do not hold. Then for some $u \in D$ there exists a $v \in N(u) \cap D$ such that $|\deg(v) - \deg(u)| \leq 1$ and for every $v \in V - D$, either $N(v) \cap D \neq \{u\}$ or $|\deg(v) - \deg(u)| \geq 2$ or both. Therefore $D - \{u\}$ is an equitable dominating set, contradiction to the minimality of D. Therefore (i) or (ii) holds.

Conversely, suppose for every $u \in D$, one of the statements (i) or (ii) holds. Suppose D is not minimal. Then there exists $u \in D$ such that $D - \{u\}$ is an equitable dominating set. Therefore there exists a $v \in D - \{u\}$ such that v equitably dominates u. That is $v \in N(u)$ and $|\deg(v) - \deg(u)| \leq 1$. Therefore u does not satisfy (i). Then u must satisfy (ii). Then there exists a $v \in V - D$ such that $N(v) \cap D = \{u\}$ and $|\deg(v) - \deg(u)| \leq 1$. Since $D - \{u\}$ is an equitable dominating set, there exists $w \in D - \{u\}$ such that w is adjacent to v and w is degree equitable with v. Therefore $w \in N(v) \cap D$, $|\deg(w) - \deg(v)| \leq 1$ and $w \neq u$, a contradiction to $N(v) \cap D = \{u\}$. Therefore D is a minimal equitable dominating set. ☐

In the following theorem, graphs with unique minimal equitable dominating sets are characterized.

Theorem 3.2. *A graph G has a unique minimal equitable dominating set if and only if the set of all equitable isolates forms an equitable dominating set.*

Proof. Sufficient condition is obvious. Let G have a unique minimal equitable dominating set D. Let $S = \{u \in V/u \text{ is an equitable isolate }\}$. Then $S \subseteq D$. We now prove that $S = D$. Suppose $D - S \neq \phi$. Let $v \in D - S$. Since v is not an equitable isolate, $V - \{v\}$ is an equitable dominating set. Hence there exists a minimal equitable dominating set $D_1 \subseteq V - \{v\}$ and $D_1 \neq D$ a contradiction to the fact that G has a unique minimal equitable dominating set. Therefore $S = D$. ☐

4. Equitable Independent Sets

Definition 4.1. Let $u \in V$. The equitable neighbourhood of u denoted by $N^e(u)$ is defined as $N^e(u) = \{v \in V/v \in N(u), |\deg(u) - \deg(v)| \leq 1\}$ and $u \in I_e \Leftrightarrow N^e(u) = \phi$.

The cardinality of $N^e(u)$ is denoted by $\deg_G^e(u)$.

Definition 4.2. The maximum and minimum equitable degree of a point in G are denoted respectively by $\Delta^e(G)$ and $\delta^e(G)$. That is $\Delta^e(G) = \max\limits_{u \in V(G)} |N^e(u)|$, $\delta^e(G) = \min\limits_{u \in V(G)} |N^e(u)|$.

Definition 4.3. A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N^e(u)$ for all $v \in S - \{u\}$.

Example 4.1.

(i) Any set S of cardinality 1 is an equitable independent set.
(ii) Every independent set is an equitable independent set.

Theorem 4.1. *Let S be a maximal equitable independent set. Then S is a minimal equitable dominating set.*

Proof. Let S be a maximal equitable independent set. Let $u \in V - S$. If $u \notin N^e(v)$ for every $v \in S$, then $S \cup \{u\}$ is an equitable independent set, a contradiction to the maximality of S. Therefore $u \in N^e(v)$ for some $v \in S$. Therefore S is an equitable dominating set. Since for any $u \in S$, $u \notin N^e(v)$ for every $v \in S - \{u\}$, either $N(u) \cap S = \phi$ or $|\deg(v) - \deg(u)| \geq 2$ for all $v \in N(u) \cap S$. Therefore S is a minimal equitable dominating set. □

Remark 4.1. Equitably independent, equitable dominating sets exist.

Definition 4.4. The maximum cardinality of an equitable independent set is denoted by β^e. The minimum cardinality of an equitably independent, equitable dominating set is denoted by i^e.

Remark 4.2. $\gamma^e \leq i^e \leq \beta^e \leq \Gamma^e$.

Acknowledgement: The authors are grateful to the referee for his valuable suggestions. This research is supported by the Government of India through a UGC Minor Research Project (project No 3130) sanctioned to the second author.

References

[1] R. B. Allan and R. C. Laskar, *On domination and independent domination numbers of a graph*, Discrete Math. **23** (1978), 73–78.

[2] D. W. Bange, A. E. Barkankas, L. H. Host and P. J. Slater, *Generalised domination and domination in graphs*, Discrete Math. **159** (1996), 1–11.

[3] C. Berge, *Theory of graphs and its applications*, Dunod, Paris, (1958).

[4] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, (1973).

[5] C. Berge and P. Duchet, *Recent problem and results about kernels in directed graphs*, Discrete Math. **86** (1990), 27–31.

[6] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, *Domination Numbers for the Queen's graph*, Bull. Inst. Comb. Appl. **10** (1994), 73–82.

[7] E. J. Cockayne, S. T. Hedetniemi, *Towards a theory of domination in Graphs*, Networks **7** (1977), 247–261.

[8] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total domination in Graphs*, **10** (1980), 211–219.

[9] E. J. Cockayne, C. W. Ko and F. B. Shepherd, *Inequalities Concerning dominating sets in graphs*, Technical Report DM-370-IR, Dept. Math., Univ. Victoria, (1985).

[1] Ramanujan Research Center in Mathematics,
Saraswathi Narayanan College,
Madurai-625 022, India.
E-mail address: sulanesri@yahoo.com

[2] Department of Mathematics,
The Madura College,
Madurai-625 011, India
E-mail address: kmdharma6902@yahoo.in