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NEW RESULTS ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are established for the asymptotic stability of certain third order nonlinear differential equations. Our results improve on Tunc's [C. Tunc, *On the asymptotic behavior of solutions of certain third-order nonlinear differential equations*, J. Appl. Math. Stoch. Anal. 2005: **1** (2005) 29–35].

1. Introduction

We consider the third-order nonlinear ordinary differential equation

$$x''' + \psi(x, x', x'')x'' + f(x, x') = p(t, x, x', x'') \tag{1.1}$$

or its equivalent system

$$x' = y, y' = z, z' = -\psi(x, y, z)z - f(x, y) + p(t, x, y, z) \tag{1.2}$$

where

$$\psi \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \ f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \text{ and } p \in C([0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \tag{1.3}$$

We shall require that $f(0, 0) = 0$, the derivatives $\dfrac{\partial\psi(x, y, z)}{\partial x} \equiv \psi_x(x, y, z)$, $\dfrac{\partial\psi(x, y, z)}{\partial z} \equiv \psi_z(x, y, z)$, $\dfrac{\partial f(x, y)}{\partial x} \equiv f_x(x, y)$ and $\dfrac{\partial f(x, y)}{\partial y} \equiv f_y(x, y)$ exist and are continuous, and the uniqueness of the solutions of (1.1) will be assumed.

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Stability and boundedness are very important problems in the theory and applications of differential equations, and an effective method for studying these problems is the second method of Lyapunov (see [1]-[11]).

Recently, Tunc [9] discussed the boundedness and asymptotic behavior of solutions of Eq. (1.1) and the following results were proved:

Theorem 1.1 (Tunc [9]). *Further to the basic assumptions on the function ψ, f and p, suppose the following:*

(i) $\int_0^x f(u,0)du > 0$ *for* $x \neq 0$,

(ii) $\lim_{|x|\to\infty} \sup \int_0^x f(u,0)du = \infty$,

(iii) $\int_0^y f(0,v)dv \geq 0$,

(iv) *the function p satisfies $|p(t,x,y,z)| \leq |e(t)|$ uniformly in t, where $e(t)$ is a continuous function of t such that* $\int_0^\infty |e(t)|dt < \infty$,

(v) *there is a positive constant $\mathcal{B}$ such that*

$$\psi(x,y,z) \geq \mathcal{B},$$

(vi) $\mathcal{B}\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)vdv\right] y \geq y \int_0^y f_x(x,v)dv$,

(vii) $4\mathcal{B}\int_0^x f(u,0)du \left\{\int_0^y [f(x,v) - f(x,0)]\, dv + \mathcal{B}\int_0^y [\psi(x,v,0) - \mathcal{B}]\, vdv\right\}$
$\geq y^2 f^2(x,0)$ *for all* $xy \neq 0$,

(viii) $y\psi_z(x,y,z) \geq 0$.

Then for any solution $(x(t), y(t), z(t))$ of system (1.2), there are positive constants c_1, c_2 and c_3 such that

$$|x(t)| \leq c_1, \quad |y(t)| \leq c_2, \quad |z(t)| \leq c_3, \quad t \geq 0.$$

Theorem 1.2 (Tunc [9]). *Suppose the following:*

(i) *there is a positive constant $\mathcal{B}$ such that the assumptions (iv)-(viii) of Theorem 1.1 hold,*

(ii) $xf(x,0) > 0$ *for* $x \neq 0$,

(iii) $\lim_{|x|\to\infty} \sup \int_0^x f(u,0)du = \infty$,

(iv) $\int_0^y f(0,v)dv \geq 0$,

(v) $\mathcal{B}\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)vdv\right] y + \psi(x,y,z) \geq y\int_0^y f_x(x,v)dv + \mathcal{B}$ *for* $y \neq 0$.

Then, every solution $(x(t), y(t), z(t))$ *of system* (1.2) *satisfies*

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} z(t) = 0.$$

Theoretically, these are interesting results since (1.1) is a rather general third-order nonlinear differential equation. For example, many third-order differential equations that have been discussed in [8] are special cases of Eq. (1.1), and some known results can be obtained using these theorems. However, it is not easy to apply Theorems 1.1 and 1.2 to these special cases to obtain new or better results since Theorems 1.1 and 1.2 have some hypotheses which are not necessary for the stability and boundedness of many nonlinear equations.

Our aim in this paper is to further study the stability and boundedness of Eq. (1.1). In the next section, we will state our results and also obtain sufficient conditions for every solution of Eq. (1.1) to be bounded by using Lyapunov's direct method. Thereafter, we will establish criteria for every solutions of Eq. (1.1) to converge to zero by employing the method introduced by Yoshizawa [11]. Finally, we will discuss the asymptotic behavior of solutions of some interesting specific cases of Eq. (1.1).

In the following discussion, we always assume (1.3) holds without further delay.

2. Main results

Our main results in this section are the following theorems.

Theorem 2.1. *Let* $\delta_0 > 0$, $a > 0$, $b > 0$, $c > 0$ *be constants such that* $ab > c$. *Assume that*

(1) $\dfrac{f(x,0)}{x} \geq \delta_0 > 0$, $x \neq 0$,

(2) $f'(x,0) \leq c$,

(3) $f_y(x,\theta y) \geq b$ *for* $0 \leq \theta \leq 1$,

(4) $\psi(x,y,z) > a$,

(5) $y\psi_z(x,y,\theta z) \geq 0$ *for* $0 \leq \theta \leq 1$,

(6) $a\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,y,0)vdv\right] y \geq y\int_0^y f_x(x,v)dv$,

(7) *the function* p *satisfies* $|p(t,x,y,z)| \leq |e(t)|$ *uniformly in* t, *where* $e(t)$ *is a continuous function of* t *such that* $\int_0^\infty |e(t)|dt < \infty$.

Then for any solution $(x(t), y(t), z(t))$ *of system* (1.2), *there are positive constants* c_1, c_2 *and* c_3 *such that*

$$|x(t)| \leq c_1, \qquad |y(t)| \leq c_2, \qquad |z(t)| \leq c_3, \ \ \text{for} \ \ t \geq 0. \tag{2.1}$$

Theorem 2.2. *Let* $\delta_0 > 0$, $a > 0, b > 0, c > 0$ *be constants such that* $ab > c$. *Assume that*

(1) $\dfrac{f(x,0)}{x} \geq \delta_0 > 0$, $x \neq 0$,

(2) $f'(x,0) \leq c$,

(3) $f_y(x, \theta y) \geq b$ *for* $0 \leq \theta \leq 1$,

(4) $\psi(x,y,z) > a$,

(5) $y\psi_z(x,y,\theta z) \geq 0$ *for* $0 \leq \theta \leq 1$,

(6) $a\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)vdv\right] y \geq y \int_0^y f_x(x,v)dv$,

(7) *the function* p *satisfies* $|p(t,x,y,z)| \leq |e(t)|$ *uniformly in* t, *where* $e(t)$ *is a continuous function of* t *such that* $\int_0^\infty |e(t)|dt < \infty$. *Then every solution* $(x(t), y(t), z(t))$ *of system* (1.2) *satisfies*

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} z(t) = 0. \tag{2.2}$$

Proof of Theorem 2.1. Clearly, Eq. (1.1) is equivalent to the system (1.2), and it suffices to show that every solution of (1.2) is bounded. To this end, consider the function

$$V(t,x,y,z) = e^{-P(t)}U(x,y,z) \tag{2.3}$$

where

$$\begin{aligned} U(x,y,z) = & \int_0^x f(u,0)du + \int_0^y \psi(x,v,0)vdv + a^{-1}\int_0^y f(x,v)dv + \frac{1}{2}a^{-1}z^2 \\ & + yz + 2a^{-1} \end{aligned}$$

and

$$P(t) = \int_0^t |e(t)|ds.$$

We claim that V is a positive function. To show this, it suffices to show that U is positive. Now, rewrite U above thus:

$$\begin{aligned} U(x,y,z) \ &= \ \frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \int_0^y [\psi(x,v,0) - a]vdv \\ & + \ a^{-1}\int_0^y [f_v(x,\theta v) - b]\, vdv + \int_0^x \left[1 - \frac{a^{-1}}{b}f'(u,0)\right] f(u,0)du + 2a^{-1} \end{aligned}$$

where $f_v(x,\theta v) = v^{-1}\{f(x,v) - f(x,0)\}$, $v \neq 0$.
On using hypotheses (1)-(5) of Theorem 2.1,

$$U(x,y,z) \geq \frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \frac{\delta_1}{2}x^2. \tag{2.4}$$

Combining (2.3) and (2.4) yields,

$$V(t,x,y,z) \geq \frac{1}{2}e^{-P(t)}\left\{a^{-1}(ay+z)^2 + \frac{a^{-1}}{b}(f(x,0)+by)^2 + \delta_1 x^2\right\}.$$

Thus, there exists a constant $K > 0$ small enough that

$$V(t,x,y,z) \geq K(x^2+y^2+z^2). \tag{2.5}$$

Hence $V(t,x,y,z)$ is a positive function.

Next, we show that the derivative of $V(t,x,y,z)$ with respect to t along the solution path of (1.2) satisfies

$$V'_{(1.2)} \equiv \left.\frac{d}{dt}V(t,x(t),y(t),z(t))\right|_{(1.2)} \leq -D_1 \tag{2.6}$$

provided that $x^2+y^2+z^2 \geq D_2$. D_1, D_2 are some positive constants.

$$V'_{(1.2)} = e^{-P(t)}\{-|e(t)|U + U'_{(1.2)}\},$$

where $U = U(x(t),y(t),z(t))$ and $U'_{(1.2)} \equiv \left.\frac{d}{dt}U(x(t),y(t),z(t))\right|_{(1.2)}$.

Then,

$$\begin{aligned}
V'_{(1.2)} =& e^{-P(t)}\left\{-|e(t)|\left[\frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \int_0^y(\psi(x,v,0)-a)v\,dv\right.\right.\\
&+ a^{-1}\int_0^y(f_v(x,\theta v)-b)v\,dv + \left.\int_0^x\left(1-\frac{a^{-1}}{b}f'(u,0)\right)f(u,0)du + 2a^{-1}\right]\\
&+ a^{-1}y\int_0^y f_x(x,v)dv - \psi_z(x,y,\theta z)yz^2 - (a^{-1}\psi(x,y,z)-1)z^2\\
&- \left.\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv\right]y + a^{-1}(z+ay)p(t,x,y,z)\right\},
\end{aligned}$$

where

$$\psi_z(x,y,\theta z) = \frac{\psi(x,y,z)-\psi(x,y,0)}{z}, \quad 0 \leq \theta \leq 1.$$

Clearly,
if $|z+ay| < 2$, then $(z+ay)p(t,x,y,z) \leq 2|p(t,x,y,z)| \leq 2|e(t)|$;
if $|z+ay| \geq 2$, then $(z+ay)p(t,x,y,z) \leq \frac{1}{2}(z+ay)^2|p(t,x,y,z)| \leq \frac{1}{2}(z+ay)^2|e(t)|$.
Hence for any t, x and y

$(z+ay)p(t,x,y,z) \leq (2+\frac{1}{2}(z+ay)^2)|p(t,x,y,z)| \leq (2+\frac{1}{2}(z+ay)^2)|e(t)|$
and so

$$
\begin{aligned}
V'_{(1.2)} \leq e^{-P(t)} \Big\{ & -|e(t)| \left[\frac{a^{-1}}{2b}(f(x,0)+by)^2 + \int_0^y (\psi(x,v,0)-a)v dv \right. \\
& \left. + a^{-1}\int_0^y [f_v(x,\theta v) - b]v dv + \int_0^x \left(1 - \frac{a^{-1}}{b} f'(u,0)\right) f(u,0) du \right] \\
& - \psi_z(x,y,\theta z) y z^2 - (a^{-1}\psi(x,y,z)-1)z^2 + a^{-1}y\int_0^y f_x(x,v)dv \\
& - \left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v dv \right] y \Big\}.
\end{aligned} \tag{2.7}
$$

Then, by noting (1)-(6), we can find an $\eta > 0$, small enough so that

$$V'_{(1.2)} \leq -e^{-P(t)} \left\{ \eta z^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \frac{1}{2}\delta_1 x^2 \right\} |e(t)|$$

where $\delta_1 = \delta_0 \left(1 - \frac{c}{ab}\right)$.
Thus, there exists a constant $D_3 > 0$ small enough such that

$$V'_{(1.2)} \leq -D_3(x^2+y^2+z^2).$$

Hence
$V'_{(1.2)} \leq -D_4$, provided $x^2+y^2+z^2 \geq D_4 D_3^{-1}$; and this completes the verification of (2.6).
Finally, we show that all solutions of (1.2) are bounded. Following [2], assume that $(x(t), y(t), z(t))$ is a solution of (1.2).
Then, there is evidently a $t_0 \geq 0$ such that

$$x^2(t_0) + y^2(t_0) + z^2(t_0) < D_2,$$

where D_2 is the constant defined earlier; for otherwise, that is, if

$$x^2(t) + y^2(t) + z^2(t) \geq D_2, \qquad t \geq 0,$$

then, by (2.6),

$$V'_{(1.2)}(t) \leq -D_1 < 0, \qquad t \geq 0,$$

and this in turn implies that $V(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, which contradicts (2.5). Hence to prove (2.1) it will suffice to show that if

$$x^2(t) + y^2(t) + z^2(t) < D_5 \quad \text{for} \quad t = T, \tag{2.8}$$

where $D_5 \geq D_2$ is a finite constant, then there is a constant $D_6 > 0$ depending on a, b, c, δ_0 and D_5 such that

$$x^2(t) + y^2(t) + z^2(t) \leq D_6 \quad \text{for} \quad t \geq T. \tag{2.9}$$

Our proof of (2.9) is based essentially on an extension of an argument in the proof of [[10], Lemma 1]. For any given constant $d > 0$ let $S(d)$ denote the surface: $x^2 + y^2 + z^2 = d$. Because V is continuous in t, x, y, z and tend to $+\infty$ as $x^2 + y^2 + z^2 \longrightarrow \infty$, there is evidently a constant $D_7 > 0$ depending on D_5 as well as on a, b, c, δ_0 such that

$$\min_{(x,y,z)\in S(D_7)} V(t, x, y, z) > \max_{(x,y,z)\in S(D_5)} V(t, x, y, z). \tag{2.10}$$

It is easy to see from (2.8) and (2.10) that

$$x^2(t) + y^2(t) + z^2(t) < D_7, \qquad t \geq T. \tag{2.11}$$

For suppose on the contrary that there is a $t > T$ such that

$$x^2(t) + y^2(t) + z^2(t) \geq D_7.$$

Then, by (2.8) and by the continuity of the quantities $x(t), y(t), z(t)$ in the argument displayed, there exists $t_1, t_2, T < t_1 < t_2$ such that

$$x^2(t_1) + y^2(t_1) + z^2(t_1) = D_5 \tag{2.12}$$

$$x^2(t_2) + y^2(t_2) + z^2(t_2) = D_7 \tag{2.13}$$

and such that

$$D_5 \leq x^2(t) + y^2(t) + z^2(t) \leq D_7, \qquad t_1 \leq t \leq t_2. \tag{2.14}$$

But writing $V(t) \equiv V(t, x(t), y(t), z(t))$, since $D_5 \geq D_2$, (2.14) obviously implies [in view of (2.7)] that

$$V(t_2) < V(t_1)$$

and this contradicts the conclusion (from (2.10),(2.12) and (2.13));

$$V(t_2) > V(t_1).$$

Hence (2.11) holds. This completes the proof of (2.1), and the theorem now follows. $\square$

Remark 2.1. Clearly, Theorem 2.1 is an improvement and extension of Theorem 1.1. In particular, from Theorem 2.1 we see that (iii) and (vii) assumed in Theorem 1.1 are not necessary, and (i), (ii) can be replaced by (1), (2) of Theorem 2.1 for the boundedness of solutions of Eq.(1.1).

Example 2.1. Consider Eq. (1.1) with

$$\psi(x,y,z) = \ln(1+x^2) + e^{yz} + 2, f(x,y) = x + \frac{x}{1+x^2}(1+y^2) + y + \frac{1}{3}y^3$$

and $p(t) = \dfrac{\sin t}{1+t^2}$.
It is easy to check that the hypotheses (1)-(5) in Theorem 2.1 are satisfied. Since $\psi(x,y,z) \geq 2$ and

$$2\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)vdv\right] y$$

$$= 2\left[\frac{x}{1+x^2}y^2 + y + \frac{1}{3}y^3 - \frac{x}{1+x^2}y^2\right] y$$

$$= 2\left(y^2 + \frac{1}{3}y^4\right) \geq y^2 + \frac{1-x^2}{(1+x^2)^2}\left(y^2 + \frac{1}{3}y^4\right) = y\int_0^y f_x(x,v)dv,$$

we see that (6) of Theorem 2.1 hold also. Hence all the hypotheses in Theorem 2.1 are satisfied, and so for every solution $x(t)$ of Eq.(1.1) there is a constant $D > 0$ such that

$$|x(t)| < D, \quad |x'(t)| < D, \quad |x''(t)| < D \text{ for } t \geq 0.$$

The following lemma is important for the proof of our next theorem.

Lemma 2.1. *Let Q be an open set in $\mathbb{R}^n$ and $I = [0,\infty)$. Consider the differential system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x}) + \mathbf{G}(t,\mathbf{x}), \tag{2.15}$$

where $\mathbf{H}$ is continuous on $Q, \mathbf{G}$ is continuous on $I \times Q$, and for any continuous and bounded function $\mathbf{x}(t)$ on $t_0 \leq t < \infty$,

$$\int_0^\infty \|\mathbf{G}(s,\mathbf{x}(s))\| ds < \infty.$$

Assume that all the solutions of (2.15) are bounded, and that there exists a non-negative continuous function $V(t,\mathbf{x})$ which satisfies locally a Lipschitz condition with respect to $\mathbf{x}$ in Q such that $V'(t,\mathbf{x}) \leq -W(\mathbf{x})$, where $W(\mathbf{x})$ is positive definite with *respect to a closed set* Ω *in* Q. *Then all the solutions of* (2.15) *approach the largest semi-invariant set contained in* Ω *of the equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x})$$

on Q.

The proof of Lemma 2.1 is found in Yoshizawa [11].

Proof of Theorem 2.2. Consider the system (1.2) and let V be defined by (2.3). Then by noting

$$\frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \int_0^y [\psi(x,v,0)-a]vdv$$
$$+a^{-1}\int_0^y [f_v(x,\theta v)-b]vdv + \int_0^x [1-\frac{a^{-1}}{b}f'(x,0)]f(u,0)du \geq 0,$$

it follows from (2.7) that

$$\left.\frac{dV}{dt}\right|_{(1.2)} \leq -e^{-P(\infty)}\left\{y\left[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v,0)vdv - a^{-1}\int_0^y f_x(x,v)dv\right]\right.$$
$$\left. + [a^{-1}\psi(x,y,z)-1]z^2\right\}.$$

Set

$$\begin{aligned} W(x,y,z) &= e^{-P(\infty)}\left\{y\left[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v,0)vdv - a^{-1}\int_0^y f_x(x,v)dv\right]\right. \\ &+ \left.[a^{-1}\psi(x,y,z)-1]z^2\right\}. \end{aligned}$$

By noting assumptions of the theorem, we see that $W(x,y,z) \geq 0$.
Now, consider the set

$$\Omega = \{(x,y,z) : W(x,y,z) = 0\}.$$

Because the function W is continuous, the set Ω is closed and W is positive definite with respect to Ω. Now, consider the system

$$x' = y, \qquad y' = z, \qquad z' = -\psi(x,y,z)z - f(x,y). \tag{2.16}$$

The asymptotic behavior of solutions of (2.16) has been discussed in [7]. With the same hypotheses we have here, it has been shown in the proof of the main theorem in [7] that $(0,0,0)$ is the largest semi-invariant set of (2.16) contained in Ω. In addition, since all the hypotheses of Theorem 2.2 are satisfied, we know that every solution of (1.2) is bounded. Now, let

$$\mathbf{x} = (x,y,z)^T, \qquad \mathbf{H}(\mathbf{x}) = (y, z, -f(x,y) - \psi(x,y,z)z)^T$$

and

$$G(t, \mathbf{x}) = (0, 0, p(t, x, y, z))^T.$$

Obviously, the system (1.2) is in the form (2.15). Then, from the above discussion, it is easy to check that all the hypotheses in Lemma 2.1 are satisfied. Hence by Lemma 2.1, every solution of (1.2) tends to the largest semi-invariant set contained in Ω of (2.16) on Q, that is $(0, 0, 0)$. This completes the proof. $\square$

Remark 2.2. Clearly, Theorem 2.2 is an improvement and extension of Theorem 1.2. In particular, from Theorem 2.2, we see that hypotheses (iii) and (vii) of Theorem 1.2 are not necessary, and also hypotheses (i), (ii) and (v) in Theorem 1.2 can be replaced by (1), (2) and (6) of Theorem 2.2 for the asymptotic stability of the trivial solutions of Eq. (1.1).

Example 2.2. Consider Eq. (1.1) with

$$\psi(x, y, z) = y \sin x + y^2 + e^{2yz} + 2, f(x, y) = y^3 + y + x + \frac{x}{1 + x^2}$$

and $p(t) = \dfrac{\sin t}{1 + t^2}$.

It is easy to check that the hypotheses (1)-(5) in Theorem 2.2 are satisfied. Since $\psi(x, y, z) > 1$ and

$$\begin{aligned} &\left[f(x, y) - f(x, 0) - \int_0^y \psi_x(x, v, 0) v dv\right] y \\ &= \left[y^3 + y - \frac{1}{3}(\cos x) y^3\right] y > y^2 \left(1 + \frac{1 - x^2}{(1 + x^2)^2}\right) = y \int_0^y f_x(x, v) dv, \end{aligned}$$

we see that (6) of Theorem 2.2 hold also. Hence all the hypotheses in Theorem 2.2 are satisfied, and so every solution $x(t)$ of Eq. (1.1) satisfies (2.2).

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