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AMPLITUDES OF LINEAR OSCILLATIONS DETERMINED BY LINEAR HOMOGENOUS DIFFERENTIAL EQUATION OF THE SECOND ORDER AND LIOUVILLE-BESGE FORMULAE

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Abstract. In this paper we present some quadraturic aspects of solving the equation by means of quadratures. It has a special comparative value when estimating Sturm's zeros, "periods" and variable amplitudes in cases which are solvable by means other than quadratures (iterations).

1. Introduction and Preliminaries

It is well known how hard it is to solve the equation of ordinary (nonharmonic) oscillations of the second order $y''(x) + a(x)y(x) = 0$, $a(x) > 0$ by means of quadratures. Long time ago, Besge and Liouville anticipated that some classes of the equation could be solved by means of quadratures, but it was not until L. M. Berkovich [3], who did it. Problems on zeros of the solution (Sturm's theorems), the problem of distance between successive zeros (being a replacement for nonexistent periods), as well as the problems of amplitudes of solutions, are still not solved in a satisfactory way (for details see [4], [5], [9], [10], [12], [13] and [14]).

It is shown in our paper [7] for the equation of linear oscillations

$$y''(x) + a(x)\,y(x) = 0,\;\; a(x) > 0 \tag{1.1}$$

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that two linearly independent particular integrals could transform, by means of iterations and for every positive and continuous coefficient $a(x)$, into a series of integrals of the coefficient

$$\left.\begin{aligned} y_1 &= \cos_{a(x)} x = 1 - \int\!\!\int a(x)\,dx^2 + \int\!\!\int a(x) \int\!\!\int a(x)\,dx^4 \dots \\ y_2 &= \sin_{a(x)} x = x - \int\!\!\int xa(x)\,dx^2 + \int\!\!\int a(x) \int\!\!\int xa(x)\,dx^4 \dots \end{aligned}\right\}. \tag{1.2}$$

It is obvious that if $a = const > 0$ the equations (1.2) transform to harmonic oscillations. If the theorem on average value of integrals is applied on the integrals in (1.2)

$$\int_0^x a(x)\,dx = a(\xi_1)\,x, \quad \int_0^x\!\!\int_0^x a(x)\,dx^2 = a(\xi_1)\,a(\xi_2)\,\frac{x^2}{2!},$$
$$\int_0^x\!\!\int_0^x xa(x)\,dx^2 = a(\xi_1)\,a(\xi_2)\,\frac{x^3}{3!}, \dots$$

the equations (1.2) yield potential series, wherefrom the Sturm's theorems are obtained in their simplest form: that the locations of zeros are in the solutions of the equations

$$\left.\begin{aligned} &\text{zeros of } y_1 \text{ in the solutions of } x\sqrt{a(x)} = (2k-1)\,\pi/2,\ k = 1, 2, 3, \dots \\ &\text{zeros of } y_2 \text{ in the solutions of } x\sqrt{a(x)} = k\pi,\ k = 0, 1, 2, 3, \dots \end{aligned}\right\}. \tag{1.3}$$

The other Sturm's theorems could be subsequently derived.

The other valuable result of the iterations and averaging is the consequence of (1.2) and (1.3), that is the solutions of (1.2) could be presented in the approximate form

$$y_1 = \cos_{a(x)} x \approx \cos\left(x\sqrt{a(x)}\right),\ y_2 = \sin_{a(x)} x \approx \frac{1}{\sqrt{a(x)}} \sin\left(x\sqrt{a(x)}\right) \tag{1.4}$$

on the right sides there are ordinary sine and cosine, but of a complex function. This could have a broader importance in the treatment of the oscillations (1.1). From the results (1.1) - (1.4), there are the following conclusions:

I. *The Sturm's functions* $\sin_{a(x)} x$ *and* $\cos_{a(x)} x$ *could, in a number of cases, be expressed by means of ordinary* $\sin g(x)$ *and* $\cos g(x)$, *where* $g(x)$ *is a complex function, continuous if* $a(x)$ *is continuous.*

II. *Something known in the Physics from long time ago: the fundamental amplitudes of the solution depend on the overall cause of the oscillations, represented by means of the coefficient* $a(x) > 0$.

2. The amplitude problem

Some 95% of oscillations in the engineering and practice are harmonic oscillations, that is the solutions of the equation

$$y''(x) + \alpha^2 y(x) = 0,\ \alpha = Const.,\ y = C_1 \cos \alpha x + C_2 \sin \alpha x.$$

When manipulating those oscillations the constant α (defining period/frequency) is incorrectly sorted into integration constants C_1 or C_2, so in a bit injudicious way the amplitude is intermingled with integration constants. Nonetheless, it is well known from Physics that the amplitude fundamentally depends on causes of the oscillations (actuators), concentrated in the above α, only secondarily depending on initial conditions, expressed through the integration constants.

Therefore, the only correct way, taking into consideration the above mentioned and the equations (1.1)- (1.4), is to look for the oscillatory solution of the linear homogeneous differential equation of the second order in the form

$$y_1 = F(g(x)) \cos g(x),\ y_2 = F(g(x)) \sin g(x) \tag{2.1}$$

Moreover, the fundamental amplitude $A = F(g(x))$ may not be the same for the both solutions y_1 and y_2, as shown in (1.4). The problem seems unsolvable, as two novel functions $g(x)$ and $F(g(x))$ are introduced, whereas there is only one function $a(x)$ in (1.1). However, it comes out this is not true.

3. Quadratural aspects

There is the experience in the Physics that if the quadrature method could be successfully applied when solving accompanying differential equation, then it means that it is a fundamental and important physical law.

In his book, L. M. Berkovich [3] refers to the classical case of Liouville-Besge equation

$$\upsilon'' + \frac{k}{(ax^2 + bx + c)^2}\upsilon = 0 \tag{3.1}$$

which could be solved by means of quadratures. He gives a generalization, still far away from the type (2.1) efforts. From (2.1), there are the derivatives

$$\begin{aligned} y_1' &= g'(x)\left[F'(g)\cos g - F(g)\sin g\right] \text{ and} \\ y_2' &= g'(x)\left[F'(g)\sin g + F(g)\cos g\right], \end{aligned}$$

the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix} = -g'(x) F^2(g) \neq 0$$

and the second derivatives

$$\begin{aligned} y_1'' &= g''(F'(g)\cos g - F(g)\sin g) + g'^2(F''(g)\cos g - 2F'(g)\sin g - F(g)\cos g) \\ y_2'' &= g''(F'(g)\sin g + F(g)\cos g) + g'^2(F''(g)\sin g + 2F'(g)\cos g - F(g)\sin g). \end{aligned}$$

After a long calculation, astonishingly simply, there are the coefficients $A(x)$ and $B(x)$ of the linear homogeneous differential equation of the second order obtained. The solutions of the equations are just (2.1).

$$W(y_1, y_2) y''(x) + A(x) y'(x) + B(x) y(x) = 0$$

where

$$\begin{aligned} A(x) &= F(g)\left(g''(x) F(g) + 2F'(g) g'^2(x)\right), \\ B(x) &= g'^3(x)\left(F(g) F''(g) - 2F'^2(g) - F^2(g)\right). \end{aligned}$$

There is a relatively simple linear homogeneous differential equation of the second order with solutions of the type (2.1) depending on two arbitrary functions $g(x)$ and $F(g(x))$

$$\begin{aligned} y''(x) &- \frac{g''(x) F(g) + 2F'(g) g'^2(x)}{g'(x)F(g)} y'(x) \\ &+ \frac{F^2(g) + 2F'^2(g) - F(g) F''(g)}{F^2(g)} g'^2(x) y(x) = 0. \end{aligned} \tag{3.2}$$

This is equivalent to the normal form of the second order equation

$$y''(x) + a(x) y'(x) + b(x) y(x) = 0$$

where

$$a(x) = -\frac{g''(x) F(g) + 2F'(g) g'^2(x)}{g'(x)F(g)} \text{ - the resistance coefficient,} \tag{3.3}$$

$$b(x) = \frac{F^2(g) + 2F'^2(g) - F(g) F''(g)}{F^2(g)} g'^2(x) \text{ - the cause.} \tag{3.4}$$

This infers the following:

The differential equation (3.1), comprising two arbitrary functions $g(x)$ and $F(x)$, under the conditions $g'(x) > 0,\ F(g(x)) > 0$ has oscillatory solutions given in (2.1), with equal but functional amplitudes $A = F(x)$.

4. Example - Special cases

I. Harmonic oscillations. If there is chosen

$$g(x) = nx,\ n \neq 0,\ g'(x) = n,\ g''(x) = 0,\ F(x) = Const. \neq 0,\ F'(x) = F''(x) = 0,$$

then there is directly $y''(x) + n^2 y(x) = 0$ and the solutions are $y_1 = \cos nx$, $y_2 = \sin nx$.

II. The equation of the oscillations with a single period, and the same but not constant amplitudes. Let there be $g(x) = x$, $g'(x) = 1$, $g''(x) = 0$. Then, (3.2) yields the following result:

The differential equation

$$y''(x) - \frac{2F'}{F} y'(x) + \frac{F^2 + 2F'^2 - FF''}{F^2} y(x) = 0 \tag{4.1}$$

where $F = F(x)$, has the oscillatory solutions

$$y_1 = F(x) \cos x,\ y_2 = F(x) \sin x \tag{4.2}$$

with the periods $T = 2k\pi$, and the amplitude $F(x) \neq 0$.

III. Nonharmonic oscillations with constant amplitudes. Let in (3.2) be $g = g(x)$, but $F = Const.$ Then, there is

$$y''(x) - \frac{g''(x)}{g'(x)} y'(x) + g'^2(x)\, y(x) = 0 \tag{4.3}$$

and it infers the following:

The equation (4.3) with two times differentiable coefficient $g(x)$, under the condition $g'(x) \neq 0$, has equally-amplitudal solutions

$$y_1 = \cos g(x),\ y_2 = \sin g(x),\ g'(x) \neq 0. \tag{4.4}$$

5. Canonical form for the general case of the equation (3.2)

Using the well-known Liouville formula

$$y(x) = e^{-\frac{1}{2}\int a(x)dx} z(x) \tag{5.1}$$

the canonical form for the equation (3.2) could be obtained, where z is a new unknown function.

After a short calculation there is

$$y(x) = e^{\frac{1}{2}\int \left[g''/g' + \frac{2F'(g)}{F(g)} g'(x)\right] dx} z(x)$$

$$y(x) = F(g)\sqrt{g'(x)}z(x) \tag{5.2}$$

Finally, there is

$$z''(x) + B(x)z(x) = 0 \tag{5.3}$$

where the coefficient is, in accordance with the fundamental theory of differential equations

$$\begin{aligned} B(x) =& b(x) - \frac{a'(x)}{2} - \frac{a^2(x)}{4} = B(F(g), g(x)) \\ =& \frac{F^2(g) + 2F'^2(g) - F(g)F''(g)}{F^2(g)} g'^2(x) \\ &+ \frac{1}{2}\left(\frac{g''(x)}{g'(x)} + \frac{2F'(g)}{F(g)}g'(x)\right)' - \frac{1}{4}\left(\frac{g''(x)}{g'(x)} + \frac{2F'(g)}{F(g)}g'(x)\right)^2. \end{aligned} \tag{5.4}$$

As the solution $y(x)$ is known from (2.1), where it is represented with (y_1, y_2), (5.1) gives

$$z = \frac{y}{F(g)\sqrt{g'(x)}}, \ g'(x) \neq 0, \ F(x) > 0$$

or

$$z_1 = \frac{\cos g(x)}{\sqrt{g'(x)}}, \ \ z_2 = \frac{\sin g(x)}{\sqrt{g'(x)}}. \tag{5.5}$$

It infers the following:

The canonical form of the differential equation (5.3) *with complex coefficient* $B = B(F(g), g(x))$, *has quadratural, that is exact solutions* (5.5).

There is also the following important and indicative:

The amplitude of the oscillatory solutions (5.3) *is*

$$A(x) = \frac{1}{\sqrt{g'(x)}} = F(g(x)), \ g'(x) \neq 0, \ F(g(x)) > 0 \tag{5.6}$$

and it depends exclusively on the cause of the oscillations $g(x)$.

6. Liouville-Besge type of quadratures

We research possibilities to solve some classes of the equation (1.1) by means of quadratures. This has been processed in detail in the monograph [3], where the complete exact solution of the equation (3.1) is obtained, and some generalizations in the form of integral-differential equations are given.

The simplest case of quadratural solution of the equation of the type (1.1) is the equation

$$y''(x) + \frac{1}{(\alpha x+\beta)^4} y(x) = 0$$

with the exact solution

$$y(x) = (\alpha x+\beta)\left[C_1 \cos\frac{1}{\alpha(\alpha x+\beta)} + C_2 \sin\frac{1}{\alpha(\alpha x+\beta)}\right]$$

whose amplitude is

$$A(x) = \alpha x + \beta.$$

This is an opportunity to check the formula (5.6). As the argument of the oscillations is

$$g(x) = \frac{1}{\alpha(\alpha x+\beta)}$$

and wherefrom

$$|g'(x)| = \frac{1}{\alpha}\left|\frac{-\alpha}{(\alpha x+\beta)^2}\right| = \frac{1}{(\alpha x+\beta)^2}$$

and

$$\sqrt{g'(x)} = \frac{1}{\alpha x+\beta};\ \alpha>0,\ \beta>0,\ x>0$$

the formula

$$A(x) = F(x) = \frac{1}{\sqrt{g'(x)}} = \alpha x+\beta$$

is correct in this case, that is the theoretically obtained formula (5.6) agrees with the simple quadratural solution.

If a generalization is made, then it could be proved that if $f(x)$ is a general dependency on x, not via $g(x)$.

There is a linear homogenous differential of the second order in the form

$$y''(x) - \left(\frac{g''(x)}{g'(x)} + \frac{2f'(x)}{f(x)}\right) y'(x) + b(x)\,y(x) = 0 \tag{6.1}$$

where $b(x)$ could be determined in a way similar to (3.4) or (5.4), with the oscillatory solutions

$$y_1 = f(x)\cos g(x),\, y_2 = f(x)\sin g(x). \tag{6.2}$$

The equation (6.1) will be canonical in a simplest case, if there is the following relation between the cause of the oscillations $g(x)$ and the strength (amplitude) of the oscillations $f(x)$

$$g(x) = \int \frac{dx}{f^2(x)},$$

what is the same as the formula (5.6). It is possible then to find general solution by means of quadratures as well, the solution being

$$y = f(x)\left[C_1 \cos \int \frac{dx}{f^2(x)} + C_2 \sin \int \frac{dx}{f^2(x)}\right]. \tag{6.3}$$

In this particular case, the equation (6.1) reads

$$y''(x) + \frac{1 - f^3(x) f''(x)}{f^4(x)} y(x) = 0. \tag{6.4}$$

In accordance with our previous results [6], if $1 - f^3(x) f''(x) > 0$, when $x \to \infty$ the equation (23) has infinite number of oscillations and Sturm's zeros in $[0, +\infty)$. If $1 - f^3(x) f''(x) < 0$, when $x \to +\infty$, then the solution has only a finite number of oscillations, since if starting from an $x \geqq x_0$ it is $\int \frac{dx}{f^2(x)} \to 0$, then it will be:

$$\cos \int \frac{dx}{f^2(x)} \to \cos 0 = 1, \text{ and } \sin \int \frac{dx}{f^2(x)} \to \sin 0 = 0.$$

This is the Prodi theorem [11], that one solution of the oscillatory equation is mandatory limited, while the other tends to zero.

For other questions of this subject see [1], [2], [8], [15] and [16].

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