Kragujevac Journal of Mathematics
Volume 35 Number 1 (2011), Pages 75–87.

AN APPLICATION OF SOFT SETS TO LATTICES

ENGLISH KUPPUSAMY RAMAMOORTHY NAGARAJAN [1] AND GANESAMOORTHI MEENAMBIGAI [2]

Abstract. Molodtsov introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainty. In this paper, we initiate the study of soft lattices by using soft set theory. The notions of soft lattices, soft distributive lattices, soft modular lattices, soft lattice ideals, soft lattice homomorphisms are introduced and several related properties and some characterization theorems are investigated.

1. Introduction

Because of various uncertainties arise in complicated problems in Economics, Engineering, Environmental Science, Medical Science and Social Science, methods of classical Mathematics may not be successfully used to solve them. Mathematical theories such as probability theory, fuzzy set theory and rough set theory were established by researchers to model uncertainties appearing in the above fields. But all these theories have their own difficulties. To overcome these difficulties, Molodtsov [2] introduced the concept of soft set as a new Mathematical tool for dealing with uncertainties. As the problem of setting the membership function does not arise in soft set theory, it can be easily applied to many different fields. Some operations on the soft set theory was studied by Maji et al. [3]. Irfan Ali et al. [4] studied some new operations on soft sets. Aktas and Cagman [5] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the concept of soft groups

Key words and phrases. Soft sets, Soft lattices, Soft distributive lattices, Soft modular lattices, Soft lattice ideals, Soft lattice homomorphisms.

2010 *Mathematics Subject Classification.* Primary: 06D50, Secondary: 06D72 .

Received: May 23, 2010.

Revised: February 14, 2011.

and derived some related properties. Feng Feng et al. [6] studied soft semi rings. Young Bae Jun [7] defined the notion of soft BCK/BCI algebras. In this paper, we deal with the algebraic structure on some collection of soft sets over a universe. We define the notions of soft lattices, soft distributive lattices and soft modular lattices. We focus on the algebraic properties of these notions. Also we give several illustrative examples. We define soft lattice homomorphism and obtain some properties. Further, we define soft lattice ideals and illustrate them by examples.

2. Some concepts in Lattices and Soft Sets

A nonempty set together with a partial order relation is called as a partially ordered set or a poset. A lattice L is a partially ordered set in which every pair of elements has the least upper bound ($\vee$) and the greatest lower bound ($\wedge$). It is denoted by $(L, \vee, \wedge)$. A nonempty subset R of L is said to be a sublattice of L if $a, b \in R$ implies $a \vee b$, $a \wedge b \in R$. A nonempty subset I of L is said to be an ideal of L if (i) $a, b \in I$ implies $a \vee b \in I$ and (ii) for any $a, b \in L$ such that $a \leq b$ and $b \in I$ implies $a \in I$. The dual concept of an ideal is that of a filter. A lattice $(L, \vee, \wedge)$ is said to be distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for all x, y, z in L. A lattice $(L, \vee, \wedge)$ is said to be modular if $x \leq z$ implies that $x \vee (y \wedge z) = (x \vee y) \wedge z$, for all y in L. A mapping f of a lattice $(L_1, \vee_1, \wedge_1)$ into a lattice $(L_2, \vee_2, \wedge_2)$ is said to be a lattice homomorphism if $f(a \vee_1 b) = f(a) \vee_2 f(b)$ and $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$, for all a, b in L_1. If such a mapping is one-one and onto, then it is called as a lattice isomorphism.

Result 2.1. A lattice is distributive if and only if, it has no sublattice isomorphic to N_5 or M_3.

Result 2.2. A lattice is modular if and only if, it has no sublattice isomorphic to N_5.

The notations, definitions and results in lattice theory used in this paper were extracted from [1].

Definition 2.1. Let U be a nonempty finite set of objects called Universe and let E be a nonempty set called parameters. An ordered pair (F, E) is said to be a Soft set over U, where F is a mapping from E into the set of all subsets of the set U. That is, $F : E \rightarrow \wp(U)$.

It has been interpreted that a soft set indeed is a parameterized family of subsets of U. The set of all Soft sets over U is denoted by $S(U)$.

Definition 2.2. Let (F, A) and (G, B) be two soft sets over the common universe U. We say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$,
(2) For all $e \in A$, $F(e) \subseteq G(e)$.

Maji et al. [3] introduced many binary operations such as intersection, union, AND-operation, and OR-operation on soft sets.

Definition 2.3. Let (F, A) and (G, B) be two soft sets over the common universe U. The union of two soft sets (F, A) and (G, B) is the soft set (H, C) where $C = A \cup B$ and H is defined as follows:

$$H(e) = \begin{cases} F(e), & \text{for } e \in A - B; \\ G(e), & \text{for } e \in B - A; \\ F(e) \cup G(e), & \text{for } e \in A \cap B. \end{cases}$$

Definition 2.4. Let (F, A) and (G, B) be two soft sets over the common universe U such that $A \cap B \neq \phi$. The intersection of (F, A) and (G, B) is the soft set (H, C) where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$, for all $e \in C$.

Definition 2.5. Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) AND (G, B) denoted by $(F, A) \bigwedge (G, B)$ and is defined by $(F, A) \bigwedge (G, B) = (H, A \times B)$ where $H((\alpha, \beta)) = F(\alpha) \cap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 2.6. Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) OR (G, B) denoted by $(F, A) \bigvee (G, B)$ and is defined by $(F, A) \bigvee (G, B) = (H, A \times B)$ where $H((\alpha, \beta)) = F(\alpha) \cup G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

In [4], Irfan Ali et al. defined the complement operator in soft sets as follows:

Definition 2.7. Let (F, A) be a soft set over U. Then the complement of (F, A) is defined as $F^c(e) = U - F(e)$, for all $e \in A$ and is denoted by (F^c, A).

Definition 2.8. The soft set (F, A) over U is said to be a null soft set if $F(e) = \emptyset$, for all $e \in A$ and is denoted by Φ .

Definition 2.9. The soft set (F, A) over U is said to be an absolute soft set if $F(e) = U$, for all $e \in A$ and is denoted by Ψ.

Lemma 2.1. *The operation "$\cup$" in $S(U)$ is idempotent and associative. That is, $(F,A)\cup(F,A)=(F,A)$ and $(F,A)\cup((G,B)\cup(H,C))=((F,A)\cup(G,B))\cup(H,C)$, for all $(F,A),(G,B)$ and $(H,C)\in S(U)$.*

Let U be a universal set and K be a nonempty collection of sets. The set of all soft sets over U with parameter set from K is denoted by $S_*(U)$.

Proposition 2.1. *$S_*(U)$ together with soft set union forms a join semi lattice.*

For each $E\in K$, the set of all soft sets over U with a fixed parameter E is denoted by $S_E(U)$.

Lemma 2.2. *In $S_E(U)$, the following holds*

(1) $(F,E)\bigcup(F,E)=(F,E)$ *and* $(F,E)\bigcap(F,E)=(F,E)$,
(2) $(F,E)\bigcup(G,E)=(G,E)\bigcup(F,E)$ *and* $(F,E)\bigcap(G,E)=(G,E)\bigcap(F,E)$,
(3) $(F,E)\bigcup((G,E)\bigcup(H,E))=((F,E)\bigcup(G,E))\bigcup(H,E)$,
(4) $(F,E)\bigcap((G,E)\bigcap(H,E))=((F,E)\bigcap(G,E))\bigcap(H,E)$,
(5) $(F,E)\bigcup((G,E)\bigcap(H,E))=((F,E)\bigcup(G,E))\bigcap((F,E)\bigcup(H,E))$,
(6) $(F,E)\bigcap((G,E)\bigcup(H,E))=((F,E)\bigcap(G,E))\bigcup((F,E)\bigcap(H,E))$,
(7) $(F,E)\bigcup((F,E)\bigcap(G,E))=(F,E)$,
(8) $(F,E)\bigcap((F,E)\bigcup(G,E))=(F,E)$,
(9) $(F,E)\bigcup\Phi=(F,E)$ *and* $(F,E)\bigcap\Phi=\Phi$,
(10) $(F,E)\bigcup\Psi=\Psi$ *and* $(F,E)\bigcap\Psi=(F,E)$,
(11) $(F,E)\bigcup(F^c,E)=\Psi$ *and* $(F,E)\bigcap(F^c,E)=\Phi$.

Proposition 2.2. *For each $E\in K$, $S_E(U)$ is a Boolean subalgebra of $S_*(U)$.*

3. Soft Lattices

In this section, we give the definition of soft lattices and some properties of soft lattices. Throughout this section, L is a lattice and A is any nonempty set. R will refer to an arbitrary binary relation between elements of A and elements of L. That is, $R\subseteq A\times L$. A set-valued function $F:A\to\wp(L)$ can be defined as $F(x)=\{y\in L/xRy\}$. The pair (F,A) is a soft set over L.

Definition 3.1. Let (F,A) be a soft set over L. Then (F,A) is said to be a *soft lattice* over L if $F(x)$ is a sublattice of L, for all $x\in A$.

The set of all soft lattices over L is denoted by $Sl(L)$.

Let us illustrate this definition using the following examples.

Example 3.1. Consider the lattice L as shown in Figure 1. Let $A = \{a, b, d\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \vee y = 1\}$. Then $F(a) = \{1, f\}, F(b) = \{1, e\}, F(d) = \{1, c, e, f\}$. Therefore, $F(x)$ is a sublattice of L, for all $x \in A$. Hence $(F, A) \in Sl(L)$.

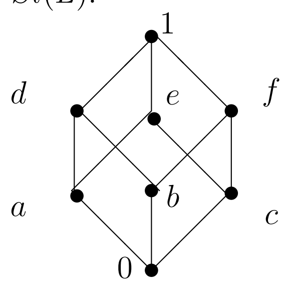

FIGURE 1.

Example 3.2. Consider the lattice L as shown in Figure 2. Let $A = \{0, a, b, c, d, 1\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \vee y = 1\}$. Then $F(0) = \{1\}, F(a) = \{1, b, c, d\}, F(b) = \{1, a, c, d\}, F(c) = \{1, a, b, d\}, F(d) = \{1, a, b, c\}, F(1) = \{0, a, b, c, d, 1\}$. In $F(a)$, $b, c \in F(a)$. But $b \wedge c = 0 \notin F(a)$. Therefore, $F(a)$ is not a sublattice of L. Similarly, $F(b), F(c), F(d)$ are not sublattices of L. Hence $(F, A) \notin Sl(L)$.

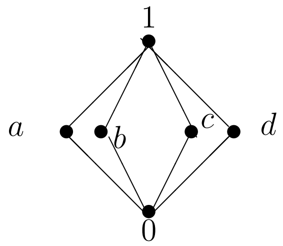

FIGURE 2.

The following example illustrates that, if (F, A) is not a soft lattice over L then there may exists a nonempty subset $B \subset A$ such that $(F/B, B)$ is a soft lattice over L.

Example 3.3. Consider the lattice L as shown in Figure 3. Let $A = \{0, a, b, c, d, 1\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = 0\}$. Then $F(0) = \{0, a, b, c, d, 1\}, F(a) = \{0, b, c, d\}, F(b) = \{0, a, c\}, F(c) = \{0, a, b\}, F(d) = \{0, a\}, F(1) = \{0\}$. Here, $F(b) = \{0, a, c\}$, $F(c) = \{0, a, b\}$ are not sublattices of L.

Therefore, $(F, A) \notin Sl(L)$. Take $B = \{0, a, d, 1\} \subset A$. Then $F_B(0) = \{0, a, b, c, d, 1\}$, $F_B(a) = \{0, b, c, d\}$, $F_B(d) = \{0, a\}$, $F_B(1) = \{0\}$. Therefore, $(F/B, B) \in Sl(L)$.

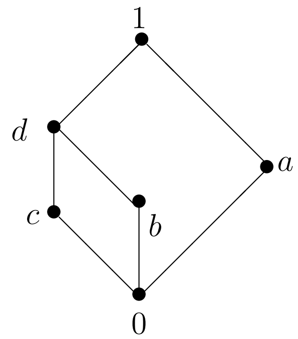

FIGURE 3.

Proposition 3.1. *Every lattice can be considered as a soft lattice.*

Proposition 3.2. *Let* $(F, A), (H, B) \in Sl(L)$ *be such that* $A \cap B \neq \emptyset$ *and* $F(e) \cap H(e) \neq \phi$ *for all* $e \in A \cap B$. *Then their intersection* $(F, A) \cap (H, B) \in Sl(L)$.

Proof. The intersection of two soft sets (F, A) and (H, B) is given by $(F, A) \cap (H, B) = (U, C)$ where $C = A \cap B \neq \emptyset$ and $U(e) = F(e) \cap H(e) \neq \emptyset$, for all $e \in C$. Let $x \in C = A \cap B$. Since $x \in A$ and $(F, A) \in Sl(L)$, $F(x)$ is a sublattice of L. Since $x \in B$ and $(H, B) \in Sl(L)$, $H(x)$ is a sublattice of L. By assumption, we have $U(x) = F(x) \cap H(x)$ is a sublattice of L. Since $x \in C$ is arbitrary, $U(x)$ is a sublattice of L, for all $x \in C$. Therefore, $(U, C) = (F, A) \cap (H, B) \in Sl(L)$. □

Proposition 3.3. *Let* $(F, A), (G, B) \in Sl(L)$. *If* $A \cap B = \emptyset$, *then* $(F, A) \cup (G, B) \in Sl(L)$.

Proof. The union of two soft sets (F, A) and (G, B) is given by $(F, A) \cup (G, B) = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & \text{for } e \in A - B; \\ G(e), & \text{for } e \in B - A; \\ F(e) \cup G(e), & \text{for } e \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, $A - B = A$, $B - A = B$, and either $e \in A$ or $e \in B$, for all $e \in A \cup B$.

$$H(e) = \begin{cases} F(e), & \text{for } e \in A; \\ G(e), & \text{for } e \in B. \end{cases}$$

Since $(F, A) \in Sl(L)$, $F(e)$ is a sublattice of L, for all $e \in A$. Since $(G, B) \in Sl(L)$, $G(e)$ is a sublattice of L, for all $e \in B$. Thus $H(e)$ is a sublattice of L, for all $e \in C$. Therefore, $(H, C) \in Sl(L)$. That is, $(F, A) \cup (G, B) \in Sl(L)$. □

Proposition 3.4. *Let $(F, A), (G, B) \in Sl(L)$ be such that $F(x) \cap G(y) \neq \phi$, for all $x \in A$, $y \in B$. Then $(F, A) \bigwedge (G, B) \in Sl(L)$.*

Proof. The AND operation of two soft sets (F, A) and (G, B) is given by $(F, A) \wedge (G, B) = (H, C)$ where $C = A \times B$ and $H((x, y)) = F(x) \cap G(y) \neq \phi$, for all $x \in A$, $y \in B$. Since $(F, A) \in Sl(L)$, $F(x)$ is a sublattice of L, for all $x \in A$. Since $(G, B) \in Sl(L)$, $G(y)$ is a sublattice of L, for all $y \in B$. Since $F(x) \cap G(y) \neq \phi$, it is a sublattice of L, for all $x \in A$, $y \in B$. That is, $H((x, y))$ is a sublattice of L for all $x \in A$, $y \in B$. Therefore, $(H, C) \in Sl(L)$. That is, $(F, A) \bigwedge (G, B) \in Sl(L)$. □

Definition 3.2. Let (F, A) and (H, K) be two soft lattices over L. Then (H, K) is a *soft sublattice* of (F, A) if

(1) $K \subseteq A$,
(2) $H(x)$ is a sublattice of $F(x)$, for all $x \in K$.

Let us illustrate this definition with the following examples.

Example 3.4. Consider the lattice L as shown in Figure 3. Let $A = \{0, a, b, c, d, 1\}$. Let $K = \{0, b, c, d\}$. Define the set-valued function $F : A \to \wp(L)$ by $F(x) = \{y \in L : xRy \Leftrightarrow x \vee y = x\}$. Then $F(0) = \{0\}, F(a) = \{0, a\}, F(b) = \{0, b\}, F(c) = \{0, c\}, F(d) = \{0, b, c, d\}, F(1) = \{0, a, b, c, d, 1\}$. Define the set-valued function $H : K \to \wp(L)$ by $H(x) = \{y \in L : xRy \Leftrightarrow x \vee y = x, x \in K\}$. Then $H(0) = \{0\}, H(b) = \{0, b\}, H(c) = \{0, c\}, H(d) = \{0, b, c, d\}$. Therefore, $(F, A), (H, K) \in Sl(L)$. Here $K \subseteq A$ and $H(x)$ is a sublattice of $F(x)$, for all $x \in K$. Therefore, (H, K) is a soft sublattice of (F, A).

Proposition 3.5. *Let $(F, A), (H, A) \in Sl(L)$. Then (F, A) is a soft sublattice of (H, A) if and only if $F(x) \subseteq H(x)$, for all $x \in A$.*

Proof. Let us assume that (F, A) is a soft sublattice of (H, A). Then $F(x) \subseteq H(x)$, for all $x \in A$. Conversely, let $F(x) \subseteq H(x)$, for all $x \in A$. Since $(F, A) \in Sl(L)$, $F(x)$ is a sublattice of L, for all $x \in A$. Since $(H, A) \in Sl(L)$, $H(x)$ is a sublattice of L, for all $x \in A$. Therefore, $F(x)$ becomes a sublattice of $H(x)$, for all $x \in A$. Also $A \subseteq A$. Thus (F, A) is a soft sublattice of (H, A). □

Corollary 3.1. *Every soft lattice is a soft sublattice of itself. That is, if (F, A) is a soft lattice over L, then (F, A) is a soft sublattice of (F, A).*

Definition 3.3. Let (F, A) be a soft lattice over L_1. Let f be a lattice homomorphism from L_1 to L_2. Then $(f(F))(x) = f(F(x))$, for all $x \in A$.

Proposition 3.6. *Let (F, A) and (H, B) be soft lattices over L_1 such that (F, A) be a soft sublattice of (H, B). If f is a homomorphism from L_1 to L_2, then $(f(F), A)$ is a soft sublattice of $(f(H), B)$.*

Proof. Given (F, A) is a soft sublattice of (H, B), then $A \subseteq B$ and $F(x)$ is a sublattice of $H(x)$, for all $x \in A$. Since f is a homomorphism from L_1 to L_2 and homomorphic image of a sublattice in L_1 is a sublattice in L_2, we have $f(F(x))$ and $f(H(y))$ are sublattices of L_2, for all $x \in A$, $y \in B$. Also, $f(F(x))$ is a sublattice of $f(H(x))$, for all $x \in A$. Hence $(f(F), A)$ is a soft sublattice of $(f(H), B)$. □

Definition 3.4. Let (F, A) and (H, B) be two soft lattices over L_1 and L_2 respectively. Let $f : L_1 \to L_2$ and $g : A \to B$. Then (f, g) is said to be a *soft lattice homomorphism* if

(1) f is a lattice homomorphism from L_1 onto L_2,
(2) g is a mapping from A onto B,
(3) $f(F(x)) = H(g(x))$, for all $x \in A$.

Then (F, A) is said to be a soft lattice homomorphic to (H, B) and it is denoted by $(F, A) \sim (H, B)$.

If f is a lattice isomorphism from L_1 onto L_2 and g is a bijection from A to B, then (f, g) is said to be a *soft lattice isomorphism*. (F, A) is soft lattice isomorphic to (H, B) and it is denoted by $(F, A) \simeq (H, B)$.

Example 3.5. Consider the lattice L_1 and L_2 as shown in Figure 4 and Figure 5 respectively. Let $A = \{0, a, b, 1\}$ and $B = \{0', 1'\}$. Define the set-valued function F by $F(x) = \{y \in L_1 : xRy \Leftrightarrow x \vee y = x, x \in A\}$. Then, $(F, A) \in Sl(L_1)$. Define the set-valued function H by $H(x) = \{y \in L_2 : xRy \Leftrightarrow x \vee y = x, x \in B\}$. Then $(H, B) \in Sl(L_2)$. Define $f : L_1 \to L_2$ by $f(0) = 0', f(a) = 0', f(b) = 1', f(1) = 1'$. Define $g : A \to B$ by $g(0) = 0', g(a) = 0', g(b) = 1', g(1) = 1'$. Then f is a lattice homomorphism from L_1 onto L_2 and g is a mapping from A onto B. Also $f(F(x)) = H(g(x))$, for all $x \in A$. Hence (F, A) is a soft lattice homomorphic to (H, B).

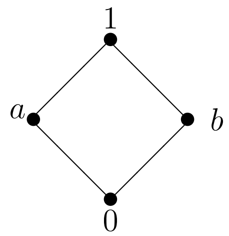

FIGURE 4.

FIGURE 5.

Definition 3.5. Let (F, A) be a soft set over L. Then (F, A) is said to be a *soft distributive lattice* over L if $F(x)$ is a distributive sublattice of L, for all $x \in A$.

Example 3.6. Consider the lattice L as shown in Figure 6. Let $A = \{a, b, c\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \vee y = x\}$. Then $F(a) = \{0, a\}, F(b) = \{0, a, b\}, F(c) = \{0, a, b, c\}$. Here $F(x)$ is a distributive sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft distributive lattice over L.

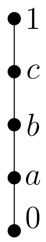

FIGURE 6.

Proposition 3.7. *Let L be a distributive lattice. Then every $(F, A) \in Sl(L)$ is a soft distributive lattice over L.*

Proof. Let $(F, A) \in Sl(L)$. Then $F(x)$ is a sublattice of L, for all $x \in A$. Since L is distributive and every sublattice of a distributive lattice is distributive, $F(x)$ is a distributive sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft distributive lattice over L. □

Remark 3.1. The converse of the above theorem is not true. That is, if (F, A) is a soft distributive lattice over L then L need not be a distributive lattice.

The following example illustrates this remark.

Example 3.7. Consider the lattice L as shown in Figure 2. Let $A = \{a, b, c, d\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = y\}$. Then $F(a) = \{0, a\}, F(b) = \{0, b\}, F(c) = \{0, c\}, F(d) = \{0, d\}$. Here $F(x)$ is a distributive sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft distributive lattice over L. But L is not a distributive lattice, since L has a sublattice isomorphic to M_3.

Theorem 3.1. *A soft lattice (F, A) over L is distributive if and only if $F(x)$ has no sublattice isomorphic to N_5 or M_3, for all $x \in A$.*

Theorem 3.2.

(1) *Let (F, A) be a soft distributive lattice over L and (H, B) be a soft sublattice of (F, A). Then (H, B) is a soft distributive lattice over L.*

(2) *Let (F, A) be a soft distributive lattice over L_1. Let (H, B) be a soft lattice homomorphic image of (F, A) over L_2. Then (H, B) is a soft distributive lattice over L_2.*

Proof. (1) Given (H, B) is a soft sublattice of (F, A) over L. Then $B \subseteq A$ and $H(x)$ is a sublattice of $F(x)$, for all $x \in B$. Since (F, A) is a soft distributive lattice over L, $F(x)$ is a distributive sublattice of L, for all $x \in A$. Since $H(x)$ is a sublattice of $F(x)$, $H(x)$ is also a distributive sublattice of L, for all $x \in B$. Therefore, (H, B) is a soft distributive lattice over L.

(2) Given (H, B) is a soft lattice homomorphic image of (F, A). Then there is a soft lattice homomorphism (f, g) where $f : L_1 \to L_2$, a lattice homomorphism and g is a map from $A$to B such that $f(F(x)) = H(g(x))$, for all $x \in A$. Since (F, A) is a soft distributive lattice, $F(x)$ is a distributive sublattice of L_1, for all $x \in A$. Since f is a homomorphism, $f(F(x))$ is also a distributive sublattice of L_2, for all $x \in A$. That is, $H(g(x))$ is a distributive sublattice of L_2, for all $x \in A$. Since g is onto, for all $y \in B$, there exists $x \in A$ such that $g(x) = y$. Therefore, $H(y)$ is a distributive sublattice of L_2, for all $y \in B$. Hence (H, B) is a soft distributive lattice over L_2. □

Definition 3.6. Let (F, A) be a soft set over L. Then (F, A) is said to be a *soft modular lattice* over L if $F(x)$ is a modular sublattice of L, for all $x \in A$.

Example 3.8. Consider the lattice L as shown in figure 1. Let $A = \{0, a, b, c, 1\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = 0\}$. Then $F(0) = L, F(a) = \{0, b, c, f\}, F(b) = \{0, a, c, e\}, F(c) = \{0, a, b, d\}, F(1) = \{0\}$. Here $F(x)$ is a modular sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft modular lattice over L.

Proposition 3.8. *Let L be a modular lattice. Then every $(F, A) \in Sl(L)$ is a soft modular lattice over L.*

Proof. Let $(F, A) \in Sl(L)$. Then $F(x)$ is a sublattice of L, for all $x \in A$. Since L is modular and every sublattice of a modular lattice is modular, $F(x)$ is a modular sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft modular lattice over L. □

Remark 3.2. The converse of the above theorem is not true. That is, if (F, A) is a soft modular lattice over L then L need not be a modular lattice.

The following example illustrates this remark.

Example 3.9. Consider the lattice L as shown in Figure 3. Let $A = \{a, b, c, d\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = y\}$. Then $F(a) = \{0, a\}, F(b) = \{0, b\}, F(c) = \{0, c\}, F(d) = \{0, b, c, d\}$. Here $F(x)$ is a modular sublattice of L, for all $x \in A$. Therefore, (F, A) is a soft modular lattice over L. But L is not a modular lattice, since L has a sublattice isomorphic to N_5.

Theorem 3.3. *A soft lattice (F, A) over L is modular if and only if $F(x)$ has no sublattice isomorphic to N_5, for all $x \in A$.*

Theorem 3.4.

(1) *Let (F, A) be a soft modular lattice over L and (H, B) be a soft sublattice of (F, A). Then (H, B) is a soft modular lattice over L.*

(2) *Let (F, A) be a soft modular lattice over L_1. Let (H, B) be a soft lattice homomorphic image of (F, A) over L_2. Then (H, B) is a soft modular lattice over L_2.*

Proof. (1) Given (H, B) is a soft sublattice of (F, A) over L. Then $B \subseteq A$ and $H(x)$ is a sublattice of $F(x)$, for all $x \in B$. Since (F, A) is a soft modular lattice over L, $F(x)$ is a modular sublattice of L, for all $x \in A$. Since $H(x)$ is a sublattice of $F(x)$, $H(x)$ is also a modular sublattice of L, for all $x \in B$. Therefore, (H, B) is a soft modular lattice over L.

(2) Given (H, B) is a soft lattice homomorphic image of (F, A). Then there is a soft lattice homomorphism (f, g) where $f : L_1 \to L_2$, a lattice homomorphism and g is a map from A to B such that $f(F(x)) = H(g(x))$, for all $x \in A$. Since (F, A) is a soft modular lattice, $F(x)$ is a modular sublattice of L_1, for all $x \in A$. Since f is a homomorphism, $f(F(x))$ is also a modular sublattice of L_2, for all $x \in A$. That is, $H(g(x))$ is a modular sublattice of L_2, for all $x \in A$. Since g is onto, for all $y \in B$, there exists $x \in A$ such that $g(x) = y$. Therefore, $H(y)$ is a modular sublattice of L_2, for all $y \in B$. Hence (H, B) is a soft modular lattice over L_2. □

Definition 3.7. Let (F, A) be a soft lattice over L. A soft set (G, B) over L is called a *soft lattice ideal* of (F, A) if

(1) $B \subseteq A$,
(2) $G(x)$ is an ideal of $F(x)$, for all $x \in B$.

Let us illustrate this definition using the following example.

Example 3.10. Consider L as shown in Figure 7. Let $A = \{0, a, b, c, 1\}$. Define the set-valued function F by $F(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = y\}$. Then $F(0) = \{0\}, F(a) = \{0, a\}, F(b) = \{0, a, b\}, F(c) = \{0, c\}, F(1) = \{0, a, b, c, 1\}$. Therefore, $(F, A) \in Sl(L)$. Let $B = \{a, b, c\}$. Define the set-valued function G by $G(x) = \{y \in L : xRy \Leftrightarrow x \wedge y = y\}$. Then $G(a) = \{0, a\}, G(b) = \{0, a, b\}, G(c) = \{0, c\}$. Therefore, (G, B) is a soft set over L. Here $B \subset A$. Also $G(x)$ is an ideal of $F(x)$, for all $x \in B$. Therefore, (G, B) is a soft lattice ideal of (F, A).

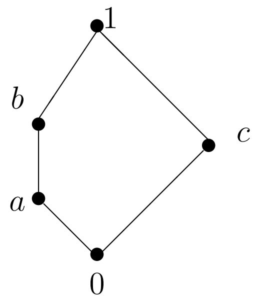

FIGURE 7.

4. CONCLUSION

In this paper, we have introduced the concept of soft lattices and have studied some of their algebraic properties. Soft distributive lattices and soft modular lattices were also introduced and characterization theorems for them have been obtained. Soft lattice ideals and soft lattice homomorphism were also introduced and their properties have been studied. An interesting topic for further study is to discuss the possible applications of ideas and methods developed in this paper.

REFERENCES

[1] B. A. Davey, H. A. Priestley, *Introduction to Lattices and order*, Cambridge University Press, India, 2009.
[2] D. Molodtsov, *Soft set theory - First results*, Comput. Math. Appl. **37** (1999) 19-31.
[3] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003) 555-562.

[4] M. Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. **57** (2009) 1547-1553.
[5] H. Aktas, N. Cagman, *Soft sets and soft groups*, Inform. Sci. **177** (2007) 2726-2735.
[6] F. Feng, Y. B. Jun, Xianzhong Zhao, *Soft semirings*, Comput. Math. Appl. **56** (2008) 2621-2628.
[7] Y. B. Jun, *Soft BCK / BCI - algebras*, Comput. Math. Appl., **56** (2008) 1408-1413.

[1] Centre for Research and Post Graduate Studies in Mathematics,
Ayya Nadar Janaki Ammal College (Autonomous), Sivakasi - 626 124,
Tamil Nadu, India.
E-mail address: `nagarajanekr@yahoo.com`

[2] Department of Mathematics,
S. V. College of Engineering and Technology, Puliangudi - 627 855,
Tamil Nadu, India.
E-mail address: `meena.moorthi@gmail.com`